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Dedicated to the memory of Professor G. M. Goluzin
(1906-1952)

ON THE GOODMAN CONJECTURE AND RELATED FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. The principal coefficient problem for p-valent functions, the Goodman conjecture, is considered for polynomial compositions. In this case, the problem is reduced to a coefficient conjecture for functions of several complex variables related to univalent functions. The proof is based on the Lyzzaik–Styer determinant theorem. Some advantages of the equivalent conjecture are discussed.

§1. Introduction

A function $f(z)$ which is regular or meromorphic in a region B is said to be p-valent in B ($p \in \mathbb{N}$) if the equation $f(z) = w$ has at most p roots in B for each complex w. Let V_p be the class of functions $f(z)$, $f(0) = 0$, that are regular and p-valent in the unit disk $D : |z| < 1$.

It has been known since 1936 that the following bound holds true for any natural numbers p and n and any function f in V_p having at most q zeros ($q \le p$):

$$|\{f\}_n| \le C(p) \sum_{m=1}^{q} |\{f\}_m| n^{2p-1}; \qquad (1)$$

here $C(p)$ depends only on p. Here and in what follows $\{f\}_n$ denotes the nth Taylor coefficient of a function f about 0. The order of magnitude occurring in (1), which is due to Littlewood [13] for $p = 1$ and to Biernacki [3; 4, Chapter 1] for $p \ge 2$, is best possible. It took nearly 70 years to show (with the participation of many eminent analysts of our century) that for $p = 1$ (the case of univalent functions) the number $C(p)$ in (1) can be replaced by 1. In this case, equality in (1) occurs if and only if $f(z)/\{f\}_1$ is the Koebe function $K(z) = z/(1-z)^2$ or one of its rotations. Concerning the famous Bieberbach conjecture on the Taylor coefficients of univalent functions (1916) and its proof, see de Branges's paper [5] and also [2]. As for sharp coefficient estimates for $p \ge 2$, the problem appears to be substantially more complicated and

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remains unsolved. In the present paper we consider the principal coefficient conjecture for p-valent functions in the case of polynomial compositions, as well as some related results and ideas.

§2. Some results of Goluzin

Let $\sum_p$ be the class of functions $F(z) = z^{-p}(1 + \alpha_1 z + \dots)$, $F(z) \neq 0$, that are meromorphic and p-valent in D. In 1940, Goluzin [7] proved the following coefficient inequality:

$$\sum_{n=1}^{\infty}(n - \lambda p)|\{[z^p F(z)]^\lambda\}_n|^2 \leq \lambda p \tag{2}$$

for every function $F \in \sum_p$ $(p \in \mathbb{N})$ and any $\lambda > 0$.

This result, a nice generalization of the Prawitz inequality for univalent functions (1927/28), is one of the important contributions to the theory of multivalent functions. The case where $\lambda = 1$ is known as the Goluzin area theorem for p-valent functions. For $p = 1$, this theorem coincides with the classic outer area theorem (1914) (see the books by Goluzin [8], Lebedev [12], and Milin [16] for more details).

With the help of inequality (2), Goluzin established sharp estimates for the initial coefficients in the class $\sum_p$ $(p \geq 2)$. On this basis, he obtained similar estimates in the class S_p consisting of the normalized functions f in V_p, $\{f\}_p = 1$, that have a zero of multiplicity p at the origin. In particular, Goluzin proved that

$$|\{f\}_{p+1}| \leq 2p \tag{3}$$

for all $p \geq 2$, provided that $f(z) = z^p + \{f\}_{p+1}z^{p+1} + \dots \in S_p$. Several years later, inequalities (1), the statement of the Bieberbach conjecture, and estimate (3) helped Goodman to formulate his principal coefficient conjecture for p-valent functions.

Equality in (3) is realized only by the function K^p and its rotations. We note that this function also plays an extremal role in other Goluzin's estimates for p-valent functions (see [7; 8, Chapter 1]). In the papers [17] by Spencer and [1] by Alenitsyn, inequality (3) and generalizations of the Goluzin area theorem were studied for mean p-valent functions.

§3. The Goodman conjecture

In 1948, Goodman conjectured (see [9]) that the sharp upper bound for $|\{f\}_n|$ for a p-valent function $f \in V_p$ can be expressed as a certain linear combination of the first p coefficients, namely,

$$|\{f\}_n| \leq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|\{f\}_k| \tag{4}$$

for $n > p$.

Later, Goodman showed that, if correct, this upper bound is sharp for every nonzero collection $\{|\{f\}_1|, \cdots, |\{f\}_p|\}$, where $f(z)$ is a polynomial in $K(z)$ of degree p. The Goodman conjecture is a generalization of the Bieberbach conjecture for $p \geq 2$. Clearly, if $\{f\}_1 = \dots = \{f\}_{p-1} = 0$ and $n = p+1$, then (4) coincides with the Goluzin estimate (3). Until the present time, even the simplest general case of (4) with $p = 2$ and $n = 3$ has been neither proved nor disproved. However, some supporting evidence has been collected for the Goodman conjecture (the details and references can be found in [11] and [14]). Lyzzaik and Styer [15] studied the Goodman conjecture in the special case of polynomials in univalent functions. For such compositions, they showed that the conjecture is equivalent to a collection of determinant inequalities for the coefficients of powers of normalized univalent functions. We use the result from [15] in the present paper to reduce the problem to some coefficient conjecture for functions of several complex variables. Then we discuss some advantages of our equivalent conjecture.

It should be mentioned that Goodman also proposed another improvement of bounds (1) for the class V_p. This approach involved the location of the zeros of p-valent functions (see [10]).

§4. The Lizzaik–Styer determinant theorem

Let M_p be the set of all functions $f \in V_p$ such that $f = P \circ \varphi$, where P is a polynomial of degree at most p and φ is a function of class $S = S_1$; see [15]. An attempt to study the Goodman conjecture merely for polynomial compositions $f \in M_p$,

$$f(z) = \sum_{m=1}^{p} b_m \varphi^m(z), \tag{5}$$

demonstrates the difficulty of the problem. The desirable result is far from being a consequence of the truth of the Bieberbach conjecture, unless f is a monomial ($b_1 = \ldots = b_{p-1} = 0$, $b_p \neq 0$ in (5)). We denote

$$E(\varphi, p, k, n) = \begin{vmatrix} \{\varphi^k\}_n & \{\varphi^{k+1}\}_n & \cdots & \{\varphi^p\}_n \\ \{\varphi^k\}_{k+1} & \{\varphi^{k+1}\}_{k+1} & \cdots & \{\varphi^p\}_{k+1} \\ \vdots & \vdots & & \vdots \\ \{\varphi^k\}_p & \{\varphi^{k+1}\}_p & \cdots & \{\varphi^p\}_p \end{vmatrix}, \tag{6}$$

where $\varphi \in S$ and $1 \leq k \leq p < n$. For $k = p$ we have $E(\varphi, p, k, n) = \{\varphi^p\}_n$. If $\varphi = K$, then

$$|E(\varphi, p, k, n)| = \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}.$$

(see Goodman [9]).

Theorem 1 [15]. *Let $n, p \in \mathbb{N}$, $n > p$. The following assertions are equivalent.*

a) $|E(\varphi, p, k, n)| \leq |E(K, p, k, n)|$ *for every $\varphi \in S$ and every integer k, $1 \leq k \leq p$, where E is defined by* (6).

b) *The Goodman conjecture* (4) *is true for all functions $f \in M_p$.*

§5. An equivalent conjecture for functions of several complex variables

Theorem 2. *The following assertions are equivalent.*

a) *The Koebe function* $K(z)$ *maximizes the modulus of the coefficient of each term of the form*

$$z_p^n \prod_{m=1}^{p-1} z_m^{k+m} \quad (p+k \leq n,\ n \in \mathbb{N})$$

of the Taylor series about $(0, \dots, 0)$ *in the class of all functions*

$$\Phi(z_1, \dots, z_p) = \prod_{m=1}^{p} \varphi^k(z_m) \prod_{1 \leq \nu < \mu \leq p} [\varphi(z_\mu) - \varphi(z_\nu)] \qquad (z_m \in D; m = 1, \dots, p), \tag{7}$$

where $p, k \in \mathbb{N}$ *and* $\varphi(z)$ *is a function of class* S.

b) *The Goodman conjecture* (4) *is true for all polynomial compositions* $f \in \bigcup_{p \in N} M_p$.

Proof. Let $\varphi \in S$ and $n, p, k \in \mathbb{N}(n \geq p+k)$. We assume that $p \geq 2$. Then (6) implies that $E(\varphi, k+p-1, k, n)$ is equal to the sum of all terms of the form

$$(-1)^{I+p-1}\{\varphi^{i_1}\}_{k+1} \dots \{\varphi^{i_{p-1}}\}_{k+p-1}\{\varphi^{i_p}\}_n,$$

where $(i_1, \dots, i_p)$ is a permutation of the indices $(k, \dots, k+p-1)$, and I is the number of inversions of the permutation $k \to i_1, \dots, k+p-1 \to i_p$.

Using the variables $z_1, \dots, z_p \in D$, we get

$$E(\varphi, k+p-1, k, n) = \left\{ \sum_{(i_1, \dots, i_p)} (-1)^{I+p-1} \prod_{m=1}^{p} \varphi^{i_m}(z_m) \right\}_{k+1, \dots, k+p-1, n},$$

where the notation $\{F(z_1, \dots, z_p)\}_{n_1, \dots, n_p}$ stands for the coefficient of the term $\prod_{m=1}^{p} z_m^{n_m}$ of the Taylor series about $(0, \dots, 0)$ of a function F of p complex variables $z_1, \dots, z_m$.

From (8) we deduce that

$$|E(\varphi, k+p-1, k, n)| = \left| \left\{ \prod_{m=1}^{p} \varphi^k(z_m) \begin{vmatrix} 1 & \varphi(z_1) & \dots & \varphi^{p-1}(z_1) \\ \vdots & \vdots & & \vdots \\ 1 & \varphi(z_{p-1}) & \dots & \varphi^{p-1}(z_{p-1}) \\ 1 & \varphi(z_p) & \dots & \varphi^{p-1}(z_p) \end{vmatrix} \right\}_{k+1, \dots, k+p-1, n} \right|.$$

Hence, using (7) and the formula for the Vandermonde determinant, we obtain

$$|E(\varphi, k+p-1, k, n)| = |\{\Phi(z_1, \dots, z_p)\}_{k+1,\dots,k+p-1,n}| \quad (p \geq 2).$$

If $p = 1$, then $\Phi = \varphi^k$ and $|\{\Phi\}_n| = |E(\varphi, k, k, n)|$.

Now, Theorem 1 allows us to complete the proof of Theorem 2. •

It is easily seen that the extremal property of the Koebe function mentioned in Theorem 2 (if valid) is much deeper than the statement of the Bieberbach conjecture. So, we do not expect that the approaches developed for $p = 1$ are still effective for $p \geq 2$. At the same time, the above coefficient problem for the functions (7) is more convenient for investigation with the help of the traditional methods than the conjecture (4) for polynomial compositions or condition (a) in Theorem 1.

i) Local properties of the Koebe function in the coefficient space. Let $\varphi(z) = K(z) + \delta(z)$, and let $r \in (0,1)$; it is assumed that $\delta(z)$ is small for $|z| \leq r$. Also, let $p, k \in \mathbb{N}$, $|z_m| \leq r$, $K_m = K(z_m)$, and $\delta_m = \delta(z_m)$ $(m = 1, \dots, p)$. Equation (7) implies that

$$\Phi(z_1, \dots, z_p) = \prod_{m=1}^{p} K_m^k \prod_{1 \leq \nu < \mu \leq p} (K_\mu - K_\nu)[1 + \Phi_1 + \Phi_2] + o\Big(\sum_{m=1}^{p} |\delta_m|^2\Big),$$

where

$$\begin{aligned} \Phi_1 &= \Phi_1(z_1, \dots, z_p) \\ &= k \sum_{m=1}^{p} \frac{\delta_m}{K_m} + \sum_{1 \leq \nu < \mu \leq p} \frac{\delta_\mu - \delta_\nu}{K_\mu - K_\nu} \\ &= O\Big(\sum_{m=1}^{p} |\delta_m|\Big), \end{aligned}$$

and

$$\begin{aligned} \Phi_2 &= \Phi_2(z_1, \dots, z_p) \\ &= k^2 \sum_{1 \leq \nu < \mu \leq p} \frac{\delta_\mu \delta_\nu}{K_\mu K_\nu} + \frac{k(k-1)}{2} \sum_{m=1}^{p} \frac{\delta_m^2}{K_m^2} \\ &\quad + k \sum_{m=1}^{p} \sum_{1 \leq \nu < \mu \leq p} \frac{\delta_m}{K_m} \left(\frac{\delta_\mu - \delta_\nu}{K_\mu - K_\nu} \right) \\ &\quad + \frac{1}{2} \sum_{\substack{1 \leq \nu < \mu \leq p \\ 1 \leq \nu' < \mu' \leq p \\ (\nu,\mu) \neq (\nu',\mu')}} \left(\frac{\delta_\mu - \delta_\nu}{K_\mu - K_\nu} \right) \left(\frac{\delta_{\mu'} - \delta_{\nu'}}{K_{\mu'} - K_{\nu'}} \right) \\ &= O\Big(\sum_{m=1}^{p} |\delta_m|^2\Big). \end{aligned}$$

This representation and Theorem 2 allow us to investigate the Goodman conjecture for polynomial compositions locally for a given variation of the Koebe function. For the details concerning univalent variations of the Koebe function (in the case where $p = k = 1$), see, e.g., [16, Chapter 3]. Also, see Gel'fer's paper [6] on a variational approach to the coefficient problem for p-valent functions.

ii) Coefficient estimates for the functions (7) with φ representable in the form of a Stieltjes integral. Goluzin (see [8, Chapter 11]) and other authors considered various classes of analytic functions $\varphi(z)$ described in terms of a Stieltjes integral

$$\int_a^b g(z,t)d\alpha(t) \quad (z \in D),$$

where a, b, and $g(z,t)$ are fixed, and $\alpha(t)$ is a monotone nondecreasing function on the interval $[a, b]$, with $\alpha(b) - \alpha(a) = 1$ (for instance). We say that these functions φ are of G-S type.

As is well known, the statement of the Bieberbach conjecture was first verified for certain functions of G-S type (starlike functions (1921), typically-real functions (1931–1932), etc. (the proofs can be found, e.g., in [8, Chapter 4]) and then, many years later, for the class S (1984). The reason is that the extremal properties of the Koebe function can be derived for certain functionals on a set of monotonic functions with the help of basic analysis only.

From this point of view, the above Taylor coefficients of the functions (7) seem to be suitable functionals, provided that each φ belongs to a given set of G-S type functions containing K. Here functions φ are not necessarily univalent. So, the coefficient condition (a) in Theorem 2 might be used for testing the Goodman conjecture on the functions of G-S type like it was done in the case of the Bieberbach conjecture.

iii) Reduction of the problem to that in terms of some logarithmic coefficients. Univalent functions satisfy a number of logarithmic inequalities of geometric nature, which are effective in applications (see the books [8, 12], and [16] for details). However, it is very difficult to relate these inequalities to conjecture (4) (in the polynomial case), or to condition (a) in Theorem 1. At the same time, equation (7) with $\varphi \in S$ can be rewritten in the following exponential form

$$\Phi(z_1, \dots, z_p) = \prod_{m=1}^{p} z_m^k \prod_{1 \le \nu < \mu \le p} (z_\mu - z_\nu) \exp\left\{k \sum_{m=1}^{p} \log \frac{\varphi(z_m)}{z_m} + \sum_{1 \le \nu < \mu \le p} U(z_\mu, z_\nu)\right\},$$

where

$$U(z,\zeta) = \log \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}.$$

Theorem 2 shows that the logarithmic and the Grunsky coefficients (the Taylor coefficients of $U(z,\zeta)$) of a univalent function φ might be used for reducing the Goodman conjecture for polynomial compositions to a "logarithmic" problem in a natural way. Actually, we mean a possible extension of Milin's exponentiation theory [16; Chapters 2, 3] to the case of $p \ge 2$ variables. We remind the reader that the exponential approach turned out to be of major importance in the proof of the Bieberbach conjecture (see [5]).

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