Controllability and Vector Potential [1]

Abstract: Kalman's fundamental notion of a controllable state space system, first described in Moscow [1], has been generalised to systems defined by higher order ordinary differential operators by J.C.Willems [8], and further to distributed systems [5]. It turns out that controllability is now identical to the notion of a vector potential in physics, or of vanishing homology in mathematics. These lectures will explain this relationship, and a few of its consequences.

Lecture 1. The solvability question for systems of partial differential equations: the Fundamental Principal of Malgrange and Palamadov [2,4]. The question dual to the solvability question.

Controllability for state space systems; its generalisation to distributed systems given as kernels of differential operators defined over the ring $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$ of constant coefficient pde; the functor $\mathsf{Hom}_A(-, \mathcal{F})$, where $\mathcal{F}$ is a space of distributions on $\mathbb{R}^n$; the description of the A-module structure of $\mathcal{D}'$, the space of distributions on $\mathbb{R}^n$, and of $\mathcal{C}^\infty$, $\mathcal{S}'$ etc.

Lecture 2. The torsion free condition for controllability; the functor $\mathsf{Hom}_A(M, -)$, i.e. dependence on $\mathcal{F}$ (Lecture 6 will continue with this theme); a calculus of kernels, the elimination problem

Lecture 3. The Popov-Belevitch-Hautus test; the Zariski topology on the set of all systems, genericity questions [6]; example of impulse controllable systems.

Lecture 4. Cohen-Macaulay rings, and the length of a generic maximal regular sequence in A; a generic under-determined system is controllable, whereas the opposite is true for over-determined systems.

Lecture 5. Discrete systems defined by partial difference equations on the lattice $\mathbb{Z}^n$, i.e. over the Laurent polynomial ring $\mathbb{C}[\sigma_1, \sigma_1^{-1}, \ldots, \sigma_n, \sigma_n^{-1}]$; degree of autonomy, its calculation in the generic case [7].

Lecture 6. Systems in other function spaces such as the space of compactly supported smooth functions, the Sobolev spaces etc.; the Nullstellensatz question for systems of partial differential equations.

Some references:

[1] R.E. Kalman, On the general theory of control systems, Proceedings, 1st World Congress of the International Federation of Automatic Control, 1960, Moscow, 481-493.

[2] B. Malgrange, Systèmes différentiels à coefficients constants, Séminaire Bourbaki, vol. 1962/63: 246.01- 246.11, 1963.

[3] U.Oberst, Multidimensional constant linear systems, Acta Applicandae Mathematicae, 1990, 20:1-175.

[4] V.P. Palamodov, A remark on exponential representation of solutions of differential equations with constant coefficients, Math. USSR Sbornik, 1968, 5: 401-416.

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[5] H. Pillai and S. Shankar, A behavioural approach to control of distributed systems, SIAM jl. Control and Optimization, 1998, 37: 388-408.

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[7] S. Shankar and P. Rocha, The generic degree of autonomy, SIAM jl. Applied Algebra Geometry, 2018, 2:410-427.

[8] J.C. Willems, The behavioral approach to open and interconnected systems, IEEE Control Systems Magazine, 2007, 27:46-99.