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ABSTRACTS

Contents

Non-algebraic Anosov flows in high dimensions

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Anosov flows and Anosov diffeomorphisms are the archetypical examples of uniformly hyperbolic dynamical systems, and, as such, have been widely studied since their introduction by D. Anosov in the 60'. There are many examples of Anosov flows on 3-manifolds exhibiting many different types of properties. The reason is the existence of 2 constructions process, by surgeries:

- the first process has been initiated by Handel and Thurston and generalized by Goodman and Fried: given any Anosov flow on a 3-manifolds, one can build infinitely many of them by surgeries along periodic orbit.
- the second, started with Franks and Williams "anomalous Anosov flows" in 1980, and then by myself with Langevin in 1994 and generalized with Beguin and Yu in 2017, allows us to build Anosov flows by gluing hyperbolic plugs.

In higher dimensions very few is known, due to a lack of examples. Indeed Franks and Williams announced in 1980 that their constructions holds in any dimensions > 3 but the argument was not precisely presented. IWe (with T. Barthelmé, A. Gogolev, and F. Rodriguez Hertz) recently noticed that indeed their argument could not work in dimensions > 3, but we could build examples following the same spirit.

I will try to present the transitive and non-transitive examples in [BBGH 2021] which are therefore the first examples of non-algebraic Anosov flows in dimension > 3.

Uniqueness of the measure of maximal entropy for geodesic flows of surfaces with caps

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The class of surfaces in this talk was introduced in the 1980s by Donnay in order to exhibit a smooth Riemannian metric on the two sphere with ergodic geodesic flow. The geodesic flows for these surfaces have unique (and therefore ergodic) measures of maximal entropy. The proof uses Climenhaga and Thompson's extension of the approach pioneered by Bowen and Franco. This is joint work with Todd Fisher and Rachel McEnroe.

Almost equivalence for transitive Anosov flows

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The talk is based on a joint preprint [2] with M. Shannon.

Two flows are almost-equivalent if one can go from one to the other by a finite number of Dehn surgeries on periodic orbits. Examples of almost equivalence go back to Fried who showed that any transitive Anosov flow is almost-equivalent to the suspension of a pseudo-Anosov homeomorphism [3], and even to Birkhoff whose construction [1] was popularized by

Fried and implies that every geodesic flow on a hyperbolic surface is almost equivalent to some suspension of an Anosov map of the 2-torus.

An open question of Ghys asks whether all transitive Anosov flows in dimension 3 are pairwise almost-equivalent. Using so-called Birkhoff sections and a result of Minakawa, we show that the answer is positive for suspension of automorphisms of the torus and for geodesic flows on hyperbolic orbifolds.

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Holomorphic foliations, structural stability, and Anosov's conjecture

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I will discuss several questions and results concerning holomorphic foliations on projective manifolds, focusing on the problems of structural stability and of the determination of the topology of leaves, notably Anosov's conjecture.

Mingled hyperbolicities: Restricted variational principles

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We quantify ergodically the lack of hyperbolicity in transitive diffeomorphisms with one-dimensional nonhyperbolic center. For that, we investigate step skew products whose fiber dynamics are circle diffeomorphisms. Such dynamics captures the key mechanisms of the dynamics of robustly transitive and nonhyperbolic maps with one-dimensional center. It also arises from the projective action of certain 2×2 elliptic matrix cocycles.

A key feature of these systems is the coexistence of saddles of different types of hyperbolicity, described in terms of fiber-contracting and -expanding regions which are mingled by the dynamics. It gives also rise to nonhyperbolic ergodic measures characterized in terms of a zero Lyapunov exponent in the circle-fiber direction. Some of those measures have positive entropy.

We describe the topological entropy of each level set of points with fiber-Lyapunov exponent α in terms of a restricted variational principle. Here α takes negative and positive values and also $\alpha = 0$. We will particularly focus on the latter case. For that we construct a nonhyperbolic ergodic measure of high entropy inspired by a periodic orbit approximation-technique of Gorodetski, Ilyashenko, Kleptsyn, and Nalski.

The talk is based on a joints works with K. Gelfert (UFRJ, Brazil) and M. Rams (IMPAN, Poland)

From Skew Products to Geometrically Integrable Maps in the Plane

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The structure of the space of C^1-smooth skew products of interval maps is described [1].

The concept of geometrically integrable self-maps of the compact plane sets is introduced. Criteria of the geometric integrability are proved and examples of these maps are considered [2].

Comparison of geometrically integrable maps and skew products properties is given.

The problem of the coexistence of periodic points periods of geometrically integrable maps is solved [2], [3].

The concept of a generalized Lorenz self-map of a compact plane set is introduced. The set of generalized Lorenz maps contains the proper subset of "two-dimensional" Lorenz maps that arise in the standard geometric Lorenz model [4].

Solution of the problem of the coexistence of periodic points periods of generalized Lorenz maps is presented [2].

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Local rigidity of abelian actions of parabolic toral automorphisms with (at least) one step-2 generator

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Two famous manifestations of local rigidity are KAM rigidity of Diophantine torus translations and smooth rigidity of hyperbolic or partially hyperbolic higher rank actions.

Damjanovic and Katok proved local rigidity for partially hyperbolic higher rank affine actions on tori. To complete the study of local rigidity of affine $\mathbb{Z}^k$ actions on the torus one needs to address the case of actions with parabolic generators.

Definition. We say that a linear map $A \in \mathrm{SL}(d, \mathbb{Z})$ is (parabolic) of step n if $(A - \mathrm{Id})^n = 0$, and $(A - \mathrm{Id})^{n-1} \neq 0$. An affine map $a(\cdot) = A(\cdot) + \alpha$ is said to be of step n if A is of step n.

We say that a $\mathbb{Z}^2$ affine action by parabolic elements is of step n if all of its elements are of step at most n.

Definition. We say that an affine $\mathbb{Z}^2$-action (a, b) is *KAM-rigid* under μ-preserving perturbations, if there exists $\sigma \in \mathbb{N}$ and $r_0 \in \mathbb{N}$ and $\varepsilon > 0$ that satisfy the following:

If $r \geq r_0$ and $(F, G) = (a + f, b + g)$ is a smooth μ-preserving $\mathbb{Z}^2$ action such that

$$(a + f) \circ (b + g) = (b + g) \circ (a + f), \tag{1}$$

$$\|f\|_r \leq \varepsilon, \quad \|g\|_r \leq \varepsilon, \quad \widehat{f} := \int_{\mathbb{T}^d} f d\lambda = 0, \quad \widehat{g} := \int_{\mathbb{T}^d} g d\lambda = 0,$$

then there exists $H = \mathrm{Id} + h \in \mathrm{Diff}^{\infty}_{\mu}(\mathbb{T}^d)$ such that $\|h\|_{r-\sigma} \leq \varepsilon$ and

$$H \circ (a + f) \circ H^{-1} = a, \quad H \circ (b + g) \circ H^{-1} = b. \tag{2}$$

We denote by $\mathcal{T}(A, B)$ the set of possible translations (α, β) for affine actions $(A+\alpha, B+\beta)$, that is

$$\mathcal{T}(A, B) := \{\alpha, \beta \in \mathbb{R}^d : (A - \mathrm{Id})\beta = (B - \mathrm{Id})\alpha\}.$$

Theorem. *Given a commuting pair (A, B) of parabolic matrices where A is step-2 ($(A-\mathrm{Id})^2 = 0$), we have the following dichotomy*

(i) For any choice of $(\alpha, \beta) \in \mathcal{T}(A, B)$, the action of (a, b) has a rank one factor that is not a (nonzero) translation and is thus not locally rigid.

(ii) For almost every choice of $(\alpha, \beta) \in \mathcal{T}(A, B)$, the action of (a, b) is ergodic and KAM-rigid under volume preserving perturbations.

This is a joint work with Danijela Damjanovic and Maria Saprykina.

Quantitative global theory of one-frequency quasiperiodic operators

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We explore Avila's global theory for one-frequency Schrödinger operators from the point of view of Aubry duality. We establish the one-to-one correspondence between the Lyapunov exponent of the Schrödinger cocycles $(\alpha, S_E^V(\cdot + i\epsilon))$ and the Lyapunov exponents of its dual cocycles. We also give a new equivalent definition of subcritical, critical, supercritical regimes. Especially, we prove that the acceleration of the Schrödinger cocycle equals to the pair of zero Lyapunov exponents of its dual cocycle. This is a joint work with Svetlana Jitomirskaya, Jiangong You and Qi Zhou.

On rationally integrable projective billiards

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Consider a billiard in a strictly convex planar domain bounded by a smooth curve. An oriented straight line intersecting the billiard is reflected at its last intersection point with the boundary according to the classical reflection law: the angle of incidence is equal to the angle of reflection.

A *caustic* of a convex billiard is a curve C whose tangent lines are reflected by the billiard to its tangent lines (e.g., a confocal ellipse in an elliptic billiard). The famous **Birkhoff Conjecture** stated in late 1920-ths deals with those bounded planar strictly convex billiards that are *Birkhoff integrable*: *admit a foliation by closed caustics in a neighborhood of the boundary from the inner side, with boundary being a leaf.* It affirms that *the only Birkhoff integrable billiards are ellipses* and the caustics are confocal ellipses. This open conjecture was studied by many mathematicians. Recent substantial progress was obtained in joint papers by V.Kaloshin and A.Sorrentino [2] and by M.Bialy and A.Mironov [1]. For its survey see [2, 1, 3].

In [5] Sergei Tabachnikov introduced projective billiards, which generalize the usual billiards on standard surfaces of constant curvature: the Euclidean plane, the hyperbolic plane and the round sphere. A *planar projective billiard* is a planar curve γ equipped with a transversal line field $\mathcal{N}$. A line intersecting γ is reflected at their intersection point P via the linear involution $\sigma_P : T_P\mathbb{R}^2 \to T_P\mathbb{R}^2$ with invariant subspaces $T_P\gamma$ and $\mathcal{N}(P)$, acting trivially on $T_P\gamma$ and as central symmetry $v \mapsto -v$ on $\mathcal{N}(P)$. In [6, p.103] Tabachnikov suggested a generalization of the Birkhoff Conjecture to strictly convex planar projective billiards, which implies its versions for billiards on surfaces of constant curvature and for Euclidean outer billiards.

Projective duality sends γ and σ_P to a curve C equipped with a family of projective involutions acting on its projective tangent lines and fixing tangency points. Tabachnikov's Conjecture is stated in dual terms. Namely, consider a *planar strictly convex closed curve C and a foliation by closed curves of its neighborhood on the concave side,* with C being its leaf. For every projective line L tangent to C at a point P consider the germ at P of involution of the line L fixing P and permuting its intersection points with each individual leaf of the foliation. Suppose that for every point $P \in C$ *the latter involution is a projective transformation of the tangent line L.* **Tabachnikov's Conjecture** affirms that under these assumptions *the curve C is an ellipse and the foliation in question is a pencil of conics.*

In the talk we present a proof of the **rational version** of the Tabachnikov's Conjecture for C^4-smooth curves: *the positive answer under the additional assumption that the foliation admits a rational first integral.*

We also prove a *local version:* in the case when C is a *germ* of real C^4-smooth (or holomorphic) planar curve and the germ of foliation admits a rational first integral. We prove that in this general case *the curve C is also a conic.* But the leaves of the foliation may be higher degree algebraic curves. We give a complete classification of germs of foliations satisfying the conditions of local Tabachnikov's Conjecture and admitting rational first integrals, up to projective transformation. Their list includes:

- pencils of conics;
- two infinite series of exotic examples, with higher degree leaves;
- two real examples with leaves of degree four;
- two real examples with leaves of degrees six.

As an application, we get the descriptions of germs of projective billiards with C^4-smooth boundaries whose billiard flows admit 0-homogeneous rational first integral in velocity.

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Anosov extension

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I will present general conditions for a Riemannian domain with spherical boundary to be embeddable into a closed manifold with Anosov geodesic flow. Joint with D. Chen and A. Erchenko.

Anti-classification results in smooth dynamics

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Squaring the circle, trisecting of an angle, finding an explicit formula for roots of a polynomial of fifth degree – none of those problems has a solution. Let us now ask one more question – can one classify all dynamical systems? Specifically, is it possible to classify all diffeomorphisms of a given manifold up to a topological conjugacy? To show that such classification exists (as, for example, in the case of diffeomorphisms of the circle) it is enough to explicitly present it. But what if it doesn't? What exactly does it mean? And how can one prove it?

In our joint work with Matt Foreman we prove that for smooth diffeomorphisms of a two-dimensional manifold there is no reasonably defined numerical invariant such that it would take the same values exactly on diffeomorphisms that are topologically conjugate. For diffeomorphisms of manifolds of dimension five and higher such classification is impossible in another, much stronger sense. In the talk we will explain the details of those statements and discuss some open problems.

On the topology of manifolds admitting diffeomorphisms with orientable expanding attractors and contracting repellers

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In the talk we consider a class of diffeomorphisms defined on a closed orientable manifold of dimension $n > 2$, whose nonwandering set consists of hyperbolic orientable basic sets of codimension one and arranges locally as the product of Cantor set and a disc of dimension $n-1$. It is established that the supporting manifold of a diffeomorphism from the class under consideration is homeomorphic to the connected sum of a finite number of manifolds homeomorphic to the torus and manifolds homeomorphic to the direct product of the sphere of dimension $n-1$ and the circle. The number of terms in a connected sum is determined by the number of basic sets and their properties.

The main result of the report was obtained in collaboration with E.V. Zhuzhoma and V.S. Medvedev.

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On gradient-like flows on manifolds of dimension four and greater

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We will say that gradient-like flow f^t belongs to a class $G(M^n)$, where M^n is connected closed oriented manifold of dimension $n \geq 3$, if:

(a) Morse index (dimension of unstable manifold) of any saddle equilibrium state of the flow f^t equals either 1 or $n-1$;

(b) invariant manifolds of different saddle equilibria do not intersect.

Denote by ν_{f^t} and μ_{f^t} the numbers of saddle and node equilibria of the flow $f^t \in G(M^n)$ and set

$$g_{f^t} = (\nu_{f^t} - \mu_{f^t} + 2)/2.$$

Everywhere below $\mathcal{S}^n_g$ stands for manifold which homeomorphic either to the sphere $\mathbb{S}^n$ if $g = 0$ or to connected sum of $g > 0$ copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Theorem 1. *Let $f^t \in G(M^n)$, $n \geq 2$. Then M^n is homeomorphic to $\mathcal{S}^n_{g_{f^t}}$.*

For $n = 2$ Theorem 1mmediately follows from [5]. For $n \geq 3$ Proposition 1ollows from [1, 2], where its analog for Morse-Smale diffeomorphisms without heteroclinic curves was obtained.

Theorem 2elow states that for manifolds $\mathcal{S}^n_g$, $n \geq 4$, the condition (b) implies the condition (a).

Theorem 2. *Let f^t be gradient-like flow on $\mathcal{S}^n_g$, $g \geq 0$, $n \geq 4$. If invariant manifolds of different saddle equilibria of f^t do not intersect, then Morse index of any saddle equilibrium equals* 1 *or* $(n-1)$*, that is $f^t \in G(\mathcal{S}^n_g)$. Moreover, there exists $k \geq 0$ such that $\nu_{f^t} = 2g + k$ and $\mu_{f^t} = k + 2$.*

For case $g = 0$ Theorem 2s proved in [4], where necessary and sufficient conditions of topological equivalence of flows from class $G(S^n)$, $n \geq 3$, where obtained. Theorem 2llows to obtain topological classification of flows from class $\mathcal{S}_g^n$, $g > 0$, in combinatorial terms using techniques of [4, 3].

For any $f^t \in G(\mathcal{S}_g^n)$ we put in correspondence a bicolor graph Γ_{f^t} which describes mutual arrangement of invariant manifolds of saddle equilibria of the flow f^t, and provide the following result.

Theorem 3. *Flows $f^t, f'^r \in G(\mathcal{S}_g^n)$ are topological equivalent iff their bicolor graphs $\Gamma_{f^t}, \Gamma_{f'^t}$ are isomorphic by means preserving colors isomorphism.*

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Sharp analysis of Maryland localization and eigenfunctions for all parameters

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Maryland model is a three-parameter family of 1D quasiperiodic Schrodinger operators, that arises from a linear version of the quantum kicked rotor problem. It features hyperbolicity of the corresponding cocycle for all parameters and is known, since Simon's proof in 1985, that it has Anderson localization for all Diophantine frequencies. It was recently proved by Jitomirskaya-Liu that it has a sharp (in all parameters) spectral transition between purely singular continuous spectrum and pure point spectrum. In this talk I will present a new approach to Anderson localization in the sharp regime using the Green's function expansion, introduce the concept of phase "anti-resonance" and show how it helps us to break the conventional resonance barrier of localization. I will also describe some novel eigenfunction structures that we discover and prove using this new approach. This work is based on joint works with Svetlana Jitomirskaya and Fan Yang.

Surgered contact flows, hyperbolicity, and orbit growth

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This is a joint project with P. Foulon and A. Vaugon.

- Foulon–Ding–Geiges–Weinstein–Handel–Thurston surgery is a contact surgery on the unit tangent bundle of a surface.
- This forces orbit complexity.
- It produces several contact 3-flows.
- From the fiber flow a flow with quadratic or exponential complexity.
- From the geodesic flow a contact structure whose every Reeb flow has positive entropy.
- If the surgered geodesic flow is hyperbolic, then it has increased orbit growth.
- There are 3-manifolds with two contact flows, one exponentially complex, one polynomially.

Quasiperiodic operators with monotone potentials

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The talk is based on joint works [1], [2] with S. Krymskii, L. Parnovski, and R. Shterenberg. We consider quasiperiodic operators on $\mathbb{Z}^d$ of the form

$$(H(x)\psi)_{\mathbf{n}} = (\Delta\psi)_{\mathbf{n}} + \varepsilon f(x + \mathbf{n}\cdot\omega)\psi_{\mathbf{n}},$$

where f is a monotone function that maps the interval $(0, 1)$ onto $(-\infty, +\infty)$ and is extended into $\mathbb{R}$ by 1-periodicity. The frequency vector ω is assumed to satisfy a Diophantine property (and, in particular, have rationally independent components). For small $\varepsilon > 0$ and under some additional monotonicity assumptions on f, we construct a diagonalization of such operator by direct analysis of the perturbation series.

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The Furstenberg theorem: adding a parameter and removing the stationarity

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The talk is based on a joint work with A. Gorodetski.

The classical Furstenberg Theorem describes the (almost sure) behaviour of a random product of independent matrices from $\mathrm{SL}(n,\mathbb{R})$; their norms turn out to grow exponentially. In our joint work [1], we study what happens if the random matrices from $\mathrm{SL}(2,\mathbb{R})$ depend on an additional parameter. It turns out that in this new situation, the conclusion changes. Namely, under some natural conditions, there almost surely exists a (random) "exceptional" set on parameters where the lower limit for the Lyapunov exponent vanishes.

Another direction of generalisation for the classical Furstenberg Theorem is removing the stationarity assumption. That is, the matrices that are multiplied are still independent, but no longer identically distributed. Though in this setting most of the standard tools are no longer applicable (no more stationary measure, no more Birkhoff ergodic theorem, etc.), it turns out that the Furstenberg theorem can (under the appropriate assumptions) still be generalised to this setting, with a deterministic sequence replacing the Lyapunov exponent. These two generalisations can be mixed together, providing the Anderson localisation conclusions for the non-stationary 1D random Schrödinger operators.

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Structural stability for dynamical systems on time scales

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We consider a system on a time scale

$$x^{\Delta} = f(t,x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{T} \tag{1}$$

where the time scale $\mathbb{T}$ is an unbounded closed subset of $\mathbb{R}$.

Definition. We say that the system (1) is *structurally stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $g(t,x): \quad |g(t,x)| < \delta, \quad |g'_x(t,x)| < \delta$ and any $t_0 \in \mathbb{T}$ there is a homeomorphism h of the space $\mathbb{R}^n$ such that

$$|\varphi_f(t,x_0) - \varphi_{f+g}(t,h(x_0))| < \varepsilon$$

for any $x_0 \in \mathbb{R}^n$, $t \in \mathbb{T}$. Here $\varphi_f(t,x_0)$ and $\varphi_{f+g}(x_0)$ are solutions of systems (1) and

$$x^{\Delta} = f(t,x) + g(t,x) \tag{2}$$

with initial conditions $x(t_0) = x_0$.

For systems of ordinary differential equations, conditions for global structural stability were obtained in [1], see also [2]. It was proved that a system is structurally stable if its linearizations are uniformly hyperbolic on families of segments.

We formulate and prove an analog of this statement for time scale systems. Although the result is very similar to that for ordinary differential equations, the proof for the time scale case is significantly different. We need to use specific approaches of time scale systems theory [3]. Remarkably, the classical results for structural stability of autonomous systems of ODEs, obtained by C.Robinson [4], are, in general, non-applicable for systems on time scales (even for the autonomous ones).

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Small denominators and large numerators of quasiperiodic Schrödinger operators

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We initiate an approach to simultaneously treat numerators and denominators of Green's functions arising from quasi-periodic Schrödinger operators, which in particular allows us to study completely resonant phases of the almost Mathieu operator.

Let $(H_{\lambda,\alpha,\theta}u)(n) = u(n+1)+u(n-1)+2\lambda\cos 2\pi(\theta+n\alpha)u(n)$ be the almost Mathieu operator on $\ell^2(\mathbb{Z})$, where $\lambda, \alpha, \theta \in \mathbb{R}$. Let

$$\beta(\alpha) = \limsup_{k\to\infty} -\frac{\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|}.$$

We prove that for any θ with $2\theta \in \alpha\mathbb{Z}+\mathbb{Z}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta(\alpha)}$. This confirms a conjecture of Avila and Jitomirskaya [The Ten Martini Problem. Ann. of Math. (2) 170 (2009), no. 1, 303–342] and a particular case of a conjecure of Jitomirskaya [Almost everything about the almost Mathieu operator. II. XIth International Congress of Mathematical Physics (Paris, 1994), 373–382, Int. Press, Cambridge, MA, 1995].

Birkhoff sections and orbit spaces of Anosov flows

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An Anosov flow in dimension 3 can be studied using transverse surfaces called Birkhoff sections. We consider the sign of the boundary components of a Birkhoff section to deduce topological properties of the flow.

Theorem (Masayuki Asaoka [1], T.M [2], Christian Bonatti 2021'). *An Anosov flow on a closed oriented 3-manifold admits a positive Birkhoff section if and only if it is* $\mathbb{R}$*-covered and positively skewed.*

I will discuss one proof of the theorem together with some relations with Fried-Goodman surgeries and Reeb flows. Notice that this theorem admits two other independent proofs found by Masayuki Asaoka [1] and Christian Bonatti.

References

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Attractors of Direct Products

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For Milnor, statistical, and minimal attractors, we construct examples of smooth flows φ on S^2 for which the attractor of the Cartesian square of φ is smaller than the Cartesian square of the attractor of φ. In the example for the minimal attractors, the flow φ also has a global physical measure such that its square does not coincide with a global physical measure of the square of φ.

We are interested in attractors definitions of which rely on a natural (in our case, Lebesgue) measure on the phase space, which allows these attractors to capture asymptotic behavior of most points while possibly neglecting what happens with a set of orbits of zero measure. One type of such attractors was introduced by J. Milnor in [2] under the name "the likely limit set". We refer to it as the Milnor attractor.

Milnor attractor, [2] The Milnor attractor $A_{Mil}(\varphi)$ of a dynamical system φ is the smallest closed set that contains the ω-limit sets of μ-almost all orbits.

Another way to define an attractor is via an "attracting" invariant measure: the attractor is its support. Most suitable are the notions of *physical* and *natural* measures, which may be viewed as analogues of SRB-measures for general, non-hyperbolic dynamical systems (see, e.g., [1]; we adapt the definitions from [1] to the case of flows). Physical measures describe the distribution of μ-a.e. orbit, while natural measures capture the limit behaviour of the reference measure.

For a flow φ on a compact manifold X with measure μ, a probability measure ν is called *physical* if there is a set B with $\mu(B) > 0$ such that for any $x \in B$ and any continuous function $f \in C(X, \mathcal{R})$ the Birkhoff time average over the orbit of x is equal to the space average w.r.t. ν:

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\varphi^t(x))dt = \int_X f\, d\nu.$$

A measure ν is called *natural* for a flow φ if there exists an open subset $U \subset X$ such that for any probability measure $\tilde{\mu}$ absolutely continuous with respect to μ and with $\operatorname{supp}(\tilde{\mu}) \subset U$ one has weak-$*$ convergence

$$\frac{1}{T} \int_0^T \varphi^t_* \tilde{\mu}\, dt \to \nu, T \to +\infty.$$

Statistical and minimal attractors are supports of these measures, respectively.

When constructing examples of dynamical systems with required properties, it is not uncommon to utilize, at least as a piece of the construction, direct products of systems in lower dimensions. It is tempting to think that the attractor of the direct product of two systems always coincides with the direct product of their attractors. Although this holds, indeed, for so-called maximal attractors, this is not true for several other types of attractors, namely, for Milnor, statistical, and minimal attractors, and also for the supports of physical measures, when the latter exist. We present examples of smooth flows on S^2 that exhibit such non-coincidence. Our examples are mostly Cartesian squares of flows. Although the square of a flow is always a flow of infinite codimension, it is interesting to find, for every type of attractor, the least codimension for the flow itself in which one can have non-coincidence between the attractor of the square and the square of the attractor.

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Probabilistic shadowing in skew products

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The talk is based on a joint work with S. B. Tikhomirov.

We investigate the probability of the event that a finite random pseudotrajectory can be effectively shadowed by an exact trajectory. The main result of the work describes a class of skew products, for which this probability tends to one as the length of a pseudotrajectory tends to infinity and the value of a maximal mistake on each step tends to zero. We also show that continuous linear skew products over a Bernoulli shift, doubling map on a circle and any Anosov linear map on a torus lie in this class. The Cramer's large deviation theorem is used in the proof.

Conditional shadowing property

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The main property of dynamical systems studied by the shadowing theory can be stated as follows. Consider a homeomorphism f of a metric space (X, dist). Let $d > 0$. A sequence $\{y_n \in X\}$ is called a d-pseudotrajectory of f if the inequalities

$$\mathrm{dist}(f(y_n), y_{n+1}) < d \tag{1}$$

hold.

One says that f has the (standard) shadowing property if for any $\varepsilon > 0$ there is a $d > 0$ such that for any d-pseudotrajectory $\{y_n \in X\}$ of f there is a point $x \in X$ for which

$$\mathrm{dist}(f^n(x), y_n) < \varepsilon.$$

Usually, the shadowing property is a corollary of some kind of hyperbolicity of f (see [1-3]). At the same time, the shadowing theory studies many properties different from the standard shadowing property that are not closely related to hyperbolicity.

Let us mention, for example, the limit shadowing property [4]; in this case, inequalities (1 are replaced by the relations

$$\mathrm{dist}(f(y_n), y_{n+1}) \to 0, \quad n \to \infty,$$

and one looks for a point x such that

$$\mathrm{dist}(f^n(x), y_n) \to 0, \quad n \to \infty.$$

Let us mention one more example of "conditional" shadowing (here the term "conditional" means that the uniform estimate (1 is replaced by particular conditions on the smallness of the values $\mathrm{dist}(f(y_n), y_{n+1})$).

In the paper [5], the authors studied shadowing of pseudotrajectories near a nonisolated fixed point p of a diffeomorphism f; in this case, the smallness of the values $\mathrm{dist}(f(y_n), y_{n+1})$ had been related to the values $\mathrm{dist}(y_n, p)$.

Finally, let us mention the research of [6] devoted to conditional shadowing for a nonautonomous system whose linear part satisfies some conditions generalizing nonuniform hyperbolicity.

In this talk, we study conditional shadowing for a nonautonomous system in a Banach space assuming that the linear part admits a family of invariant subspaces (scale) with different behavior of trajectories. Conditions of shadowing are formulated in terms of smallness of the projections of one-step errors to the scale and of smallness of Lipschitz constants of the projections of nonlinear terms.

We also give conditions under which a system has the conditional property of inverse shadowing (dual to the shadowing property).

The main results of the talk are published in [7].

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3-diffeomorphisms with dynamics "one-dimensional surfaced attractor-repeller"

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We study the structural stability of three-dimensional diffeomorphisms with source-sink dynamics. Here the role of source and sink is played by one-dimensional hyperbolic repeller and attractor. It is well known that in the case when the repeller and the attractor are solenoids (not embedded in the surface), the diffeomorphism is not structurally stable. The author proves that in the case when the attractor and the repeller are canonically embedded in a surface, the diffeomorphism is also not structurally stable.

Measures maximizing the entropy for Kan endomorphisms

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In 1994, Ittai Kan provided the first example of maps with intermingled basins. The Kan example corresponds to a partially hyperbolic endomorphism defined on a surface, with the boundary exhibiting two intermingled hyperbolic physical measures. Both measures are supported on the boundary, and they also maximize the topological entropy. In this talk, we give the existence of a third hyperbolic measure supported in the interior of the cylinder that maximizes the entropy. I also will give this statement for a larger class of invariant measures of large class maps including perturbations of the Kan example.

Smooth conjugacy for codimension 1 Anosov diffeomorphisms

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In this talk we shall present joint work with A. Gogolev on some rigidity results in the setting of codimension one Anosov diffeomorphisms on manifols of dimension larger than or equal to 3. This result is a further development of our program with Andrey of classiying smooth conjugacy via matching of jacobians. Even though the data is much coarser than the usual matching of derivatives it results in a more stable theory.

(Dis)continuity of Lyapunov exponents

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Joint work with J. Buzzi and S. Crovisier.

Suppose f is a topologically transitive C^∞ surface diffeomorphism. The maps which associate to an ergodic invariant measure μ the entropy of μ and the top Lyapunov exponent of μ are not continuous in the weak star topology. We compare their discontinuities.

On the abominable properties of the Almost Mathieu operator with Liouville frequencies

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We show that, for sufficiently well approximable frequencies, several spectral characteristics of the Almost Mathieu operator can be as poor as at all possible in the class of all discrete Schroedinger operators. For example, the modulus of continuity of the integrated density of states may be no better than logarithmic. Other characteristics to be discussed are homogeneity, the Parreau-Widom property, and (for the critical AMO) the Hausdorff content of the spectrum. Based on joint work with A. Avila, Y. Last, and Q. Zhou

Shadowing for finite pseudotrajectories with decreasing size of error

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We consider the shadowing property of pseudotrajectories with decreasing errors for a linear skew product. The probabilistic properties of finite pseudotrajectories are studied. It is shown that for pseudotrajectories with errors decreasing exponentially, the typical dependence between the length of the pseudotrajectory and the shadowing accuracy is polynomial. The proof is based on the large deviation principle and the gambler's ruin problem. The talk starts with the overview of shadowing for finite pseudotrajectories and its probabilistic aspects. The work is supported by RSCF grant 21-11-00047.

Codimension one basic sets of A-flows

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The talk is based on results obtained jointly with V. Medvedev.

Introduction. Dynamical systems satisfying an Axiom A (in short, A-systems) were introduced by S.Smale. By definition, a non-wandering set of A-system is the topological closure of periodic orbits endowed with a hyperbolic structure. Due to Smale's Spectral Decomposition Theorem, the non-wandering set of any A-system is a disjoint union of closed, invariant, and topologically transitive sets called *basic sets*. E.Zeeman proved that any n-manifold, $n \geq 3$, supporting nonsingular flows supports an A-flow with a one-dimensional nontrivial basic set. It is natural to consider the existence of two-dimensional (automatically non-trivial) basic sets on n-manifolds. Mainly, we consider A-flows on closed 3-manifolds M^3. We prove that any closed orientable 3-manifolds supports A-flows with two-dimensional attractors. Our main attention concerns to embedding of non-mixing attractors and its basins (stable manifolds) in M^3.

Main results.

Theorem 1. Let Ω be a codimension one basic set of A-flow f^t on a closed n-manifold M^n, $n \geq 3$. Then Ω is either an attractor or repeller.

Theorem 2. Let f^t be an A-flow on an orientable closed 3-manifold M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor Λ_a. Then there is a compactification $M(\Lambda_a) = W^s(\Lambda_a) \cup_{i=1}^{k} l_i$ of the basin $W^s(\Lambda_a)$ by the family of circles $l_1, \ldots, l_k$ such that

- $M(\Lambda_a)$ is a closed orientable 3-manifold;
- the flow $f^t|_{W^s(\Lambda_a)}$ is extended continuously to the nonsingular flow $\tilde{f}^t$ on $M(\Lambda_a)$ with the non-wandering set $NW(\tilde{f}^t) = \Lambda_a \cup_{i=1}^{k} l_i$ where $l_1, \ldots, l_k$ are repelling isolated periodic trajectories of $\tilde{f}^t$;
- the family $L = \{l_1, \ldots, l_k\} \subset M(\Lambda_a)$ is a fibered link in $M(\Lambda_a)$.

Theorem 3. Let $\{l_1, \ldots, l_k\} \subset M^3$ be a fibered link in a closed orientable 3-manifold M^3. Then there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor and the repelling isolated periodic trajectories $l_1, \ldots, l_k$.

Corollary. Given any closed orientable 3-manifold M^3, there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a two-dimensional attractor.

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