"Numbers and functions"
Memorial conference for 80th birthday of
Alexey Nikolaevich Parshin

November 28 -- December 2, 2022

Organizers

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow

Steklov International Mathematical Center, Moscow

The conference is supported by
the Ministry of Science and Higher Education of the Russian Federation
(the grant to the Steklov International Mathematical Center,
agreement no. 075-15-2022-265).

Jean-Michel Bismut. Coherent sheaves, Chern character, and RRG

Let X be a compact complex manifold. On X, one can consider holomorphic vector bundles, and also coherent sheaves. When X is projective, the corresponding Grothendieck groups coincide.

When X is non-projective, a result of Voisin shows that in general, coherent sheaves may not have finite locally free resolutions.

In our talk, we will focus on two results.

1. The construction of a Chern character for coherent sheaves with values in Bott–Chern cohomology, which strictly refines on de Rham cohomology. This will be done using a fundamental construction of Block.

2. The proof of a Riemann–Roch–Grothendieck formula for direct images of coherent sheaves. It relies in particular on the theory of the hypoelliptic Laplacian.

Our results refine on earlier work by Levy, Toledo–Tong, and Grivaux.

This is joint work with Shu Shen and Zhaoting Wei, available in `https://arxiv.org/abs/2102.08129`.

Fedor A. Bogomolov. Some problems of geometry of projective curves

In my talk I will discuss some problems and results of the theory of projective curves defined over number fields and finite fields. In particular I plan to address the structure and distribution of points of finite order.

Christopher Deninger. Dynamical systems for arithmetic schemes

For any normal scheme X of finite type over $\operatorname{Spec}\mathbb{Z}$ we construct a continuous time dynamical system whose periodic orbits come in compact packets that are in bijection with the closed points of X. All periodic orbits in a given packet have the same length equal to the logarithm of the order of the residue field of the corresponding closed point. For $\operatorname{Spec}\mathbb{Z}$ itself we get a dynamical system whose periodic orbits are closely related to the prime numbers. The construction uses new ringed spaces obtained by sheafifying rational Witt vector rings. In the zero-dimensional case there is a close relation to work of Kucharczyk and Scholze who realized Galois groups of fields containing the maximal cyclotomic extension of the rational number field as étale fundamental groups of ordinary topological spaces. A p-adic variant of our construction turns out to be closely related to the Fargues-Fontaine curve of p-adic Hodge theory.

Sergei V. Konyagin. A construction of A. Schinzel — many numbers in a short interval without small prime factors

Hardy and Littlewood (1923) conjectured that for any integers $x, y \geq 2$

$$\pi(x+y) \leq \pi(x) + \pi(y). \tag{1}$$

Let us call a set $\{b_1, \ldots, b_k\}$ of integers admissible if for each prime p there is some congruence class mod$p$ which contains none of the integers b_i. The prime k-tuple conjecture states that if a set $\{b_1, \dots, b_k\}$ is admissible, then there exist infinitely many integers n for which all the numbers $n + b_1, \dots, n + b_k$ are primes.

Let x be a positive integer and $\rho^*(x)$ be the maximum number of integers in any interval $(y, y + x]$ (with no restriction on y) which are relatively prime to all positive integers $\leq x$. The prime k-tuple conjecture implies that

$$\max_{y \geq x}(\pi(x+y) - \pi(y)) = \limsup_{y \geq x}(\pi(x+y) - \pi(y)) = \rho^*(x).$$

Hensley and Richards (1974) proved that

$$\rho^*(x) - \pi(x) \geq (\log 2 - o(1))x(\log x)^{-2} \quad (x \to \infty).$$

Therefore, (1) is not compatible with the prime k-tuple conjecture. Using a construction of Schinzel we show that

$$\rho^*(x) - \pi(x) \geq ((1/2) - o(1))x(\log x)^{-2} \log\log\log x \quad (x \to \infty).$$

Leonid V. Kuz'min. Arithmetic of certain ℓ-extensions ramified at three places

Let ℓ be a regular odd prime, k the ℓ th cyclotomic field and $K = k(\sqrt[\ell]{a})$, where a is a positive integer such that there are exactly three places not over ℓ that ramify in K_∞/k_∞. Here K_∞ (resp. k_∞) denotes the cyclotomic $\mathbb{Z}_\ell$-extension of K (resp. of k).

Under assumption that a has exactly three prime divisors p_1, p_2, p_3 we study the ℓ-class group $\mathrm{Cl}_\ell(K)$ of K and the Iwasawa module (the Tate module) of K.

We prove that for $\ell > 3$ the order of $\mathrm{Cl}_\ell(K)$, which we denote by ℓ^r satisfies the condition $r \geqslant 2$. Calculation of cohomologies of some groups of unit yields that $r \geqslant \ell - 1$ or r is odd. This implies $r > 2$ for $\ell > 3$. If $\ell = 3$ we describe the structure of the Tate module $T_\ell(K_\infty)$.

Some of these results are generalized to the case, when a has two natural divisors p and q such that q remains prime in K_∞ and p splits into two factors.

Dmitry R. Lebedev. On a matrix element representation of the GKZ hypergeometric functions

This talk is based on my joint paper with A.A. Gerasimov and S.V. Oblezin (arXiv: 2209.02516 v1[math.RT] 6 Sep 2022).

We develop a representation theory approach to the study of generalized hypergeometric functions of Gelfand, Kapranov and Zelevisnky (GKZ). We show that the GKZ hypergeometric functions may be identified with matrix elements of non-reductive Lie algebras L_N of oscillator type. The Whittaker functions associated with principal series representations of $gl_N(R)$ being special cases of GKZ hypergeometric functions, thus admit along with a standard matrix element representations associated with reductive Lie algebra $gl_{\ell+1}(R)$, another matrix element representation in terms of $L_{\ell(\ell+1)}$.

Андрей Е. Миронов. Об уравнениях для первых интегралов интегрируемых бильярдов

Мы расскажем о методе нахождения дифференциальных уравнений на первые интегралы интегрируемых бильярдов. Мы применяем этот метод для исследования проволочного бильярда (wire billiards), для нахождения поверхностей в $\mathbb{R}^3$ с (локальным) первым бильярдным интегралом и для нахождения кусочно гладкой поверхности, гомеоморфной тору, с двумя независимыми первыми бильярдными интегралами.

Юрий В. Нестеренко. Алгебраическая независимость и экспоненциальная функция

Практически каждое доказательство в теории трансцендентных чисел использует исключение переменных. Это относится к классическим результатам о трансцендентности e (Ш. Эрмит), к алгебраической независимости значений так называемых E-функций (К. Зигель и А.Б. Шидловский), к решению 7-й проблемы Гильберта (А.О. Гельфонд, Т. Шнейдер). Все эти утверждения использовали либо исключение переменных с помощью однородных линейных форм, либо исключение одной переменной с помощью двух многочленов от неё. Ещё в начале 50-х годов прошлого века А.О. Гельфонд указывал на необходимость развития общей теории исключения применительно к задачам о трансцендентности и алгебраической независимости чисел. Цель настоящего доклада привлечь внимание слушателей к реализации в этого пожелания Гельфонда, а также к некоторым результатам, полученным в последовавшие годы с его помощью, и открытым вопросам.

Пусть K — конечное расширение поля рациональных чисел. Для каждого однородного несмешанного идеала I кольца $R = K[x_0, x_1, \dots, x_m]$ с помощью формы Чжоу этого идеала можно определить ряд его характеристик: *размерность* $\dim I$, *степень* $\deg I$, *логарифмическую высоту* $h(I)$ и *величину идеала* $|I(\omega)|$ в произвольной точке ω проективного m-мерного пространства над $\mathbb{C}$. Три последние характеристики аналогичны соответствующим характеристикам для многочленов. В частности, $\deg I$, $h(I)$ и $\ln |I(\omega)|$ ведут себя почти линейно при разложении I в пересечение примарных компонент. Процесс исключения переменных можно реализовать как индуктивную оценку снизу величины $|I(\omega)|$ в зависимости от характеристик $\dim I$, $\deg I$, $h(I)$. Индукция проводится по размерности идеала. В частности, получаемая в конце индукции, такая оценка для главных идеалов позволяет получить оценку снизу для многочленов — их образующих и доказать, что эти многочлены в точке ω отличны от нуля. Другими словами, координаты точки ω однородно алгебраически независимы над K. Мы укажем ряд конкретных примеров, связанных с такой схемой рассуждений в случае экспоненциальной функции.

Дмитрий О. Орлов. Конечномерные ДГ алгебры и их свойства

Ivan A. Panin. On Grothendieck–Serre conjecture in mixed characteristic for $SL_{1,D}$

Let R be an unramified regular local ring of mixed characteristic, D an Azumaya R-algebra, K the fraction field of R, $Nrd : D^\times \to R^\times$ the reduced norm homomorphism. Let $a \in R^\times$ be a unit. Suppose the equation $Nrd = a$ has a solution over K, then it has a solution over R.

Particularly, we prove the following. Let R be as above and a, b, c be units in R. Consider the equation $T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = c$. If it has a solution over K, then it has a solution over R.

Vladimir L. Popov. Group embeddings related to automorphism groups of algebraic varieties

The following questions are in the focus of exploration of automorphism groups of algebraic varieties, which is the last decade trend:

(Q1) Is a given group embeddable in the automorphism group of an algebraic variety?

(Q2) If yes, what are the properties of such varieties? Do they exist in some distinguished classes of varieties (e.g., rational, nonrational, affine, complete, etc.)? What are the “extreme"values of the parameters of such varieties (e.g., the minimum of their dimensions)?

(Q3) Conversely, in which groups can the automorphism groups of algebraic varieties of some type be embedded (e.g., are these groups linear)?

The topics of the talk belong to this blend of abstract algebra and algebraic geometry.

Alexei A. Rosly. A Physist's View on some of Parshin's Ideas

I would like to take this opportunity to recollect some ideas and constructions from the past, which were non-separated (in the sense of Hausdorff) from Parshin's work. I believe, some initiatives were not properly developed, although they deserved to.

Armen G. Sergeev. Spin^c-structures and Seiberg-Witten equations

In the study of Riemann surfaces the key role is played by the complex structure compatible with Riemannian metric and Cauchy–Riemann operator related to this structure. However, in the case of 4-dimensional Riemannian manifolds the subclass of the manifolds having the complex structures is comparatively narrow and it is hard to understand general properties of Riemannian 4-manifolds investigating only this subclass. So in the study of such manifolds two natural questions arise: the first one — what can replace the complex structure on 4-dimensional Riemannian manifolds, and the second one — which linear differential operator should play the role of $\bar{\partial}$-operator in the 4-dimensional case.

To answer the first question it is proposed to replace the complex structure by the $Spin^c$-structure existing on any 4-dimensional Riemannian manifold. To answer the second question we replace the $\bar{\partial}$-operator on the 4-dimensional Riemannian manifold by the Dirac operator associated with the given $Spin^c$-structure. Having the $Spin^c$-structure one can introduce the Seiberg–Witten action functional. The local minima of this functional satisfy the Seiberg–Witten equations being the main subject of our talk.

These equations, found at the end of XXth century, are one of the principal discoveries in topology and geometry of 4-dimensional Riemannian manifolds. As the Yang–Mills equations they are the limiting case of more general supersymmetric Yang–Mills equations. But in contrast with conformally invariant Yang–Mills equations the Seiberg–Witten equations are not invariant under the change of scale. So in order to derive a “useful information” from them it is necessary to introduce the scale parameter λ and take the limit for $\lambda \to \infty$. This limit is called adiabatic and is another main subject of our talk.

Constantin A. Shramov. Conic bundles

Consider a conic bundle over a smooth incomplete curve C, i.e. a smooth surface S with a proper surjective morphism to C such that the push-forward of the structure sheaf of S coincides with the structure sheaf of C, and the anticanonical class of S is ample over C. If the base field is perfect, a conic bundle always extends to a conic bundle over a completion of C. I will tell about a necessary and sufficient condition for the existence of such an extension in the case of an arbitrary base field. The talk is based on a joint work in progress with V. Vologodsky.

Yum-Tong Siu. Global non-deformability of irreducible compact Hermitian symmetric manifolds

The technique of higher order jets and jet differentials has been a powerful tool in complex geometry. We discuss its application to the global non-deformability problem. The problem of global non-deformability of the complex projective space was posed in 1958 by Kodaira and Spencer and later generalzied to irreducible compact Hermitian symmetric manifolds. It has been a longstanding open conjecture, known for the complex projective space and the hyperquadric or for Kähler deformation. We use high order jets to handle the Grassmannians and the other open cases.

Alexander L. Smirnov. A non-commutative tensor square of $\mathbb{Z}$ and the Riemann Hypothesis

For effective applications of the absolute geometry in arithmetic we need a nontrivial absolute tensor square of $\mathbb{Z}$. However all the known constructions lead to trivial squares. To improve the situation one may try to use a non-commutative tensor square instead. But there is no consideration to imitate in this case. In the talk we are going to present a relation between the RH and an NC-tensor product.

Vyacheslav I. Yanchevskiĭ. Henselian division algebras, G-involutions, and reduced unitary Whitehead groups for anisotropic outer forms of type A_n.

Let K be an infinite field. There are many important examples of infinite projectively simple groups (i.e., groups without non-central normal subgroups) supplied by linear algebra. For example, $SL_n(K)$, $n > 1$, $Sp_n(K, f)$ (where $Sp_n(K, f)$ are symplectic groups of alternating forms f), and etc.

A very useful extension of the range of examples of infinite projectively simple groups was the transition to linear algebraic groups, which led to new interesting conjectures and results. This approach made it possible to identify common properties that reflect the phenomenon of projective simplicity.

Let G be a linear algebraic group defined over a field K, G_K be the group of its K-rational points. Recall that a group G is anisotropic over K if it has no proper parabolic subgroups defined over K. Here a parabolic subgroup is a subgroup containing a Borel subgroup. Denote by G_K^+ the normal subgroup of G_K generated by rational over K elements of unipotent radicals of K-defined parabolic subgroups. In this situation, J. Tits established the following important fact (1964).

Theorem. *Let K contain at least 4 elements. Then any subgroup of G_K normalized by the group G_K^+ either contains G_K^+ or is central. In particular, G_K^+ is projectively simple.*

Thus, a new class of projectively simple groups arises. It is natural to assume that the structure of the group G_K is known if $G_K = G_K^+$. For special groups G and many fields K this fact was known by the time of the proof of the theorem, and therefore the following assumption seemed quite natural.

Conjecture (Kneser–Tits). *For a simply connected simple group G, which is defined and isotropic over the field K, $G_K^+ = G_K$.*

Note that the Kneser–Tits conjecture is obviously true in the case when K is algebraically closed. One also note that E. Cartan established the validity of the conjecture in the case when $K = \mathbb{R}$ and G is a simple simply connected algebraic group. For a long time it was believed that the Kneser–Tits conjecture was true since it is confirmed in a number of special cases. However, in 1975, V.P. Platonov showed that in the general case the conjecture is false. The latter led to Tits's definition of groups of algebraic K-groups $W(K, G) = G_K/G_K^+$ (for further details see [1].

Let G be a simply connected K-defined simple algebraic group. Then it belongs to one of the types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2. Among these types, the most interesting (and difficult to study) are groups of type A_n. The outer forms of groups of this type are limited to the following special unitary groups

$$SU_m(D, f) = \{u \in U_m(D, f)\colon Nrd_{M_m(D)}(a) = 1\},$$

where D is a division algebra of index d endowed with a unitary involution τ (i.e., with a nontrivial restriction on the center D), and K coincides with the field of τ-invariant elements of the center D, f is a non-degenerate m-dimensional Hermitian form, $U_m(D, f)$ is a unitary group of the form f, and $n = md - 1$.

In the isotropic case the form f is isotropic and there is an extensive bibliography devoted to the calculation of such groups. Passing to the anisotropic situation, we note that the Hermitian form f must be anisotropic. Despite the fact that the first papers on this topic date back to the early 2000s, the study of such groups is still difficult to approach. Since these groups will play a key role in the report, we will give their precise definition.

Definition. *The group $SUK_1^{an}(D, \tau) = SU_1(D, f)/U_1(D, f)'$, where $U_1(D, f)'$ is the commutant of the group $U_1(D, f)$, is called reduced unitary Whitehead group for the anisotropic form f.*

The first main results related to the calculation of non-trivial reduced Whitehead groups were obtained in frame of the class of Henselian division algebras and used the idea of reducing the problem of calculating these groups to the definition of some special subgroups of the multiplicative groups for their residue algebras. The structure of finite-dimensional central Henselian algebras was firstly obtained by Platonov and Yanchevskiĭ in 1985.

In a recent paper by the speaker [2] a scheme was proposed for calculating the groups $SUK_1^{an}(D, \tau)$ for the so-called cyclic involutions τ. The aim of the talk is to generalize the results from [2] related to the case of G-involutions for solvable groups G.

References.

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Yuri G. Zarhin. Non-isogenous elliptic curves

Let $E_f : y^2 = f(x)$ and $E_h : y^2 = h(x)$ be elliptic curves over a field K of characteristic zero that are defined by cubic polynomials $f(x)$ and $h(x)$ with coefficients in K.

Suppose that one of the polynomials is *irreducible* and the other *reducible*. We prove that if E_f and E_h are isogenous over an algebraic closure $\bar{K}$ of K then they both are isogenous over $\bar{K}$ to the elliptic curve

$$y^2 = x^3 - 1.$$

References.

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