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ТЕОРИЯ ВЕРОЯТНОСТЕЙ И ЕЕ ПРИМЕНЕНИЯ

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THE FIRST PASSAGE TIME DENSITY OF BROWNIAN MOTION AND THE HEAT EQUATION WITH DIRICHLET BOUNDARY CONDITION IN TIME DEPENDENT DOMAINS

В книге [Jimyeong Lee, “First passage time densities through Hölder curves”, ALEA Lat. Am. J. Probab. Math. Stat., 15:2 (2018), 837–849] доказано, что плотность момента первого пересечения границы одномерным стандартным броуновским движением будет непрерывной, когда граница непрерывна по Гёльдеру с показателем больше 1/2. С целью распространить результат [Jimyeong Lee, “First passage time densities through Hölder curves”, ALEA Lat. Am. J. Probab. Math. Stat., 15:2 (2018), 837–849] на многомерные области мы показываем, что существует непрерывная функция плотности момента первого пересечения подвижных границ в $\mathbf{R}^d$, $d \geqslant 2$, стандартным d-мерным броуновским движением при C^3-диффеоморфизме. Как и в [Jimyeong Lee, “First passage time densities through Hölder curves”, ALEA Lat. Am. J. Probab. Math. Stat., 15:2 (2018), 837–849], используя свойство локального времени стандартного d-мерного броуновского движения и уравнение теплопроводности с граничным условием Дирихле, мы находим достаточное условие существования непрерывной функции плотности.

Ключевые слова и фразы: момент первого пересечения границы, уравнение теплопроводности, броуновское движение, граничные условия Дирихле.

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1. Introduction. First passage time (FPT) problem, which is also called boundary crossing problem, is the one of classical subjects in probability which has also many applications to other fields, for example, finance and biology. There are a bunch of articles studying this problem, but especially we mention one of them, [1], which is about we can have a continuous first-passage-time density function of one dimensional standard Brownian motion when the boundary is Hölder continuous with exponent greater than 1/2.

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The purpose of this paper is that we extend the result and the general strategy of [1] into the multidimensional domain, precisely, to find a continuous density function of the first hitting time in a time varying domain in $\mathbf{R}^d$, $d \geqslant 2$, by investigating a relation between the first passage time density and the derivative at the boundary of the solution of the heat equation with Dirichlet boundary condition. Thus this paper is concerned with standard d-dimensional Brownian motion killed on the boundary of a deterministic moving domain which corresponds to the heat equation with Dirichlet boundary condition. In [2], this it is studied for reflected Brownian motion whose analytic counterpart is the heat equation with Neumann boundary condition.

Unfortunately, in general, it is hard to obtain the explicit form of the density function. For one dimensional Brownian motion, some analytic solutions are introduced in [1]. For two dimensional Brownian motion, in [3], the analytical solution of the Laplace transform of FPT distributions and its inverse is done numerically and other existing literatures including applications to quantitative finance are also summarized.

The organization of this paper is as follows: In section 2, we set up the regularity of time dependent domain and state the main theorem containing the existence of the continuous density function which is proportional to the normal derivative at the boundary of the solution of the heat equation with Dirichlet boundary condition. In section 3, we prove the weaker form of the main theorem to apply PDE techniques and Proposition 2, called the jump relation for our time varying domain that allows to have a implicit formula of the continuous density function that is an integral equation of Volterra type in Proposition 5. In section 4, we prove the main theorem by comparing between the solution obtained by the probabilistic construction, called Feynman–Kac formula, and the one obtained by Green's formula.

2. Preliminaries: problem setting and main theorem. We start with the domain Ω which is a bounded connected open set in $\mathbf{R}^d$, $d \geqslant 2$. We denote by $\partial\Omega$ and $\overline{\Omega}$ the boundary of Ω and the closure of Ω. The domain Ω changes in time with respect to a continuous velocity vector field $v\colon \mathbf{R}^d \times \mathbf{R}_+ \to \mathbf{R}^d$ such that there is a set of integral curves $\{\theta_s^t\colon \mathbf{R}^d \to \mathbf{R}^d\}_{s\leqslant t}$ which satisfies $\partial\theta_s^t x/\partial t = v(\theta_s^t x, t)$ and $\theta_s^s x = x$ for all $x \in \mathbf{R}^d$. For the existence of θ_s^t, we assume that for any finite interval $I \subset \mathbf{R}_+$, there is $L > 0$ such that $|v(x_1, t) - v(x_2, t)| \leqslant L|x_1 - x_2|$ for all x_1, x_2 in $\mathbf{R}^d$ and all $t \in I$. From now on, we restrict time domain to a finite interval $[0, T]$ for fixed $T > 0$. Then we have the following statement.

Proposition 1. *There is a unique homeomorphism* $\theta_s^t\colon \mathbf{R}^d \to \mathbf{R}^d$ *for any* $0 \leqslant s \leqslant t \leqslant T$, *which satisfies* $\partial\theta_s^t x/\partial t = v(\theta_s^t x, t)$ *and* $\theta_s^s x = x$ *for all* $x \in \mathbf{R}^d$.

Proof. See [4, Chap. 1, Theorem 2.1] or [5, Chap. 4, Proposition 1.1].

Thus we can write $\theta_s^t x = x + \int_s^t v(\theta_s^\tau x, \tau)\, d\tau$ and let us write θ_t^s as the inverse of θ_s^t. We denote by Ω_t the image of Ω under θ_0^t, by abuse of notation,

$\theta_s^t\colon \Omega_s \to \Omega_t$ is a homeomorphism and then we can also extend the domain of θ_s^t into $\overline{\Omega_s}$ so that θ_s^t is a boundary-preserving mapping from $\overline{\Omega_s}$ to $\overline{\Omega_t}$.

Now we consider the initial-boundary value problem for the heat equation in moving domains $\{\Omega_t\}_{t\geqslant 0}$ as follows:

$$\begin{cases} u_t = \dfrac{1}{2}\Delta_x u,\ 0 < t \leqslant T,\ x \in \Omega_t, \\ u(x,t) = 0,\ 0 \leqslant t \leqslant T,\ x \in \partial\Omega_t, \\ u(x,0) = u_0(x),\ x \in \Omega_0 = \Omega. \end{cases} \tag{2.1}$$

To obtain a solution of (2.1), we have to find suitable conditions for the initial function u_0, the moving velocity v and the boundaries $\partial\Omega_t$. In the following sections, we will approximate the Dirac delta function with a sequence of C_c^∞-functions thus we assume that u_0 is C_c^∞ and whose support is contained in Ω. Moreover, we suppose that $\partial\Omega$ is C^3, that is, if for each point $x^0 \in \partial\Omega$ there exist $r > 0$ and a C^3-function $F\colon \mathbf{R}^{d-1} \to \mathbf{R}$ such that — upon relabeling and reorienting the coordinates axes if necessary — we have

$$\Omega \cap \mathcal{B}(x^0, r) = \{x \in \mathcal{B}(x^0, r)\colon x_d > F(x_1, \dots, x_{d-1})\}, \tag{2.2}$$

where $\mathcal{B}(x, r) = \{y \in \mathbf{R}^d\colon |x-y| < r\}$ is the open ball with a center $x \in \mathbf{R}^d$ and radius $r > 0$ throughout the paper. In addition, let us assume that v is C^3 so that $\theta_s^t x (= \theta(x,t))$ is a function of x and t which is C^3, thus $\theta_s^t\colon \overline{\Omega_s} \to \overline{\Omega_t}$ is C^3-diffeomorphism (c.f. of [5, Chap. 4]). Let us denote by the parabolic cylinder and the lateral boundary

$$D_T := \bigcup_{0<t\leqslant T} \Omega_t \times \{t\}, \qquad S_T := \bigcup_{0\leqslant t\leqslant T} \partial\Omega_t \times \{t\}. \tag{2.3}$$

Then we have the following existence theorem.

Theorem 1 (existence)**.** *There exists a unique solution u of* (2.1) *such that $u \in C_{x,t}^{2,1}(\overline{D_T})$.*

Proof. See [6, Chap. 3, Theorem 7].

Remark 1. The assumption that $\partial\Omega$ and v are C^3 is to have $\nabla_x u(\,\cdot\,, t)$ is bounded on Ω_t so that it can be continuously extended to $\overline{\Omega_t}$ which is essential in Theorem 2 below. For the proof of Propositions 2 and 3, it is enough that $\partial\Omega$ is C^2 and v is C^2 with respect to spatial variable so that θ_s^t is C^2-diffeomorphism.

Let us call $\mathbf{P}_{r,s}$, $r \in \mathbf{R}$, $s \geqslant 0$, the law on $C([s,\infty))$ of the standard d-dimensional Brownian motion B_t, $t \geqslant s$, which starts from r at time s, i.e., $B_s = r$. For each $t > s$, the law of B_t is absolutely continuous with respect to the Lebesgue measure and has a density $G_{s,t}(r,\cdot\,)$, which is the Gaussian

$$G(\,\cdot\,, t; r, s) = \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^d \exp\left\{-\frac{(\,\cdot - r)^2}{2(t-s)}\right\}.$$

For $s \geqslant 0$ and $r \in \Omega_s$, we define

$$\tau_{r,s}^{\Omega,v} := \inf\{t \geqslant s \colon B_t \in \partial\Omega_t\} \quad \text{and equals } \infty \text{ if the set is empty}, \tag{2.4}$$

where $B_s = r$ and denote by $dF_{r,s}(y,q)$, $y \in \partial\Omega_q$, the distribution of $\tau_{r,s}^{\Omega,v}$ induced by $\mathbf{P}_{r,s}$. For $s = 0$, we use abbreviated forms $\mathbf{P}_r$, $\mathbf{E}_r$, τ_r, $dF_r(y,q)$ instead of $\mathbf{P}_{r,0}$, $\mathbf{E}_{r,0}$, $\tau_{r,0}^{\Omega,v}$, $dF_{r,0}(y,q)$, respectively, whenever it is needed. In addition, for $r_0 \in \Omega$ and $t > 0$, let us call $d\mu_{r_0}(\,\cdot\,, t)$ the positive measure on Ω_t such that

$$\int_{\Omega_t} d\mu_{r_0}(x,t) f(x) = \mathbf{E}_{r_0}[f(B_t); \tau_{r_0}^{\Omega,v} \geqslant t], \tag{2.5}$$

for all $f \in C_c^\infty(\mathbf{R}^d)$ with $\operatorname{supp} f \Subset \Omega_t$.

Since $\partial\Omega_t$ is C^3, we have the outward pointing unit normal vector field $n = n_{x,t}$ at $x \in \partial\Omega_t$ and let us denote $\partial f(x,t)/\partial n := n \cdot \nabla_x f(x,t)$ for $f \in C_x^1(\overline{D_T})$. Throughout the paper, we write $d\mathcal{H}^{d-1}$ as the $(d-1)$-dimensional Hausdorff measure on $\mathbf{R}^d$. The main result in the paper is the following statement.

Theorem 2. *Under the same assumption as in Theorem 1, we have, for any $r_0 \in \Omega_0 = \Omega$,*

(1) $d\mu_{r_0}(x,t) = G_{0,t}^{\Omega,v}(r_0,x)\,dx$, *where for all $x \in \Omega_t$,*

$$G_{0,t}^{\Omega,v}(r_0,x) = G_{0,t}(r_0,x) - \int_{[0,t)} \int_{\partial\Omega_s} G_{s,t}(y,x)\,dF_{r_0}(y,s); \tag{2.6}$$

(2) $dF_{r_0}(y,s)$ *has a continuous density function p such that* $dF_{r_0}(y,s) = p(y,s)\,d\mathcal{H}^{d-1}(y)\,ds$;

(3) *it holds*:

$$p(x,t) = -\frac{1}{2}\frac{\partial}{\partial n} G_{0,t}^{\Omega,v}(r_0,x)$$

for all $t > 0$ and $x \in \partial\Omega_t$;

(4) $G_{0,t}^{\Omega,v}(r_0,x)$ *solves*

$$w_t = \frac{1}{2}\Delta_x w, \qquad x \in \Omega_t, \quad t > 0, \tag{2.7}$$

$$\lim_{\Omega_t \ni x \to y} w(x,t) = 0, \qquad y \in \partial\Omega_t, \quad t > 0, \tag{2.8}$$

$$\lim_{(x,t)\to(y,0)} w(x,t) = \delta_{r_0}(y), \qquad y \in \Omega_0. \tag{2.9}$$

3. The weaker form of Theorem 2. Let us first prove item (1) of Theorem 2. By (2.5) and the strong Markov property of Brownian motion,

$$\int_{\Omega_t} d\mu_{r_0}(x,t) f(x) = \mathbf{E}_{r_0}[f(B_t); \tau_{r_0}^{\Omega,v} \geqslant t]$$
$$= \mathbf{E}_{r_0}[f(B_t)] - \mathbf{E}_{r_0}[f(B_t); \tau_{r_0}^{\Omega,v} < t] = \int_{\Omega_t} f(x) G_{0,t}(r_0,x)\,dx$$

$$
\begin{aligned}
&- \int_{[0,t)} \int_{\partial\Omega_s} \mathbf{E}_{y,s}[f(B_t) \mid B_s = y,\, \tau_{r_0}^{\Omega,v} = s]\, dF_{r_0}(y,s) \\
&= \int_{\Omega_t} f(x) G_{0,t}(r_0,x)\, dx - \int_{\Omega_t} f(x) \int_{[0,t)} \int_{\partial\Omega_s} G_{s,t}(y,x)\, dF_{r_0}(y,s)\, dx \\
&= \int_{\Omega_t} f(x) \Big(G_{0,t}(r_0,x) - \int_{[0,t)} \int_{\partial\Omega_s} G_{s,t}(y,x)\, dF_{r_0}(y,s) \Big)\, dx. \qquad (3.1)
\end{aligned}
$$

Thus the proof is done. Before proving the other items of Theorem 2, we study the weaker form of it which we will specify in what follows since the initial datum of item (4) is the Dirac delta function which is hard to control directly: so we let it as C_c^∞ and apply approximation as usual. By item (1) of Theorem 2, we have $G_{0,t}^{\Omega,v}$, so let us define

$$
u(x,t) := \int_{\Omega_0=\Omega} u_0(\xi) G_{0,t}^{\Omega,v}(\xi,x)\, d\xi \qquad (3.2)
$$

for given $u_0 \in C_c^\infty(\Omega;\mathbf{R}_+)$ and all $(x,t) \in D$.

We prove the following weaker form of Theorem 2.

Theorem 3. *Under the same assumption as in Theorem 1, we have*

(1) *the function u defined in (3.2) is a unique solution of Theorem 1;*

(2) *moreover, for all $t > 0$,*

$$
p_{u_0}(x,t) := -\frac{1}{2} \frac{\partial u}{\partial n}(x,t)
$$

satisfies

$$
\begin{aligned}
p_{u_0}(x,t) = &- \int_{\Omega_0} u_0(\xi) \frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0)\, d\xi \\
&+ \int_0^t \int_{\partial\Omega_s} \frac{\partial G}{\partial n_{x,t}}(x,t;y,s) p_{u_0}(y,s)\, d\mathcal{H}^{d-1}(y)\, ds. \qquad (3.3)
\end{aligned}
$$

Before going to the proof of Theorem 3, we will prove Propositions 2 and 3, as mentioned in Remark 1, we assume that $\partial\Omega$ is C^2 and v is C^2 with respect to a spatial variable. First, the following Lemmas 1 and 2 are needed for Proposition 2.

Lemma 1. *There is $C > 0$ such that, for all $0 \leqslant t \leqslant T$ and all $x \in \partial\Omega_t$, we have*

$$
|\langle y - x, n_{x,t} \rangle| \leqslant C|y-x|^2,
$$

if $y \in \partial\Omega_t$ is sufficiently close to x.

Proof. First we fix $0 \leqslant t \leqslant T$ and $x \in \partial\Omega_t$. There is a local representation $F \in C^2$ at x, without loss of generality, we may assume that $x = 0$ and $\mathbf{D}F(x) = 0$ such that all $y \in \partial\Omega_t$ sufficiently close to x can be written as

$y = (y_1, \ldots, y_d) = (y_1, \ldots, y_{d-1}, F(y_1, \ldots, y_{d-1}))$. By Taylor's theorem, we have

$$\begin{aligned}|\langle y - x, n_{x,t}\rangle| &= |\langle y, n_{x,t}\rangle| = |F(y_1, \ldots, y_{d-1})| \\ &\leqslant \|\mathbf{D}^2 F\|_\infty \sum_{i=1}^{d-1} |y_i|^2 \leqslant \|\mathbf{D}^2 F\|_\infty |y|^2.\end{aligned}$$

Thus the boundaries $\{\partial\Omega_t\}_{0 \leqslant t \leqslant T}$ are compact and diffeomorphic to each other under θ so that the second differential of a local representation is uniformly bounded. The proof is complete.

Lemma 2. *Given $\varepsilon > 0$, there is $\delta > 0$ such that, for all $0 < t \leqslant T$ and all $x \in \partial\Omega_t$,*

$$\left| \int_{\partial\Omega_s} \left(\frac{1}{\sqrt{2\pi(t-s)}} \right)^{d-1} \exp\left\{ -\frac{|x-y|^2}{2(t-s)} \right\} d\mathcal{H}^{d-1}(y) - 1 \right| < \varepsilon$$

if $t - \delta < s < t$.

Proof. We fix $0 < t \leqslant T$ and $x \in \partial\Omega_t$. For a local representation $F \in C^2$ at x, without loss of generality, we may assume that $x = 0$ and $\mathbf{D}F(x) = 0$ such that all $y \in \partial\Omega_t$ sufficiently close to x can be written as $y = (y_1, \ldots, y_d) = (y_1, \ldots, y_{d-1}, F(y_1, \ldots, y_{d-1}))$. Let us choose γ and η such that $1/4 < \gamma < 1/2$ and $\eta > \sup_{\overline{D}_T} |v|$. We define

$$E_s^1 := \partial\Omega_s \cap \mathcal{B}(x, \eta(t-s)^\gamma), \quad E_s^2 := \partial\Omega_s - E_s^1.$$

Then $\theta_t^s x \in E_s^1$ for all s sufficiently close to t. By a change of variables, we obtain

$$\begin{aligned}I &= \int_{E_s^1} \left(\frac{1}{\sqrt{2\pi(t-s)}} \right)^{d-1} \exp\left\{ -\frac{|\theta_s^t y|^2}{2(t-s)} \right\} |\mathrm{Jac}(\theta_s^t y)| \, d\mathcal{H}^{d-1}(y) \\ &= \int_{\theta_s^t(E_s^1)} \left(\frac{1}{\sqrt{2\pi(t-s)}} \right)^{d-1} \exp\left\{ -\frac{|y|^2}{2(t-s)} \right\} d\mathcal{H}^{d-1}(y) \\ &= \int_{P(\theta_s^t(E_s^1))} \left(\frac{1}{\sqrt{2\pi(t-s)}} \right)^{d-1} \\ &\quad \times \exp\left\{ -\frac{\sum_{i=1}^{d-1} |y_i|^2 + |F(y_1, \ldots, y_{d-1})|^2}{2(t-s)} \right\} \sqrt{|A|} \, dy_1 \cdots dy_{d-1},\end{aligned}$$

where Jac is a Jacobian matrix and P is a projection such that $P(y_1, \ldots, y_d) = (y_1, \ldots, y_{d-1})$ and A is a $(d-1) \times (d-1)$-matrix whose entry a_{ij} is given by $\delta_{ij} + (\partial F / \partial y_i)(\partial F / \partial y_j)$.

If $y \in E_s^1$, then

$$\begin{aligned}|\theta_s^t y|^2 &= |y|^2 + 2\left| \left\langle y, \int_s^t v(\theta_s^\tau y, \tau) \, d\tau \right\rangle \right| + \left| \int_s^t v(\theta_s^\tau y, \tau) \, d\tau \right|^2 \\ &\leqslant |y|^2 + 2\eta^2 (t-s)^{1+\gamma} + \eta^2 (t-s)^2\end{aligned}$$

and

$$|\mathrm{Jac}(\theta_s^t y)| = \left|\left[\delta_{ij} + \int_s^t \frac{\partial}{\partial y_j}\big(v^{(i)}(\theta_s^\tau y, \tau)\big)\, d\tau\right]\right|.$$

Thus for given $\varepsilon > 0$, there is $\delta_1 > 0$ which does not depend on the choice of (x,t) such that, for all $t-\delta_1 < s < t$,

$$\left| I - \int_{E_s^1} \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{|y|^2}{2(t-s)}\right\} d\mathcal{H}^{d-1}(y)\right| < \frac{\varepsilon}{4}.$$

If $(y_1,\dots,y_{d-1}) \in P(\theta_s^t(E_s^1))$, then $|\theta_t^s(y_1,\dots,y_{d-1},F(y_1,\dots,y_{d-1}))| \leqslant \eta(t-s)^\gamma$. So we have $|(y_1,\dots,y_{d-1})| \leqslant 2\eta(t-s)^\gamma$ for all s sufficiently close to t, thus

$$|F(y_1,\dots,y_{d-1})| \leqslant \|\mathbf{D}^2F\|_\infty 4\eta^2(t-s)^{2\gamma}$$

and then

$$-\frac{|F(y_1,\dots,y_{d-1})|^2}{2(t-s)} \geqslant -8\eta^4\|\mathbf{D}^2F\|_\infty^2(t-s)^{4\gamma-1}.$$

By the mean value theorem, we get

$$\left|\frac{\partial F}{\partial y_i}\right| \leqslant \|\mathbf{D}^2F\|_\infty |(y_1,\dots,y_{d-1})| \leqslant \|\mathbf{D}^2F\|_\infty 2\eta(t-s)^\gamma.$$

Since the second differential of a local representation is uniformly bounded, for given $\varepsilon > 0$, there is $\delta_2 > 0$ which does not depend on the choice of (x,t) such that for all $t-\delta_2 < s < t$,

$$\left| I - \int_{P(\theta_s^t(E_s^1))} \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{\sum_{i=1}^{d-1}|y_i|^2}{2(t-s)}\right\} dy_1\cdots dy_{d-1}\right| < \frac{\varepsilon}{4}.$$

Moreover, if $|(y_1,\dots,y_{d-1})| \leqslant \frac{\eta}{2}(t-s)^\gamma$, then $(y_1,\dots,y_{d-1}) \in P(\theta_s^t(E_s^1))$ for all s sufficiently close to t, so for given $\varepsilon > 0$, there is $\delta_3 > 0$ which does not depend on the choice of (x,t) such that, for all $t-\delta_3 < s < t$,

$$\left| \int_{P(\theta_s^t(E_s^1))} \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{\sum_{i=1}^{d-1}|y_i|^2}{2(t-s)}\right\} dy_1\cdots dy_{d-1} - 1\right| < \frac{\varepsilon}{4}.$$

If we take $\delta_4 = \min\{\delta_1,\delta_2,\delta_3\}$, we conclude that for given $\varepsilon > 0$, there is $\delta_4 > 0$ which does not depend on the choice of (x,t) such that, for all $t-\delta_4 < s < t$,

$$\left| \int_{E_s^1} \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{|y|^2}{2(t-s)}\right\} d\mathcal{H}^{d-1}(y) - 1\right| < \frac{3\varepsilon}{4}.$$

For all $y \in E_s^2$, we have

$$\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{|y|^2}{2(t-s)}\right\} \leqslant \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{\eta^2}{2(t-s)^{1-2\gamma}}\right\}$$

and so for given $\varepsilon > 0$, there is $\delta_5 > 0$ which does not depend on the choice of (x, t) such that, for all $t - \delta_5 < s < t$,

$$\left|\int_{E_s^2} \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{d-1} \exp\left\{-\frac{|y|^2}{2(t-s)}\right\} d\mathcal{H}^{d-1}(y)\right| < \frac{\varepsilon}{4}.$$

Therefore, if we let δ be the minimum value of δ_i's, the proof is complete.

The following Proposition 2, we call it a jump relation, is the most significant property of the single-layer potential, whose one dimensional version is proved in [7] and also applied in the analysis of [1]. We develope it into a multidimensional case under the assumption discussed above and give a rigorous calculation.

Proposition 2 (jump relation)**.** *For $\varphi \in C(S_T)$, we have*

$$\lim_{h\to 0^+} \int_0^t \int_{\partial\Omega_s} n_{x,t} \cdot \nabla_x G(x - hn_{x,t}, t; y, s)\varphi(y, s)\, d\mathcal{H}^{d-1}(y)\, ds = \varphi(x, t) + \int_0^t \int_{\partial\Omega_s} \frac{\partial G}{\partial n_{x,t}}(x, t; y, s)\varphi(y, s)\, d\mathcal{H}^{d-1}(y)\, ds,$$

for all $0 < t \leqslant T$ and $x \in \partial\Omega_t$.

Proof. Let us fix $0 < t \leqslant T$, $x \in \partial\Omega_t$. Let γ, η, E_s^1, E_s^2 be same as in the proof of Lemma 2. Without loss of generality, we may assume that $\varphi \geqslant 0$. We can write

$$\int_0^t \int_{\partial\Omega_s} n_{x,t} \cdot \nabla_x G(x - hn_{x,t}, t; y, s)\varphi(y, s)\, d\mathcal{H}^{d-1}(y)\, ds = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{t-\delta}^t \int_{E_s^1} \frac{h}{t-s}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^d \times \exp\left\{-\frac{|x - hn_{x,t} - y|^2}{2(t-s)}\right\}\varphi(y, s)\, d\mathcal{H}^{d-1}(y)\, ds,$$

$$I_2 = \int_{t-\delta}^t \int_{E_s^1} \frac{\langle y - x, n_{x,t}\rangle}{t-s}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^d \times \exp\left\{-\frac{|x - hn_{x,t} - y|^2}{2(t-s)}\right\}\varphi(y, s)\, d\mathcal{H}^{d-1}(y)\, ds,$$

$$I_3 = \int_{t-\delta}^{t} \int_{E_s^2} \frac{\langle y - x + hn_{x,t}, n_{x,t} \rangle}{t-s} \left(\frac{1}{\sqrt{2\pi(t-s)}} \right)^d \times \exp\left\{ -\frac{|x - hn_{x,t} - y|^2}{2(t-s)} \right\} \varphi(y,s)\, d\mathcal{H}^{d-1}(y)\, ds,$$

$$I_4 = \int_0^{t-\delta} \int_{\partial\Omega_s} n_{x,t} \cdot \nabla_x G(x - hn_{x,t}, t; y, s) \varphi(y,s)\, d\mathcal{H}^{d-1}(y)\, ds.$$

For I_1, we can rewrite as follows:

$$J(s,h) = \int_{E_s^1} \left(\frac{1}{\sqrt{2\pi(t-s)}} \right)^{d-1} \exp\left\{ -\frac{|x-y|^2}{2(t-s)} \right\} \times \exp\left\{ \frac{h\langle x - y, n_{x,t} \rangle}{t-s} \right\} \varphi(y,s)\, d\mathcal{H}^{d-1}(y),$$

$$I_1 = \int_{t-\delta}^{t} \frac{h}{t-s} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{h^2}{2(t-s)} \right\} J(s,h)\, ds.$$

By Lemma 1, there is $C > 0$ such that

$$\frac{|\langle x - y, n_{x,t} \rangle|}{t-s} \leqslant \frac{|\langle x - \theta_s^t y + \theta_s^t y - y, n_{x,t} \rangle|}{t-s} \leqslant \frac{C}{(t-s)^{1-2\gamma}}, \tag{3.4}$$

for all s sufficiently close to t and all $y \in E_s^1$.

Then by (3.4),

$$J(s,0) \exp\left\{ -\frac{Ch}{(t-s)^{1-2\gamma}} \right\} \leqslant J(s,h) \leqslant J(s,0) \exp\left\{ \frac{Ch}{(t-s)^{1-2\gamma}} \right\}. \tag{3.5}$$

By (3.5) and a change of variable $z = h/\sqrt{t-s}$, we have

$$\int_{h/\sqrt{\delta}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{ -\frac{|z|^2}{2} \right\} \exp\{-Cz(t-s)^{2\gamma-1/2}\} J\left(t - \frac{h^2}{z^2}, 0 \right) dz \leqslant I_1 \leqslant \int_{h/\sqrt{\delta}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{ -\frac{|z|^2}{2} \right\} \exp\{Cz(t-s)^{2\gamma-1/2}\} J\left(t - \frac{h^2}{z^2}, 0 \right) dz. \tag{3.6}$$

By Lemma 2, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|J(s,0) - \varphi(x,t)| < \varepsilon, \tag{3.7}$$

for all $t - \delta < s < t$. Therefore, it follows by (3.6) and (3.7) that

$$\int_{h/\sqrt{\delta}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{ -\frac{|z + C\delta^{2\gamma-1/2}|^2}{2} \right\} \exp\left\{ \frac{C^2\delta^{4\gamma-1}}{2} \right\} (\varphi(x,t) - \varepsilon)\, dz \leqslant I_1 \leqslant \int_{h/\sqrt{\delta}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{ -\frac{|z - C\delta^{2\gamma-1/2}|^2}{2} \right\} \exp\left\{ \frac{C^2\delta^{4\gamma-1}}{2} \right\} (\varphi(x,t) + \varepsilon)\, dz$$

and so we have

$$\left|\lim_{\delta\to 0^+}\lim_{h\to 0^+} I_1 - \varphi(x,t)\right| < \varepsilon.$$

For I_2, by (3.4), (3.5), and (3.7), we have

$$|I_2| \leqslant \int_{t-\delta}^{t} \frac{C_1}{(t-s)^{3/2-2\gamma}} \exp\left\{\frac{C^2\delta^{4\gamma-1}}{2}\right\}(\varphi(x,t)+\varepsilon)\,ds \leqslant C_2\delta^{2\gamma-1/2}$$

and so it follows that $\lim_{\delta\to 0^+}\lim_{h\to 0^+}|I_2| = 0$.

For I_3, for all sufficiently small $h>0$ such that $(2h/\eta)^{1/\gamma} < \delta$, we can decompose I_3 into $I_{3,1} + I_{3,2}$ as

$$\begin{aligned} I_{3,1} = &\int_{t-\delta}^{t-(2h/\eta)^{1/\gamma}} \int_{E_s^2} \frac{\langle y - x + hn_{x,t}, n_{x,t}\rangle}{t-s}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^d \\ &\times \exp\left\{-\frac{|x-hn_{x,t}-y|^2}{2(t-s)}\right\}\varphi(y,s)\,d\mathcal{H}^{d-1}(y)\,ds, \\ I_{3,2} = &\int_{t-(2h/\eta)^{1/\gamma}}^{t} \int_{E_s^2} \frac{\langle y - x + hn_{x,t}, n_{x,t}\rangle}{t-s}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^d \\ &\times \exp\left\{-\frac{|x-hn_{x,t}-y|^2}{2(t-s)}\right\}\varphi(y,s)\,d\mathcal{H}^{d-1}(y)\,ds. \end{aligned}$$

If $y\in E_s^2$ and $t-\delta < s < t-(2h/\eta)^{1/\gamma}$, then $|x-y| \geqslant \eta(t-s)^\gamma > 2h$, so we have

$$\begin{aligned} |I_{3,1}| &\leqslant C_3\int_{t-\delta}^{t-(2h/\eta)^{1/\gamma}} \frac{1}{(t-s)^{d/2}}\int_{E_s^2}\frac{|x-y|}{t-s}\exp\left\{-\frac{|x-y|^2}{8(t-s)}\right\}d\mathcal{H}^{d-1}(y)\,ds \\ &\leqslant C_3\int_{t-\delta}^{t-(2h/\eta)^{1/\gamma}} \frac{1}{\eta(t-s)^{\gamma+d/2}}\int_{E_s^2}\frac{|x-y|^2}{t-s}\exp\left\{-\frac{|x-y|^2}{8(t-s)}\right\}d\mathcal{H}^{d-1}(y)\,ds \\ &\leqslant C_4\int_{t-\delta}^{t-(2h/\eta)^{1/\gamma}} \int_{E_s^2}\frac{1}{(t-s)^{1-\gamma+d/2}}\exp\left\{-\frac{C_5}{(t-s)^{1-2\gamma}}\right\}d\mathcal{H}^{d-1}(y)\,ds \end{aligned}$$

and so it follows that $\lim_{\delta\to 0^+}\lim_{h\to 0^+}|I_{3,1}| = 0$.

Let $y\in\partial\Omega_s$. For all sufficiently small $h>0$, if $t-(2h/\eta)^{1/\gamma} \leqslant s \leqslant t$ and $\gamma < \beta < 1/2$, then

$$\begin{aligned} |x-hn_{x,t}-y| &\geqslant |x-hn_{x,t}-\theta_s^t y| - |\theta_s^t y - y| \geqslant d(x-hn_{x,t},\partial\Omega_t) - \eta(t-s) \\ &\geqslant h - C_6h^2 - \eta(t-s) \geqslant (t-s)^\beta, \end{aligned}$$

where $d(x-hn_{x,t},\partial\Omega_t) = \inf\{|x-hn_{x,t}-y| \colon y\in\partial\Omega_t\} \geqslant h - C_6h^2$ is by Lemma 1.

Hence we get

$$\begin{aligned}|I_{3,2}| &\leqslant C_7 \int_{t-(2h/\eta)^{1/\gamma}}^{t} \frac{1}{(t-s)^{\beta+d/2}} \int_{E_s^2} \frac{|x-hn_{x,t}-y|^2}{t-s} \\ &\quad \times \exp\left\{-\frac{|x-hn_{x,t}-y|^2}{2(t-s)}\right\} d\mathcal{H}^{d-1}(y)\, ds \\ &\leqslant C_8 \int_{t-(2h/\eta)^{1/\gamma}}^{t} \int_{E_s^2} \frac{1}{(t-s)^{1-\beta+d/2}} \exp\left\{-\frac{1}{2(t-s)^{1-2\beta}}\right\} d\mathcal{H}^{d-1}(y)\, ds\end{aligned}$$

and so it follows that $\lim_{h\to 0^+} |I_{3,2}| = 0$.

For I_4, by Lebesgue's dominated convergence theorem, it follows that

$$\lim_{h\to 0^+} I_4 = \int_0^{t-\delta} \int_{\partial\Omega_s} \frac{\partial G}{\partial n_{x,t}}(x,t;y,s)\varphi(y,s)\, d\mathcal{H}^{d-1}(y)\, ds.$$

Finally, by combining all estimates above, we obtain

$$\left| \lim_{\delta\to 0^+} \lim_{h\to 0^+} (I_1+I_2+I_3+I_4) - \varphi(x,t) - \int_0^t \int_{\partial\Omega_s} \frac{\partial G}{\partial n_{x,t}}(x,t;y,s)\varphi(y,s)\, d\mathcal{H}^{d-1}(y)\, ds \right| < \varepsilon.$$

Since ε is arbitrary, the proof is complete.

Remark 2. In Chapter 5 of [6], the jump relation is proved for time independent domains and in Chapter 3 of [8], it is also proved when the domain D is given by $D = \{(Z,t)\colon z_d > f(z_1,\dots,z_{d-1},t)\}$, where $f\colon \mathbf{R}^d \to \mathbf{R}$ satisfies

$$\begin{gathered}|f(x,t)-f(y,t)| \leqslant a_1|x-y|, \qquad x,y\in\mathbf{R}^{d-1}, \quad t\in\mathbf{R}, \\ f(x,t) = I_{1/2}(b(x,\cdot))(t) = \int_{\mathbf{R}} |s-t|^{-1/2} b(x,s)\, ds,\end{gathered}$$

where $x\in\mathbf{R}^{d-1}$ is fixed and $b(x,\cdot)$ is of bounded mean oscillation on $\mathbf{R}$.

The following proposition depends on a local property of Brownian motion and the regularity of the boundary whose one dimensional version is done in [1], which is about the accessibility of the boundary necessary to show Dirichlet boundary condition in Theorem 3.

Proposition 3. *If the starting point of Brownian motion is close to* X, *the first passage time converges to* 0. *Precisely*,

$$\lim_{\Omega_0\ni\xi\to\xi_0\in\partial\Omega_0} \mathbf{P}_\xi[\tau_\xi^{\Omega,v} > s] = 0$$

for all $s > 0$.

Proof. Without loss of generality, we may assume that $0 \in \partial\Omega_0$ and the tangent plane of $\partial\Omega_0$ at 0 and e_d is the outward unit normal vector at 0. A standard d-dimensional Brownian motion $B_t = (B_t^{(1)}, \ldots, B_t^{(d)})$ which starts at $\xi = (\xi^{(1)}, \ldots, \xi^{(d)})$. Since θ_t^0 is a diffeomorphim, B_t is out of Ω_t if and only if $\theta_t^0 B_t = B_t - \int_0^t v(\theta_t^s B_t, s)\, ds$ is out of Ω_0. We have $a > 0$ such that $\mathcal{B}(0,a) \cap \partial\Omega_0 = \{x \in \mathbf{R}^d \colon x^{(d)} = F(x^{(1)}, \ldots, x^{(d-1)})\}$ for some $F \in C^2$. Thus there is $c > 0$ which does not depend on a such that if $x \in \mathcal{B}(0,a) \cap \partial\Omega_0$, then $|x^{(d)}| \leqslant ca^2$. Since $\sup|v| < \eta$, we have

$$\inf\{t \geqslant 0 \colon (\theta_t^0 B_t)^{(d)} = ca^2\} \leqslant \inf\{t \geqslant 0 \colon B_t^{(d)} - \eta t = ca^2\}$$

and

$$\begin{aligned}\inf\{t \geqslant 0 \colon \big|\big((\theta_t^0 B_t)^{(1)}, \ldots, (\theta_t^0 B_t)^{(d-1)}\big)\big| = a\} \\ \geqslant \inf\{t \geqslant 0 \colon \big|\big(B_t^{(1)}, \ldots, B_t^{(d-1)}\big)\big| = a - \eta t\} \\ \geqslant U^{(1)}_{(a-\eta t)/\sqrt{d-1}} \wedge \cdots \wedge U^{(d-1)}_{(a-\eta t)/\sqrt{d-1}},\end{aligned}$$

where

$$U^{(i)}_{(a-\eta t)/\sqrt{d-1}} = \inf\left\{t \geqslant 0 \colon |B_t^{(i)}| = \frac{a - \eta t}{\sqrt{d-1}}\right\} \quad \text{for} \quad 1 \leqslant i \leqslant d-1.$$

It satisfies that $a/\sqrt{d-1} > ca^2$ for all sufficiently small $a > 0$ and for such fixed a, we have $a/\sqrt{d-1} - |\xi_i| \geqslant ca^2 - \xi_d$ for all ξ sufficiently close to 0 and all $1 \leqslant i \leqslant d-1$.

For any standard one-dimensional Brownian motion $\widetilde{B}_t$ which starts at 0, we have $\limsup_{t\downarrow 0}(\widetilde{B}_t/\sqrt{t}) = \infty$, then there is a sequence $t_k \downarrow 0$ such that $2\eta t_k \leqslant \widetilde{B}_{t_k}$. Thus if we fix a sufficiently small, then we obtain for all ξ sufficiently close to 0,

$$\inf\{t \geqslant 0 \colon \widetilde{B}_t = ca^2 + \eta t - \xi_d\} \leqslant \inf\left\{t \geqslant 0 \colon \widetilde{B}_t = \frac{a - \eta t}{\sqrt{d-1}} - \xi_i\right\}$$

and since $\eta \geqslant \eta/\sqrt{d-1}$, we also get

$$\inf\{t \geqslant 0 \colon \widetilde{B}_t = ca^2 + \eta t - \xi_d\} \leqslant \inf\left\{t \geqslant 0 \colon \widetilde{B}_t = \frac{\eta t - a}{\sqrt{d-1}} - \xi_i\right\}.$$

The variable $U^{(i)}_{(a-\eta t)/\sqrt{d-1}}$ has the same law with

$$\inf\left\{t \geqslant 0 \colon \widetilde{B}_t = \frac{a - \eta t}{\sqrt{d-1}} - \xi_i\right\} \wedge \inf\left\{t \geqslant 0 \colon \widetilde{B}_t = \frac{\eta t - a}{\sqrt{d-1}} - \xi_i\right\}$$

and $\inf\{t \geqslant 0 \colon \widetilde{B}_t = ca^2 + \eta t - \xi_d\}$ has the same law with $\inf\{t \geqslant 0 \colon B_t^{(d)} - \eta t = ca^2\}$, therefore, we conclude that

$$\inf\{t \geqslant 0 \colon (\theta_t^0 B_t)^{(d)} = ca^2\} \leqslant \inf\{t \geqslant 0 \colon \big|(\theta_t^0 B_t)^{(1)}, \ldots, (\theta_t^0 B_t)^{(d-1)})\big| = a\}.$$

Since we can fix a arbitrary small so that for any $s > 0$, the first passage time can be smaller than s as the starting point ξ is sufficiently close to 0. The proof is complete.

Proof of Theorem 3. Using the invariance of the law of the Brownian motion under time reversal, we have

$$u(x,t) = \int_{\Omega_0} u_0(\xi) G_{0,t}^{\Omega,v}(\xi,x)\, d\xi = \mathbf{E}_x[u_0(B_t); \tau_x^{\Omega_t,-v} \geqslant t],$$

where $\tau^{\Omega_t,-v}$ denote the first passage time when the time varying domain starts with Ω_t and changes with respect to $-v$. Using this equality, we also have

$$|u(x,t)| = \left|\mathbf{E}_x[u_0(B_t); \tau_x^{\Omega_t,-v} \geqslant t]\right| \leqslant \|u_0\|_\infty \mathbf{P}_x[\tau_x^{\Omega_t,-v} \geqslant t].$$

Thus by Proposition 2, we have $u(x,t) \to 0$ since x approaches to $\partial\Omega_t$.

Let us prove that u satisfies the initial data u_0, that is, $\lim_{(x,t)\to(y,0)} u(x,t) = u_0(y)$ for all $y \in \Omega_0$. Fix $y \in \Omega_0$ and, without loss of generality, we may assume that $y = 0$ by shifting the origin. For any $x \in \Omega_0$ and any positive $\lambda > 0$,

$$\begin{aligned}\mathbf{P}_x[\tau_x^{\Omega_t,-v} < t] &\leqslant \mathbf{P}_x\Big[\max_{s\in[0,t]} |\theta_t^s B_s - x| \geqslant d(x,\partial\Omega_t)\Big] \\ &\leqslant \mathbf{P}_x\Big[\max_{s\in[0,t]} |B_s| \geqslant d(x,\partial\Omega_t) - |x| - \eta t\Big] \\ &= \mathbf{P}_x\Big[\max_{s\in[0,t]} \exp\{\lambda|B_s|\} \geqslant \exp\{\lambda(d(x,\partial\Omega_t) - |x| - \eta t)\}\Big].\end{aligned} \tag{3.8}$$

Since $\exp\{\lambda|B_t|\}$ is a positive submartingale, we can apply Doob's inequality, then

$$\begin{aligned}&\mathbf{P}_x\Big[\max_{s\in[0,t]} \exp\{\lambda|B_s|\} \geqslant \exp\{\lambda(d(x,\partial\Omega_t) - |x| - \eta t)\}\Big] \\ &\qquad \leqslant \frac{\mathbf{E}_x[\exp(\lambda|B_t|)]}{\exp\{\lambda(d(x,\partial\Omega_t) - |x| - \eta t)\}}.\end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we obtain

$$\lim_{(x,t)\to(y,0)} \mathbf{P}_x[\tau_x^{\Omega_t,-v} < t] \leqslant \exp\{-\lambda d(y,\partial\Omega_0)\}$$

so that the left-hand side vanishes since $\lambda > 0$ is arbitrary. Thus we deduce that

$$\lim_{(x,t)\to(y,0)} \int_{\Omega_0} u_0(\xi) G_{0,t}^{\Omega,v}(\xi,x)\, d\xi = \lim_{(x,t)\to(y,0)} \mathbf{E}_x[u_0(B_t)] = u_0(y).$$

Since the Gaussian kernel satisfies the heat equation, we conclude that u is a unique solution of Theorem 1.

To show (3.3), let us fix $(x,t) \in D_T$. Applying Green's formula, we have

$$\int_{\Omega_s} u(y,s)\Delta_y G(x,t;y,s) - G(x,t;y,s)\Delta_y u(y,s)\,dy \\ = \int_{\partial\Omega_s} u(y,s)\,\frac{\partial G}{\partial n_{y,s}}(x,t;y,s) - G(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y). \quad (3.10)$$

Since u is 0 on the boundary, (3.10) becomes

$$\int_{\Omega_s} u(y,s)\Delta_y G(x,t;y,s) - G(x,t;y,s)\Delta_y u(y,s)\,dy \\ = \int_{\Omega_s} -2(uG)_s\,dy = -2\frac{\partial}{\partial s}\int_{\Omega_s} uG\,dy \\ = \int_{\partial\Omega_s} -G(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y). \quad (3.11)$$

By integrating the last two terms of (3.11) with respect to s from 0 to t, we obtain another different representation of u as follows:

$$u(x,t) = \int_{\Omega_0} u_0(\xi)G(x,t;\xi,0)\,d\xi \\ + \frac{1}{2}\int_0^t \int_{\partial\Omega_s} G(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds. \quad (3.12)$$

Taking normal derivatives of both sides in (3.12) and applying the jump relation, we get

$$\frac{1}{2}\frac{\partial u}{\partial n}(x,t) = \int_{\Omega_0} u_0(\xi)\,\frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0)\,d\xi \\ + \frac{1}{2}\int_0^t \int_{\partial\Omega_s} \frac{\partial G}{\partial n_{x,t}}(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds$$

which implies (3.3). The proof is complete.

4. Proof of Theorem 2. Comparing the definition (3.2) of u and (3.12), using (2.6), we see the following equality:

$$\int_{[0,t)}\int_{\partial\Omega_s} G_{s,t}(y,x)\int_{\Omega_0} u_0(\xi)\,dF_\xi(y,s)\,d\xi \\ = -\frac{1}{2}\int_0^t \int_{\partial\Omega_s} G(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds.$$

Let us denote $dF_{u_0}(y,s) := \int_{\Omega_0} u_0(\xi)\,dF_\xi(y,s)\,d\xi$.

Proposition 4. *For dF_{u_0} defined above, we have*

$$dF_{u_0}(y,s) = -\frac{1}{2}\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds.$$

For the proof of Proposition 4, we introduce the mass lost $\Delta_I^{\Omega,v}(u)$, $I = [t_1,t_2] \subset [0,T]$, $t_1 \leqslant t_2$, defined by

$$\Delta_I^{\Omega,v}(u) = \int_{\Omega_{t_1}} u(x,t_1)\,dx - \int_{\Omega_{t_2}} u(x,t_2)\,dx.$$

If we see the right-hand side of (3.12), we can extend u to $\overline{u}$ defined in $\{(x,t)\colon x \in \mathbf{R}^d,\ 0 < t \leqslant T\}$ as

$$\overline{u}(x,t) = \int_{\Omega_0} u_0(\xi)G(x,t;\xi,0)\,d\xi + \frac{1}{2}\int_0^t \int_{\partial\Omega_s} G(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds.$$

Then this satisfies the heat equation with $\lim_{(x,t)\to(y,0)} \overline{u}(x,t) = 0$ for all $y \in \Omega_0^c$ and also satisfies $\overline{u}(x,t) = 0$ for all $0 < t \leqslant T$ and all $x \in \partial\Omega_t$. Moreover, by the properties of the Gaussian kernel, we have

$$\lim_{|x|\to\infty} \sup_{0<t\leqslant T} |\overline{u}(x,t)| = 0.$$

It follows that $\overline{u}(x,t) = 0$ in $\{(x,t)\colon x \in \Omega_t^c,\ 0 < t \leqslant T\}$ by the weak maximum (minimum) principle. Thus we assume that u is defined in $\{(x,t)\colon x\in\mathbf{R}^d,\ 0 \leqslant t \leqslant T\}$ such that it is 0 in $\{(x,t)\colon x \in \Omega_t^c,\ 0 \leqslant t \leqslant T\}$.

Heuristically

$$\Delta_I^X(u) = -\int\int_{t_1}^{t_2} u_t(x,t)\,dt\,dx = -\int\int_{t_1}^{t_2} \frac{1}{2}\Delta_x u(x,t)\,dx\,dt = -\frac{1}{2}\int_{t_1}^{t_2}\int_{\partial\Omega_t} \frac{\partial u}{\partial n}(x,t)\,d\mathcal{H}^{d-1}(x)\,dt.$$

Since we do not control $\Delta_x u$ at the moving boundary, we cannot make this argument rigorously. Thus we use a different approach.

Proof of Proposition 4. It suffices to show that

$$-\frac{1}{2}\int_I\int_{\partial\Omega_t} \frac{\partial u}{\partial n}(x,t)\,d\mathcal{H}^{d-1}(x)\,dt = \Delta_I^{\Omega,v}(u) = \int_I\int_{\partial\Omega_t} dF_{u_0}(x,t).$$

If we integrate both sides of (3.12), then

$$\int_{\mathbf{R}^d} u(x,t)\,dx = \int_{\mathbf{R}^d}\int_{\Omega_0} u_0(\xi)G(x,t;\xi,0)\,d\xi\,dx + \frac{1}{2}\int_{\mathbf{R}^d}\int_0^t\int_{\partial\Omega_s} G(x,t;y,s)\,\frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds.$$

Applying Fubini's theorem, we get

$$\int_{\Omega_t} u(x,t)\,dx = \int_{\mathbf{R}^d} u(x,t)\,dx$$
$$= \int_{\Omega_0} u_0(\xi)\,d\xi + \frac{1}{2}\int_0^t \int_{\partial\Omega_s} \frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds.$$

Thus we get the first equality of the proposition,

$$\Delta^X_{[t_1,t_2]}(u) = -\frac{1}{2}\int_{t_1}^{t_2}\int_{\partial\Omega_s} \frac{\partial u}{\partial n}(y,s)\,d\mathcal{H}^{d-1}(y)\,ds.$$

By (2.5) and item (1) of Theorem 2, we get

$$\mathbf{P}_\xi[\tau^X_\xi \geqslant t] = \int_{\Omega_t} G^{\Omega,v}_{0,t}(\xi,x)\,dx.$$

For $0 = t_1 < t_2$, using Fubini's theorem again, we get

$$\Delta^{\Omega,v}_I(u) = \int_{\Omega_0} u_0(\xi)\,d\xi - \int_{\Omega_{t_2}}\int_{\Omega_0} u_0(\xi) G^{\Omega,v}_{0,t_2}(\xi,x)\,d\xi\,dx$$
$$= \int_{\Omega_0} u_0(\xi)\,d\xi - \int_{-\infty}^0 u_0(\xi)]\mathbf{P}_\xi[\tau^{\Omega,v}_\xi \geqslant t_2]\,d\xi$$
$$= \int_{-\infty}^0 u_0(\xi)\mathbf{P}_\xi[0 \leqslant \tau^X_\xi < t_2]\,d\xi.$$

For $0 < t_1 < t_2$, similarly,

$$\Delta^{\Omega,v}_I(u) = \int_{\Omega_{t_1}}\int_{\Omega_0} u_0(\xi) G^{\Omega,v}_{0,t_1}(\xi,x)\,d\xi\,dx - \int_{\Omega_{t_2}}\int_{\Omega_0} u_0(\xi) G^{\Omega,v}_{0,t_2}(\xi,x)\,d\xi\,dx$$
$$= \int_{\Omega_0} u_0(\xi)\mathbf{P}_\xi[t_1 \leqslant \tau^{\Omega,v}_\xi < t_2]\,d\xi.$$

Then, for $I_\varepsilon = [t_2, t_2+\varepsilon]$, we get

$$\lim_{\varepsilon\to 0} \Delta^{\Omega,v}_{I_\varepsilon}(u) = \lim_{\varepsilon\to 0} -\frac{1}{2}\int_{I_\varepsilon}\int_{\partial\Omega_t} \frac{\partial u}{\partial n}(x,t)\,d\mathcal{H}^{d-1}(x)\,dt$$
$$= \lim_{\varepsilon\to 0}\int_{\Omega_0} u_0(\xi)\mathbf{P}_\xi[t_2 \leqslant \tau^{\Omega,v}_\xi < t_2+\varepsilon]\,d\xi$$
$$= \int_{\Omega_0} u_0(\xi)\mathbf{P}_\xi[\tau^{\Omega,v}_\xi = t_2]\,d\xi.$$

Finally we conclude that

$$\Delta^{\Omega,v}_I(u) = \int_{\Omega_0} u_0(\xi)\mathbf{P}_\xi[t_1 \leqslant \tau^X_\xi \leqslant t_2]\,d\xi = \int_I\int_{\partial\Omega_s} dF_{u_0}(y,s)\,ds.$$

The proof is complete.

By approximating the initial delta measure of Theorem 2, we prove Proposition 5.

Proposition 5. *Let us assume the starting point of Brownian motion* $r_0 = 0 \in \Omega_0$ *and let us choose a sequence* $\{h_m\} \subset C_c^\infty(\mathbf{R}^d; \mathbf{R}_+)$ *with* $\operatorname{supp} h_m = \mathcal{B}(0, 1/m) \Subset \Omega_0$ *and* $\|h_m\|_1 = 1$*. For each* h_m*, there exists a corresponding solution* u_m *with* $-(1/2)\partial u_m(x,t)/\partial n =: p_m(x,t)$ *in the sense of Theorem 3. Then we have the following statements*:

(1) *there is a unique* $p \in C(S_T)$ *with* $p(\,\cdot\,, 0) = 0$ *such that for all* $t > 0$ *and all* $x \in \Omega_t$,

$$p(x,t) = -\frac{\partial G}{\partial n_{x,t}}(x,t;0,0) + \int_0^t \int_{\partial\Omega_s} \frac{\partial G}{\partial n_{x,t}}(x,t;y,s) p(y,s)\, d\mathcal{H}^{d-1}(y)\, ds; \tag{4.1}$$

(2) *a sequence* p_m *defined above converges to* p *in* sup *norm.*

Proof. For all T sufficiently small, by Lemmas 1 and 2, we obtain that there is $C_1 > 0$ such that for all $0 < t \leqslant T$ and all $x \in \partial\Omega_t$,

$$\left| \int_0^t \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau)\, d\mathcal{H}^{d-1}(y)\, d\tau \right| \leqslant C_1 (t-\tau)^{2\gamma - 1/2}.$$

For $T_s > 0$, we define $\mathcal{F}\colon C(S_{T_s}) \to C(S_{T_s})$ such that, for $q \in C(S_{T_s})$,

$$(\mathcal{F}q)(x,t) = -\frac{\partial G}{\partial n_{x,t}}(x,t;0,0) + \int_0^t \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau) q(y,\tau)\, d\mathcal{H}^{d-1}(y)\, d\tau,$$

and $(\mathcal{F}q)(\,\cdot\,, 0) = 0$. If we choose T_s sufficiently small, then $\mathcal{F}$ is a contraction mapping so $\mathcal{F}$ has a unique fixed point. Let us call this p_{T_s}.

Now we have p_{T_s} for some $T_s > 0$. For $T^\star > T_s$, let us denote $S_{[T_s, T^\star]} := \bigcup_{T_s \leqslant t \leqslant T^\star} \partial\Omega_t \times \{t\}$.

We define $\mathcal{K}\colon C(S_{[T_s,T^\star]}) \to C(S_{[T_s,T^\star]})$ as, for $q \in C(S_{[T_s,T^\star]})$,

$$\begin{aligned}(\mathcal{K}q)(x,t) = &-\frac{\partial G}{\partial n_{x,t}}(x,t;0,0) \\ &+ \int_0^{T_s} \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau) p_{T_s}(y,\tau)\, d\mathcal{H}^{d-1}(y)\, d\tau \\ &+ \int_{T_s}^t \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau) q(y,\tau)\, d\mathcal{H}^{d-1}(y)\, d\tau.\end{aligned}$$

Then, for $q_1, q_2 \in C(S_{[T_s,T^\star]})$, we have

$$\|\mathcal{K}q_1 - \mathcal{K}q_2\|_\infty \leqslant C_2(t - T_s)^{2\gamma-1/2} \|q_1 - q_2\|_\infty \leqslant C_2 (T^\star - T_s)^{2\gamma - 1/2} \|q_1 - q_2\|_\infty.$$

Similarly, if we choose $T^\star$ such that $C_2(T^\star - T_s)^{2\gamma-1/2} < 1$, then $\mathcal{K}$ is a contraction mapping so that $\mathcal{K}$ has a unique fixed point.

Therefore, if we have p defined S_{T_s}, p_{T_s}, then we can extend this to time $T_s + C_3$, where C_3 is a constant. Thus if we repeat this step inductively, we have p defined on S_T which satisfies (4.1).

We now prove that p_m converges to p in sup norm for all sufficiently small $T_s > 0$. By (3.3),

$$p_m(x,t) = -\int_{\Omega_0} h_m(\xi)\,\frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0)\,d\xi + \int_0^t \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau)p_m(y,\tau)\,d\mathcal{H}^{d-1}(y)\,d\tau. \tag{4.2}$$

For $0 < t \leqslant T_s$, taking the difference between (4.1) and (4.2), we get

$$|p_m(x,t) - p(x,t)| \leqslant \left|\int_{\Omega_0} h_m(\xi)\left(\frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0) - \frac{\partial G}{\partial n_{x,t}}(x,t;0,0)\right)d\xi\right| + \left|\int_0^t \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau)(p_m(y,\tau) - p(y,\tau))\,d\mathcal{H}^{d-1}(y)\,d\tau\right|.$$

Let us denote $\|p_m - p\|_{T_s} := \sup_{\tau\in[0,T_s]} \sup_{y\in\partial\Omega_\tau} |p_m(y,\tau) - p(y,\tau)|$. For the second term of the right-hand side, we have

$$\left|\int_0^t \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau)(p_m(y,\tau) - p(y,\tau))\,d\mathcal{H}^{d-1}(y)\,d\tau\right| \leqslant C_4 \int_0^t \frac{\|p_m - p\|_{T_s}}{(t-\tau)^{3/2-2\gamma}}\,d\tau = C_5 t^{2\gamma-1/2}\|p_m - p\|_{T_s} \leqslant C_5 T_s^{2\gamma-1/2}\|p_m - p\|_{T_s}.$$

Let us choose $T_s > 0$ such that $C_5 T_s^{2\gamma-1/2} < 1$. Then

$$(1 - C_5 T_s^{2\gamma-1})\|p_m - p\|_{T_s} \leqslant \sup_{0<t\leqslant T_s} \sup_{x\in\partial\Omega_t} \left|\int_{\Omega_0} h_m(\xi)\left(\frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0) - \frac{\partial G}{\partial n_{x,t}}(x,t;0,0)\right)d\xi\right| \leqslant \sup_{0<t\leqslant T_s} \sup_{x\in\partial\Omega_t} \sup_{|\xi|\leqslant 1/m} \left|\frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0) - \frac{\partial G}{\partial n_{x,t}}(x,t;0,0)\right|.$$

We have

$$|\partial_i^2 G(x,t;\xi,0)| = \left|\left\{\left(\frac{\xi_i - x_i}{t}\right)^2 - \frac{1}{t}\right\}\left(\frac{1}{\sqrt{2\pi t}}\right)^d \exp\left\{-\frac{|x-\xi|^2}{2t}\right\}\right| \leqslant \frac{C_6}{|x-\xi|^{d+2}},$$

$$|\partial_i^2 G(x,t;\xi,0)| \leqslant \frac{C_7}{t^{d/2+1}},$$

$$
\begin{aligned}
|\partial_j \partial_i G(x,t;\xi,0)| &= \left| \frac{(\xi_i - x_i)(\xi_j - x_j)}{t^2} \left(\frac{1}{\sqrt{2\pi t}} \right)^d \exp\left\{ -\frac{|x-\xi|^2}{2t} \right\} \right| \\
&\leqslant \frac{C_8}{|x-\xi|^{d+2}}, \\
|\partial_j \partial_i G(x,t;\xi,0)| &\leqslant \frac{C_9}{t^{d/2+1}}.
\end{aligned} \tag{4.3}
$$

For all sufficiently small T_s and all sufficiently large m, by the mean value theorem and (4.3), there exists $C_{10} > 0$ such that, for all $|\xi| \leqslant 1/m$,

$$
\left| \frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0) - \frac{\partial G}{\partial n_{x,t}}(x,t;0,0) \right| \leqslant \frac{C_{10}|\xi|}{(|x| - 1/m)^{d+2}} \leqslant \frac{C_{10}}{m(|x|-1/m)^{d+2}}.
$$

Therefore, we conclude that p_m converges to p in sup norm for all sufficiently small $T_s > 0$.

To extend from T_s to $T^\star$, assuming that p_m converges to p in $C_{S_{T_s}}$ for some $T_s > 0$ and writing $\|p_m - p\|_{[T_s,T^\star]} = \sup_{\tau \in [T_s,T^\star]} \sup_{y \in \partial\Omega_\tau} |p_m(y,\tau) - p(y,\tau)|$, for $T_s \leqslant t \leqslant T^\star$, we deduce that by the mean value theorem and (4.3),

$$
\begin{aligned}
|p_m(x,t) - p(x,t)| &\leqslant \left| \int_{\Omega_0} h_m(\xi) \left(\frac{\partial G}{\partial n_{x,t}}(x,t;\xi,0) - \frac{\partial G}{\partial n_{x,t}}(x,t;0,0) \right) d\xi \right| \\
&\quad + \left| \int_0^{T_s} \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau)(p_m(y,\tau) - p(y,\tau))\, d\mathcal{H}^{d-1}(y)\, d\tau \right| \\
&\quad + \left| \int_{T_s}^{t} \int_{\partial\Omega_\tau} \frac{\partial G}{\partial n_{x,t}}(x,t;y,\tau)(p_m(y,\tau) - p(y,\tau))\, d\mathcal{H}^{d-1}(y)\, d\tau \right| \\
&\leqslant \frac{C_{11}}{mT_s^{d/2+1}} + C_1 \int_0^{T_s} \frac{\|p_m - p\|_{T_s}}{(t-\tau)^{3/2-2\gamma}}\, d\tau + C_1 \int_{T_s}^{t} \frac{\|p_m - p\|_{[T_s,T^\star]}}{(t-\tau)^{3/2-2\gamma}}\, d\tau \\
&\leqslant \frac{C_{11}}{mT_s^{d/2+1}} + C_1 \int_0^{T_s} \frac{\|p_m - p\|_{T_s}}{(T_s-\tau)^{3/2-2\gamma}}\, d\tau \\
&\quad + C_{12}\|p_m - p\|_{[T_s,T^\star]}(t - T_s)^{2\gamma-1/2} \\
&\leqslant \frac{C_{11}}{mT_s^{d/2+1}} + C_{13} T_s^{2\gamma-1/2}\|p_m - p\|_{T_s} \\
&\quad + C_{12}\|p_m - p\|_{[T_s,T^\star]}(T^\star - T_s)^{2\gamma-1/2}.
\end{aligned}
$$

Let us choose $T^\star > T_s$ such that $C_{12}(T^\star - T_s)^{2\gamma-1/2} < 1$, then we have

$$
(1 - C_{12}(T^\star - T_s)^{\gamma-1/2})\|p_m - p\|_{[T_s,T^\star]} \leqslant \frac{C_{11}}{mT_s^{d/2+1}} + C_{13} T_s^{2\gamma-1/2}\|p_m - p\|_{T_s}.
$$

The right term vanishes when m goes to ∞ so that p_m converges to p in $C(S_{T_s+C_{14}})$ for some constant $C_{14} > 0$. By repeating this argument inductively, it follows that p_m converges to p in sup norm. The proof is complete.

Now we can show that p is the continuous density function of dF_{r_0}. By Proposition 5, we have

$$\lim_{m\to\infty}\int_{\Omega_0} h_m(\xi)\mathbf{P}_\xi(\tau_\xi^{\Omega,v}\in I)\,d\xi$$
$$=\lim_{m\to\infty}\int_I\int_{\partial\Omega_t} p_m(x,t)\,dt=\int_I\int_{\partial\Omega_t} p(x,t)\,dt.$$

For $I=[0,t]\subset[0,T]$, therefore, we obtain that

$$\int_{[0,t]}\int_{\partial\Omega_s} dF_{r_0}(y,s)=\mathbf{P}_{r_0}(\tau_{r_0}^{\Omega,v}\leqslant t)=\int_0^t\int_{\partial\Omega_s} p(y,s)\,ds,$$

which implies that p is the continuous density function of $dF_{r_0}(y,s)$, thus item (2) is proved.

By Theorem 1 and the properties of the Gaussian kernel, $G_{0,t}^{\Omega,v}(r_0,x)$ solves (2.7)–(2.9). Hence $G^{\Omega,v}$ is the Green function of the heat equation with Dirichlet boundary condition which implies item (4). Furthermore, $G_{0,t}^{\Omega,v}(r_0,x)$ can be written as

$$G_{0,t}^{\Omega,v}(r_0,x)=G_{0,t}(r_0,x)-\int_0^t\int_{\partial\Omega_\tau} G_{\tau,t}(y,x)p(y,\tau)\,d\mathcal{H}^{d-1}(y)\,d\tau.$$

Taking a normal derivative with respect to x, applying the jump relation and (4.1), we have

$$\begin{aligned}\frac{\partial}{\partial n}G_{0,t}^{\Omega,v}(r_0,x)&=\frac{\partial}{\partial n}G_{0,t}(r_0,x)-p(x,t)\\&\quad-\int_0^t\int_{\partial\Omega_\tau}\frac{\partial}{\partial n}G_{\tau,t}(y,x)p(y,\tau)\,d\mathcal{H}^{d-1}(y)\,d\tau\\&=-2p(x,t).\end{aligned}$$

Thus

$$p(x,t)=-\frac{1}{2}\frac{\partial}{\partial n}G_{0,t}^{\Omega,v}(r_0,x),$$

so it proves item (3) of Theorem 2.

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