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LIMIT THEOREMS FOR THE ESTERMANN ZETA FUNCTION. III

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Аннотация

A joint limit theorem in the sense of the weak convergence of probability measures on the complex plane for Estermann zeta-functions is obtained.

1 Introduction

This paper is a continuation of [2] and [3], therefore first of all we recall the results obtained in the above papers.

Let, for $\alpha \in \mathbb{C}$,

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha$$

denote the generalized divisor function. The Estermann zeta-function $E(s;k/l,\alpha)$, $s=\sigma+it$, with parameters k/l, $(k,l)=1$, and α is defined, for $\sigma > \max(1, 1+\Re\alpha)$ by

$$E(s;k/l,\alpha) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\},$$

and by analytic continuation elsewhere. It has two simple poles $s=1$ and $s=1+\alpha$ if $\alpha \neq 0$, and one double pole $s=1$ if $\alpha = 0$.

In [2] we started with the characterization of the function $E(s;k/l,\alpha)$ by limit theorems in the sense of the weak convergence of probability measures. Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and define, for $T>0$,

$$\nu_T^t(...) = \frac{1}{T}\text{meas}\{t \in [0,T] : \ldots\},$$

where in place of dots a condition satisfied by t is to be written, and the sign t in ν_T^t only shows that the measure is taken over $t \in [0,T]$. Let $\mathcal{B}(S)$ be the class of Borel sets of the space S. We also need the following topological structure. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. Then with the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and define, for $m \in \mathbb{N}$,

$$\omega(m) = \prod_{p^r \| m} \omega^r(p),$$

where $p^r \| m$ means that $p^r \mid m$ but $p^{r+1} \nmid m$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define, for $\sigma > 1/2$, a complex-valued random element $E(\sigma; k/l, \alpha; \omega)$ by the formula

$$E(\sigma; k/l, \alpha; \omega) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^\sigma} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

Since $E(s; k/l, \alpha) = E(s - \alpha; k/l, -\alpha)$, we suppose without loss of generality throughout the paper, as in [2] and [3], that $\Re\alpha \le 0$.

Theorem A. [2]. *Supose that $\sigma > 1/2$ and $\Re\alpha \le 0$. Then the probability measure*

$$\nu_T^t(E(\sigma + it; k/l, \alpha) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the random element $E(\sigma; k/l, \alpha; \omega)$ as $T \to \infty$

In [3], Theorem A has been generalized to the space of meromorphic functions. For a region G on the complex plane, denote by $M(G)$ the space of meromorphic on G functions equipped with the topology of uniform convergence on compacta. The space $H(G)$ of analytic on G functions forms a subspace of $M(G)$. Let $D = \{s \in \mathbb{C} : \sigma > 1/2\}$, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D)$-valued random element $E(s; k/l, \alpha; \omega)$ by the formula

$$E(s; k/l, \alpha; \omega) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^s} \exp\left\{2\pi m \frac{k}{l}\right\}.$$

Theorem B. [3]. *Suppose that $\Re\alpha \le 0$. Then the probability measure*

$$\nu_T^\tau(E(s + i\tau; k/l, \alpha) \in A), \quad A \in \mathcal{B}(M(D)),$$

converges weakly to the distribution of the random element $E(s; k/l, \alpha; \omega)$ as $T \to \infty$.

We recall that the distribution of $E(s;k/l,\alpha;\omega)$ is the probability measure

$$m_H(\omega \in \Omega : E(s;k/l,\alpha;\omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

The aim of this paper is to obtain a joint limit theorem on the complex plane for Estermann zeta-functions. Let, for $\sigma > \max(1, 1+\Re\alpha_j)$,

$$E(s;k_j/l_j,\alpha_j) = \sum_{m=1}^{\infty} \frac{\sigma_{\alpha_j}(m)}{m^s} \exp\{2\pi m k_j/l_j\}, \quad j=1,\dots,r.$$

Denote by

$$\mathbb{C}^r = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r}$$

the Cartesian product, and define the probability measure

$$P_{T;\sigma_1,\dots,\sigma_r} = \nu_T^t((E(\sigma_1+it;k_1/l_1,\alpha_1),\dots,E(\sigma_r+it;k_r/l_r,\alpha_r)) \in A), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Moreover, for $\min\limits_{1\le j\le r} \sigma_j > 1/2$ and $\omega \in \Omega$, let

$$E(\sigma_1,\dots,\sigma_r;\omega) = (E(\sigma_1;k_1/l_1,\alpha_1;\omega),\dots,E(\sigma_r;k_r/l_r,\alpha_r;\omega)),$$

where

$$E(\sigma_j;k_j/l_j,\alpha_j;\omega) = \sum_{m=1}^{\infty} \frac{\sigma_{\alpha_j}(m)\omega(m)}{m^{\sigma_j}} \exp\{2\pi m k_j/l_j\}, \quad j=1,\dots,r.$$

Then $E(\sigma_1,\dots,\sigma_r;\omega)$ is a $\mathbb{C}^r$-valued random element define on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. The main result of this paper is the following statement.

THEOREM 1. *Let* $\min\limits_{1\le j\le r} \sigma_j > 1/2$ *and* $\Re\alpha_j \le 0$, $j=1,\dots,r$. *Then the probability measure* $P_{T;\sigma_1,\dots,\sigma_r}$ *converges weakly to the distribution of the random element* $E(\sigma_1,\dots,\sigma_r;\omega)$ *as* $T\to\infty$.

The proof of Theorem 1 is based on Theorem A and Prokhorov's theorems.

2 Application of Prokhorov's theory

In the theory of the weak convergence of probability measures an important role is played by the relative compactness of families of probability measures. By the definition, a family of probability measures $\{P\}$ on $(S,\mathcal{B}(S))$ is relatively compact if every sequence of $\{P\}$ contains a weakly convergent subsequence. Clearly, if the probability measure P_n converges weakly to some measure P as $n\to\infty$, then the family $\{P_n\}$ is relatively compact. However, the relative compactness of $\{P_n\}$

does not imply in general the weak convergence of P_n, but, under some additional condition, does. Therefore, it is important to know how to obtain the relative compactness of a given family. Yu. V. Prokhorov observed [5] that for this a notion of the tightness can be used. By the definition, the family of probability measures $\{P\}$ is tight if, for arbitrary $\varepsilon > 0$, there exists a compact subset $K \subset S$ such that $P(K) > 1-\varepsilon$ for all $P \in \{P\}$.

LEMMA **1.** (First Prokhorov's theorem) *If the family of probability measures $\{P\}$ is tight, then it is relatively compact.*

LEMMA **2.** (Second Prokhorov's theorem) *Suppose that S is a separable complete metric space. If the family of probability measure $\{P\}$ on $(S, \mathcal{B}(S))$ is relatively compact, then it is tight.*

Proof of Prokhorov's theorems can be found, for example, in [1].

LEMMA **3.** *The family of probability measures $\{P_{T;\sigma_1,\dots,\sigma_r}\}$ is tight.*

PROOF. Let, for $\sigma_j > 1/2$ and $\Re\alpha_j \le 0$,

$$P_{T;\sigma_j}(A) = \nu_T^t((E(\sigma_j + it; k_j/l_j, \alpha_j)) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \quad j = 1, \dots, r.$$

Then, by Theorem A, we have that the measure $P_{T;\sigma_j}$ converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{\sigma_{\alpha_j}(m)\omega(m)}{m^{\sigma_j}} \exp\{2\pi i m k_j/l_j\}$$

as $T \to \infty$, $j = 1, \dots, r$. Hence, the family of probability measures $\{P_{T;\sigma_j}\}$ is relatively compact, $j = 1, \dots, r$. Since $\mathbb{C}$ is a separable complete metric space, in view of Lemma 2, the family $\{P_{T;\sigma_j}\}$ is tight, $j = 1, \dots, r$. Therefore, for arbitrary $\varepsilon > 0$, there exists a compact subset $K_j \subset \mathbb{C}$ such that

$$P_{T;\sigma_j}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r, \tag{1}$$

for all $T > 0$.

Suppose that the random variable θ is defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$ and uniformly distributed on $[0,1]$. Let, for $\sigma_j > 1/2$,

$$E_{T,j}(\sigma_j) = E(\sigma_j + iT\theta; k_j/l_j, \alpha_j), \quad j = 1, \dots, r,$$

and

$$E_T(\sigma_1, \dots, \sigma_r) = (E_{T,1}(\sigma_1), \dots, E_{T,r}(\sigma_r)).$$

We put $K = K_1 \times \dots \times K_r$. Then, clearly, K is a compact subset of the space $\mathbb{C}^r$. Taking into account inequality (1), we obtain that

$$P_{T;\sigma_1,\dots,\sigma_r}(\mathbb{C}^r \setminus K) = \mathbb{P}(E_T(\sigma_1, \dots, \sigma_r) \in \mathbb{C}^r \setminus K) =$$

$$
\begin{aligned}
&= \mathbb{P}\Big(\bigcup_{j=1}^{r}(E_{T,j}(\sigma_j) \in \mathbb{C}^r \setminus K_j)\Big) \le \\
&\le \sum_{j=1}^{r}\mathbb{P}(E_{T,j}(\sigma_j) \in \mathbb{C} \setminus K_j) = \sum_{j=1}^{r} P_{T;\sigma_j}(\mathbb{C} \setminus K_j) < \varepsilon
\end{aligned}
$$

for all $T > 0$. Therefore, $P_{T;\sigma_1,\dots,\sigma_r}(K) > 1-\varepsilon$, for all $T > 0$, i. e., the family $\{P_{T;\sigma_1,\dots,\sigma_r}\}$ is tight. □

COROLLARY **1.** *The family of probability measures* $\{P_{T;\sigma_1,\dots,\sigma_r}\}$ *is relatively compact.*

PROOF. The corollary is a direct consequence of Lemmas 1 and 3. □

3 Limit theorem for a linear combination

In this section we consider the linear combination

$$E(s) = \sum_{j=1}^{r} u_j E(s+\sigma_{0j}; k_j/l_j, \alpha_j)$$

where $u_j, j = 1, \dots, r$, are arbitrary complex numbers, $\sigma > \max\limits_{1\le j\le r} \sigma_{1j} = \max\limits_{1\le j\le r}(1/2 - \sigma_{0j})$, $\sigma_{0j} > 1/2$. So we have that $\sigma_{1j} < 0,\ j = 1, \dots, r$.

Let

$$E(\sigma,\omega) = \sum_{j=1}^{r} u_j E(s+\sigma_{0j}; k_j/l_j, \alpha_j; \omega).$$

If $\sigma > \max\limits_{1\le j\le r} \sigma_{1j}$, then $E(\sigma,\omega)$ is a complex-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

THEOREM 2**.** *Let* $\sigma > \max\limits_{1\le j\le r} \sigma_{1j}$*. Then the probability measure*

$$P_{T;\sigma}(A) = \nu_T^t(E(\sigma+it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the random element $E(\sigma,\omega)$ *as* $T \to \infty$.

PROOF. For the proof, we will use a way similar to that of the proof of Theorem A. Therefore, we will give only sketches of the proofs of statements.

Let, for positive integers N and n, and a fixed $\sigma_0 > 1/2$,

$$E_{N,n}(\sigma+it; k_j/l_j, \alpha_j) = \sum_{m=1}^{N} \frac{\sigma_{\alpha_j}(m)}{m^{\sigma+it}} \exp\left\{2\pi i \frac{k_j}{l_j}\right\} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\}, \quad j = 1, \dots, r,$$

and

$$E_{N,n}(\sigma+it;k_j/l_j,\alpha_j;\omega)=\sum_{m=1}^{N}\frac{\sigma_{\alpha_j}(m)}{m^{\sigma+it}}\exp\left\{2\pi i\frac{k_j}{l_j}\right\}\exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\},$$

$$j=1,\dots,r,\quad \omega\in\Omega,$$

and define

$$E_{N,n}(\sigma+it)=\sum_{j=1}^{r}u_jE_{N,n}(\sigma+\sigma_{0j}+it;k_j/l_j,\alpha_j),$$

and

$$E_{N,n}(\sigma+it;\omega)=\sum_{j=1}^{r}u_jE_{N,n}(\sigma+\sigma_{0j}+it;k_j/l_j,\alpha_j;\omega).$$

Consider the weak convergence of two probability measures

$$P_{T,N,n;\sigma}(A)=\nu_T^t(E_{N,n}(\sigma+it)\in A),\quad A\in\mathcal{B}(\mathbb{C}),$$

and

$$\widehat{P}_{T,N,n;\sigma}(A)=\nu_T^t(E_{N,n}(\sigma+it;\omega)\in A),\quad A\in\mathcal{B}(\mathbb{C}).$$

□

LEMMA **4.** *The probability measures $P_{T,N,n;\sigma}$ and $\widehat{P}_{T,N,n;\sigma}$ both converge weakly to the same probability measure on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ as $T\to\infty$.*

PROOF. Since logarithms of prime numbers are linearly independent over the field of rational numbers, we easily find that the probability measure

$$Q_T(A)=\nu_T^t((p^{-it}:p \text{ is prime})\in A),\quad A\in\mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H as $T\to\infty$. Moreover, there exist a continuous functions h and $\widehat{h}:\Omega\to\mathbb{C}$ such that

$$h((p^{-it}:p \text{ is prime}))=E_{N,n}(\sigma+it)$$

and

$$\widehat{h}((p^{-it}:p \text{ is prime}))=E_{N,n}(\sigma+it;\omega).$$

Therefore, we obtain that $P_{T,N,n;\sigma}$ and $\widehat{P}_{T,N,n;\sigma}$ converge weakly to m_Hh^{-1} and $m_H\widehat{h}^{-1}$, respectively, as $T\to\infty$. However, the invariance of the Haar measure m_H show that $m_Hh^{-1}=m_H\widehat{h}^{-1}$.

Now let

$$E_n(\sigma+it;k_j/l_j,\alpha_j)=\sum_{m=1}^{\infty}\frac{\sigma_{\alpha_j}(m)}{m^{\sigma+it}}\exp\left\{2\pi im\frac{k_j}{l_j}\right\}\exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\},\quad j=1,\dots,r,$$

and

$$E_n(\sigma+it;k_j/l_j,\alpha_j;\omega)=\sum_{m=1}^{\infty}\frac{\sigma_{\alpha_j}(m)}{m^{\sigma+it}}\exp\left\{2\pi im\frac{k_j}{l_j}\right\}\exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\},$$

$$j=1,\dots,r,\quad \omega\in\Omega.$$

Then, for each $j=1,\dots,r$, the series for $E_n(\sigma+it;k_j/l_j,\alpha_j)$ and $E_n(\sigma+it;k_j/l_j,\alpha_j;\omega)$ absolutely converge for $\sigma>1/2$. This was observed in [2]. Next, define the functions

$$E_n(\sigma+it)=\sum_{j=1}^{r}u_jE_n(\sigma+\sigma_{0j}+it;k_j/l_j,\alpha_j),\quad j=1,\dots,r,$$

and

$$E_n(\sigma+it;\omega)=\sum_{j=1}^{r}u_jE_n(\sigma+\sigma_{0j}+it;k_j/l_j,\alpha_j;\omega),\quad j=1,\dots,r,\ \ \omega\in\Omega,$$

and the probability measures

$$P_{T,n;\sigma}(A)=\nu_T^t(E_n(\sigma+it)\in A),\quad A\in\mathcal{B}(\mathbb{C}),$$

and

$$\widehat{P}_{T,n;\sigma}(A)=\nu_T^t(E_n(\sigma+it)\in A),\quad A\in\mathcal{B}(\mathbb{C}).$$

□

LEMMA **5.** *Let $\sigma>\max\limits_{1\le j\le r}\sigma_{1j}$. Then the probability measures $P_{T,n;\sigma}$ and $\widehat{P}_{T,n;\sigma}$ both converge weakly to the same probability measure on $\mathbb{C},\mathcal{B}(\mathbb{C})$ as $T\to\infty$.*

PROOF A scheme of the proof is the same as that of Lemma 2 in [2]. By Lemma 4 we have that the measures $P_{T,N,n;\sigma}$ and $\widehat{P}_{T,N,n;\sigma}$ both converge weakly to some measure $P_{N,n;\sigma}$ as $T\to\infty$. The first step is to prove that the family of probability measures $\{P_{N,n;\sigma}\}$ is tight for fixed n. Let θ be the random variable defined in Section 2, and let

$$X_{T,N,n}(\sigma)=E_{N,n}(\sigma+iT\theta).$$

Then, in view of Lemma 4,

$$X_{T,N,n}(\sigma)\xrightarrow[T\to\infty]{\mathcal{D}}X_{N,n}(\sigma),\tag{2}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $X_{N,n}(\sigma)$ is a complex-valued random element with the distribution $P_{N,n;\sigma}$. By the definition of $E_{N,n}(\sigma+it)$ we find that

$$\sup_{N\ge1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T|E_{N,n}(\sigma+it)|\mathbf{d}t\le R<\infty.$$

Hence, taking $M = R/\varepsilon$, denoting $K_\varepsilon = \{s \in \mathbb{C} : |s| \leq M\}$ and using (2), we obtain that

$$\mathbb{P}(X_{N,n}(\sigma) \in K_\varepsilon) \geq 1 - \varepsilon$$

for all $N \in \mathbb{N}$. This gives the tightness of the family $\{P_{N,n}\}$.

Since $E_{N,n}(\sigma+it; k_j/l_j, \alpha_j)$ is a partial sum of $E_n(\sigma+it; k_j/l_j, \alpha_j)$, we have, for $\sigma > 1/2$,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |E_{N,n}(\sigma+it; k_j/l_j, \alpha_j) - E_n(\sigma+it; k_j/l_j, \alpha_j)| \mathbf{d}t = 0, \quad j = 1, \dots, r.$$

Hence, obviously, for $\sigma > 1/2$,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |E_{N,n}(\sigma+it) - E_n(\sigma+it)| \mathbf{d}t = 0. \tag{3}$$

Let

$$X_{T,n}(\sigma) = E_n(\sigma + iT\theta).$$

Then (3), for every $\varepsilon > 0$, implies

$$\lim_{N\to\infty} \limsup_{T\to\infty} \mathbb{P}(|X_{T,N,n}(\sigma) - X_{T,N}(\sigma)| \geq \varepsilon) = 0. \tag{4}$$

Since the family $\{P_{N,n}\}$ is tight, by Lemma 1 it is relatively compact. Let $\{P_{N_1,n;\sigma}\} \subset \{P_{N,n;\sigma}\}$ be such that $P_{N_1,n;\sigma}$ converges weakly to $P_{n;\sigma}$, say, as $N_1 \to \infty$, i. e., $X_{N_1,n}(\sigma) \xrightarrow[N_1\to\infty]{\mathcal{D}} P_{n;\sigma}$. This, (2), (4) and Theorem 4.2 from [1] now yield the relation

$$X_{T,n}(\sigma) \xrightarrow[N_1\to\infty]{\mathcal{D}} P_{n;\sigma}. \tag{5}$$

Thus, we proved that $P_{T,n;\sigma}$ converges weakly to $P_{n;\sigma}$ as $T \to \infty$. Moreover, (5) shows that the measure $P_{n;\sigma}$ is independent of $\{P_{N_1,n;\sigma}\}$. Hence,

$$X_{N,n}(\sigma) \xrightarrow[N\to\infty]{\mathcal{D}} P_{n;\sigma}. \tag{6}$$

Now let

$$\widehat{X}_{T,N,n}(\sigma) = E_{N,n}(\sigma + iT\theta; \omega)$$

and

$$\widehat{X}_{T,n}(\sigma) = E_n(\sigma + iT\theta; \omega).$$

Then the above arguments with (6) show that the measure $\widehat{P}_{T,n;\sigma}$ also converge weakly to $P_{n;\sigma}$ as $T \to \infty$. □

Proof. [Proof of Theorem 2] Define once one probability measure

$$\widehat{P}_{T,\sigma}(A) = \nu_T^t(E(\sigma + it; \omega) \in A), \quad A \in \mathcal{B}(\mathbb{C}).$$

First we will prove that, for $\sigma > \max\limits_{1\le j\le r} \sigma_{1j}$, the probability measures $P_{T;\sigma}$ and $\widehat{P}_{T;\sigma}$ both converge weakly to the same probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. This can be done in the same way as in the proof of Lemma 5, using Lemma 5 and the relations

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T |E(\sigma+it) - E_n(\sigma+it)|\mathbf{d}t = 0$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T |E(\sigma+it,\omega) - E_n(\sigma+it;\omega)|\mathbf{d}t = 0,$$

where $\sigma > \max\limits_{1\le j\le r} \sigma_{1j}$. The later relations follow from the analogical relations for the functions $E(\sigma+it; k_j/l_j, \alpha_j)$, $E_n(\sigma+it; k_j/l_j, \alpha_j)$ and $E(\sigma+it; k_j/l_j, \alpha_j; \omega)$, $E_n(\sigma+it; k_j/l_j, \alpha_j; \omega)$, for $\sigma > 1/2$, obtained in [2], $j = 1, \dots, r$, and the definitions of the functions $E(\sigma+it)$, $E_n(\sigma+it)$, $E(\sigma+it;\omega)$ and $E_n(\sigma+it;\omega)$. Therefore, it remains to identify the limit measure P_σ.

Let, for $t \in \mathbb{R}, a_t = \{p^{-it} : p \text{ is prime}\}$ and let $\varphi_t(\omega) = a_t\omega, \omega \in \Omega$. Then $\{\varphi_t : t \in \mathbb{R}\}$ is an ergodic one-parameter group of measurable transformations on Ω [4].

If $A \in \mathcal{B}(\mathbb{C})$ is a continuity set of the measure P_σ, then, for $\sigma > \max\limits_{1\le j\le r} \sigma_{1j}$,

$$\lim_{T\to\infty} \nu_T^t(E(\sigma+it;\omega) \in A) = P_\sigma(A) \tag{7}$$

for almost all $\omega \in \Omega$. Suppose that the set A is fixed, and define a random variable η on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ by the formula

$$\eta(\omega) = \begin{cases} 1 & \text{if} \quad E(\sigma,\omega) \in A, \\ 0 & \text{if} \quad E(\sigma,\omega) \neq A. \end{cases}$$

Then we have that the expectation

$$\mathbb{E}(\eta) = \int\limits_\Omega \eta \mathbf{d}m_H = m_H(\omega \in \Omega : E(\sigma,\omega) \in A) \overset{def}{=} P_{E;\sigma}(A), \tag{8}$$

where $P_{E;\sigma}$ is the distribution of the random element $E(\sigma,\omega)$. In view of the ergodicity of $\{\varphi_t : t \in \mathbb{R}\}$, the process $\eta(\varphi_t(\omega))$ is ergodic. Thus, the Birkhoff-Khintchine theorem yields

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \eta(\varphi_t(\omega))\mathbf{d}t = \mathbb{E}\eta \tag{9}$$

for almost all $\omega \in \Omega$. However,

$$\frac{1}{T}\int_0^T \eta(\varphi_t(\omega))\mathbf{d}t = \nu_T^t(E(\sigma+it,\omega)\in A).$$

Hence, and from (8) and (9) we deduce that

$$\lim_{T\to\infty} \nu_T^t(E(\sigma+it,\omega)\in A) = P_{E;\sigma}(A),$$

for almost all $\omega \in \Omega$. This together with (7) shows that $P_\sigma(A) = P_{E;\sigma}(A)$ for all continuity sets A of P_σ. Hence the theorem follows. □

4 Proof of Theorem 1

In view of Corollary 1, there exists a sequence $T_1 \to \infty$ such that $P_{T_1;\sigma_1,\dots,\sigma_r}$ converges weakly to some probability measure $P_{\sigma_1,\dots,\sigma_r}$ on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $T_1 \to \infty$. Hence, there exists a $\mathbb{C}^r$-valued random element

$$E = E(\sigma_1,\dots,\sigma_r) = (e_1(\sigma_1),\dots,e_r(\sigma_r))$$

defined on a certain probability space, and $P_{\sigma_1,\dots,\sigma_r}$ is its distribution. In other words,

$$E_{T_1}(\sigma_1,\dots,\sigma_r) \xrightarrow[T_1\to\infty]{\mathcal{D}} E(\sigma_1,\dots,\sigma_r). \tag{10}$$

Let $\sigma_j = \sigma + \sigma_{0j}$, where $\sigma > \max\limits_{1\le j\le r} \sigma_{1j}$. Then (10) and the continuity of the function $u:\ \mathbb{C}^r \to \mathbb{C}$ given by

$$u(s_1,\dots,s_r) = \sum_{j=1}^r u_j s_j$$

show that

$$u(E_{T_1}(\sigma_1,\dots,\sigma_r)) \xrightarrow[T_1\to\infty]{\mathcal{D}} u(E(\sigma_1,\dots,\sigma_r)). \tag{11}$$

Here

$$u(E_{T_1}(\sigma_1,\dots,\sigma_r)) = \sum_{j=1}^r u_j E_{T_{1,j}}(\sigma+\sigma_{0j})$$

and

$$u(E(\sigma_1,\dots,\sigma_r)) = \sum_{j=1}^r u_j e_j(\sigma+\sigma_{0j}).$$

In view of the definition of $E_T(\sigma_1,\dots,\sigma_r)$ and $E(s)$, we rewrite (11) in the from

$$E(\sigma+iT_1\theta) \xrightarrow[T_1\to\infty]{\mathcal{D}} u(E(\sigma_1,\dots,\sigma_r)). \tag{12}$$

On the other hand, by Theorem 2

$$E(\sigma+iT_1\theta)\xrightarrow[T_1\to\infty]{\mathcal{D}}E(\sigma,\omega).$$

Now this and (12) imply

$$E(\sigma,\omega)\overset{\mathcal{D}}{=}u(E(\sigma_1,\dots,\sigma_r)).$$

Taking in this relation $\sigma=0$ (this is possible because $\sigma_{1j}<0$, $j=1,\dots,r$) we find that

$$\sum_{j=1}^{r}u_jE\left(\sigma_{0j};\frac{k_j}{l_j},\alpha_j,\omega\right)\overset{\mathcal{D}}{=}\sum_{j=1}^{r}u_je_j(\sigma_{0j}). \tag{13}$$

Note that here $u_1,\dots,u_r$ are arbitrary complex numbers.

It is known [1] that the family of $\mathbb{R}^{2r}$ generated by all hyperplanes forms a determining class, hence also in $\mathbb{C}^r$. Therefore, in virtue of (13) we have that the $\mathbb{C}^r$-valued random elements $(E(\sigma_{01};k_1/l_1,\alpha_1;\omega),\dots,E(\sigma_{0r};k_r/l_r,\alpha_r;\omega))$ and $(e_1(\sigma_{01}),$
$\dots,e_r(\sigma_{0r}))$ have the same distribution. However, for $j=1,\dots,r$, $\sigma_{0j}>1/2$. Therefore, remembering the definition of $E(\sigma_1,\dots,\sigma_r;\omega)$, we obtain that, for $\min\limits_{1\le j\le r}\sigma_j>1/2$,

$$E(\sigma_1,\dots,\sigma_r;\omega)\overset{\mathcal{D}}{=}E(\sigma_1,\dots,\sigma_r).$$

Hence, in view of (10)

$$E_{T_1}(\sigma_1,\dots,\sigma_r)\xrightarrow[T_1\to\infty]{\mathcal{D}}E(\sigma_1,\dots,\sigma_r;\omega).$$

In the latter relation, the random element $E(\sigma_1,\dots,\sigma_r;\omega)$ is independent of the sequence T_1. Since the family of probability measures $\{P_{T;\sigma_1,\dots,\sigma_r}\}$ is relatively compact, hence the theorem follows.

5 Concluding remarks

The proof of Theorem 1 is based on the modified Cramér-Wald criterion. An another, direct, way of the proof is also possible. First, a r-dimensional limit theorem for Dirichlet polynomials on the weak convergence of the probability measures

$$\nu_T^t((E_{N,n}(\sigma_1+it;k_1/l_1,\alpha_1),\dots,E_{N,n}(\sigma_r+it;k_r/l_r,\alpha_r))\in A),\quad A\in\mathcal{B}(\mathbb{C}^r),$$

and

$$\nu_T^t((E_{N,n}(\sigma_1+it;k_1/l_1,\alpha_1;\omega),\dots,E_{N,n}(\sigma_r+it;k_r/l_r,\alpha_r;\omega))\in A),\quad A\in\mathcal{B}(\mathbb{C}^r),$$

to the same measure as $T \to \infty$ must be proved. Using the latter theorem, a r-dimensional limit theorem for absolutely convergent Dirichlet series on the weak convergence of the probability measures

$$\nu_T^t((E_n(\sigma_1+it;k_1/l_1,\alpha_1),\ldots,E_n(\sigma_r+it;k_r/l_r,\alpha_r))\in A), \quad A\in\mathcal{B}(\mathbb{C}^r),$$

and

$$\nu_T^t((E_n(\sigma_1+it;k_1/l_1,\alpha_1;\omega),\ldots,E_n(\sigma_r+it;k_r/l_r,\alpha_r;\omega))\in A), \quad A\in\mathcal{B}(\mathbb{C}^r),$$

to the same measure as $T \to \infty$ must be obtained. This theorem implies the weak convergence of the probability measures $P_{T;\sigma_1,\ldots,\sigma_r}$ and

$$\nu_T^t((E(\sigma_1+it;k_1/l_1,\alpha_1;\omega),\ldots,E(\sigma_r+it;k_r/l_r,\alpha_r;\omega))\in A), \quad A\in\mathcal{B}(\mathbb{C}^r),$$

to the same measure as $T \to \infty$. Now, an application of elements of the ergodic theory gives the assertion of Theorem 1. Note that this direct way of the proof is more clear, however, it uses more complicated formulas.

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