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НЕКОТОРЫЕ ЗАМЕЧАНИЯ О ДИСТАНЦИЯХ В ПРОСТРАНСТВАХ АНАЛИТИЧЕСКИХ ФУНКЦИЙ В ОГРАНИЧЕННЫХ ОБЛАСТЯХ С ГРАНИЦЕЙ ИЗ C^2 И В ДОПУСТИМЫХ ОБЛАСТЯХ[1 2]

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Аннотация

Воспроизводящая формула Бергмана и различные оценки для проектора Бергмана с положительным воспроизводящим ядром, а также точные оценки типа Форелли-Рудина для ядра Бергмана играют важную роль в некоторых новых экстремальных задачах, связанных с так называемой функцией дистанции в пространствах аналитических функций в различных областях в $\mathbb{C}^n$.

В этой работе, опираясь на известные теоремы вложения для пространств аналитических функций в ограниченных областях с границей из C^2 и в допустимых областях, мы получили новые результаты связанные с экстремальной задачей для пространств типа Бергмана аналитических функций.

Также мы приводим некоторые утверждения для пространств Неванлинны и BMOA, для аналитических пространств Бесова в областях с границей из C^2 и в допустимых областях.

Отметим также, что проблемам, связанные с регулярностью проектора Бергмана, которую мы часто используем в доказательствах, уделяется большое внимание. Многие оценки для воспроизводящих операторов и их ядер привлекают внимание математиков уже более 40 лет.

Структура этой работы такая же, как и в предыдущих работах по данной теме: сначала мы излагаем некоторые полученные ранее факты, связанные с проектором Бергмана, а потом, опираясь на них, доказываем оценки для функции дистанции.

Основываясь на известных результатах для классических пространств аналитических функций в различных областях в $\mathbb{C}^n$, мы получаем несколько новых утверждений для функции дистанции на произведении строгопсевдовыпуклых областей с гладкой границей, в ограниченных областях с границей из C^2 и в допустимых областях.

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Кроме того, нами получены некоторые точные результаты на произведении строгопсевдовыпуклых областей с гладкой границей в $\mathbb{C}^n$, расширяющие наши результаты для строгопсевдовыпуклых областей.

Ключевые слова: оценки дистанций, аналитические функции, ограниченные области, допустимые области, псевдовыпуклые области.

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SOME REMARKS ON DISTANCES IN SPACES OF ANALYTIC FUNCTIONS IN BOUNDED DOMAINS WITH C^2 BOUNDARY AND ADMISSIBLE DOMAINS[3] [4]

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Abstract

The so-called Bergman representation formula (reproducing formula) and various estimates for Bergman projection with positive reproducing kernel and sharp Forelli-Rudin type estimates of Bergman kernel are playing a crucial role in certain new extremal problems related to so-called distance function in analytic function spaces in various domains in C^n.

In this paper based on some recent embedding theorems for analytic spaces in bounded domains with C^2 boundary and admissible domains new results for Bergman-type analytic function spaces related with this extremal problem will be provided. Some(not sharp) assertions for BMOA and Nevanlinna spaces, for analytic Besov spaces in any D bounded domain with C^2 boundary or admissible domains in C^n will be also provided.

We remark for readers in addition the problems related to regularity of Bergman projection which we use always in proof in various types of domains with various types of boundaries (or properties of boundaries) are currently and in the past already are under intensive attention. Many estimates for reproducing operators and there kernels and L^p boundedness of Bergman projections have been also the object of considerable interest for more than 40 years. These tools serve as the core of all our proofs. When the boundary of the domain D is sufficiently smooth decisive results were obtained in various settings. Our intention in this paper is the same as a previous our papers on this topic, namely we collect some facts from earlier investigation concerning Bergman projection and use them for our purposes in estimates of $dist_Y(f, \mathcal{X})$ function (distance function).

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Based on our previous work and recent results on embeddings in classical analytic spaces in domains of various type in C^n we provide several new general assertions for distance function in products of strictly pseudoconvex domains with smooth boundary, general bounded domains with C^2 boundary and in admissible domains in various spaces of analytic functions of several complex variables.These are first results of this type for bounded domains with C^2 boundary and admissible domains. In addition to our results we add some new sharp results for special kind of domains so called products of strictly pseudoconex domains with smooth boundary in C^n extendind our sharp results in strictly pseudoconex domains.

Keywords: Distance estimates, analytic function, bounded domains, admissible domains, pseudoconvex domains.

Bibliography: 22 titles.

1. Introduction

The so-called Bergman representation formula (reproducing formula) and various estimates for Bergman projection with positive reproducing kernel and sharp Forelli-Rudin type estimates of Bergman kernel are playing a crucial role in certain new extremal problems related to so-called distance function in analytic function spaces in various domains in C^n.(see, for example, [15], [16], [17], [18], [9] and various references there).

In this paper based on some recent embedding theorems for analytic spaces in bounded domains with C^2 boundary and admissible domains (see, for example, [1], [2], [3])new results for Bergman-type analytic function spaces related with this extremal problem will be provided.Some(not sharp) assertions for BMOA and Nevanlinna spaces, for analytic Besov spaces in any D bounded domain with C^2 boundary or admissible domains in C^n will be also provided.

We remark for readers in addition the problems related to regularity of Bergman projection which we use always in proof in various types of domains with various types of boundaries (or properties of boundaries) are currently and in the past already were under intensive attention. (see, for example, [7], [5] and references there).

Many estimates for reproducing operators and there kernels and L^p boundedness of Bergman projections have been also the object of considerable interest for more than 40 years. These tools serve as the core of all our proofs. When the boundary of the domain D is sufficiently smooth decisive results were obtained in various settings (see [7], [5] and references there).

Our intention in this paper is the same as a previous our papers on this topic, namely we collect some facts from earlier investigation concerning Bergman projection and use them for our purposes in estimates of $dist_Y(f, \mathcal{X})$ function (distance function).

Following our previous paper [16], [15], [9] we easily see to obtain a sharp result for distance function we need only several tools and the scheme here is the following.

First we need an embedding of our quazinormed analytic space (in any domain) into another one ($X \subset Y$), this immediately pose a problem of $dist_Y(f, X) = \inf_{g \in X} \|f-g\|_Y$ for all $f \in Y \backslash X$, then we need the Bergman reproducing formula for all f function from Y space. Then finally we use the boundedness of Bergman type projections with $|K(z, w)|$ positive kernel acting from X to X together with sharp estimates of Bergman kernel. These three tools were used in general Siegel domain of second type, polydisk and unit ball in [16], [15], [17], [18](see also various references there). We continue to use these tools providing new sharp (and not sharp) results in various spaces of analytic functions, in analytic spaces in pseudoconvex domains, in spaces in any bounded domain D with C^2 boundary and in so-called admissible domains also.

The plan of this note is the following. We provide first two typical results on this topic with complete proofs taken from one of our previous recent paper but in more general setting of products of strictly pseudoconvex domains with smooth boundary and then in next sections discuss its various modification and extensions in more general settings based on same ideas in a sketchy form.

We denote as c or as C with various indexes various positive constants below. We denote by λ a characteristic function of a set everywhere below.

2. A distance theorem for analytic Bergman spaces in products of bounded strictly pseudoconvex domains with smooth boundary

Functional spaces and their properties on m products of strictly pseudoconvex domains with smooth boundary were studied in various papers (see, for example, [22],[21] and references there). The goal of this section is to provide new sharp distance theorem in m- products of bounded strictly pseudoconvex domains with a smooth boundary in C^n but with a complete proof only for $m = 1$ case.

Proofs of $m = 1$ and general case are very similar and we choose it to shorten the exposition. Moreover proofs related with this particular case can be seen in[9]. The reason we do this to heavily shorten this and last sections where series of other similar type results with similar proofs based also on recent work from [1], [2], [3] in more general settings namely in bounded domains with C^2 boundary and admissible domains will be formulated by us. So we first consider Bergman type spaces in $D \subset \mathbb{C}^n$, where D is smoothly bounded relatively compact strictly pseudoconvex domain, providing sharp results in this case in this section. In one of the next sections of this paper this results will be partially generalized, but proofs mostly will be omitted there since it is based on this proof. The theory of bounded strictly pseudoconvex domains in C^n with smooth boundary and function spaces of several complex variables on them is a subject of very active study (see, for example, [9], [13], [11], [14] and various references there).

Our proofs in this section are heavily based on estimates from [14], where even more general situation was considered when our domains are embedded in so-called Stein manifolds. For definition of Bergman spaces and other objects we refer the reader to [14].

We define Bergman spaces as usual as follows.

$$A^p_\alpha(D) = \left\{ f \in H(D) : \int_D |f(z)|^p \delta_D(z)^{\alpha-1} dV(z) < \infty \right\} \; 0 < p < \infty, \; \alpha > 0,$$

where $H(D)$ is a space of all analytic functions in D and $dV(z)$ (or $dv(z)$) is a Lebegues measure in D and $\delta_D(z)$ is a distance from z to boundary of D [14].

Since $|f(z)|^p$ is subharmonic (even plurisubharmonic) for a holomorphic f in D, we have $A^p_s(D) \subset A^\infty_t(D)$ for $0 < p < \infty$, $sp > n$ and $t = s$. Also, $A^p_s(D) \subset A^1_s(D)$ for $0 < p \leqslant 1$ and $A^p_s(D) \subset A^1_t(D)$ for $p > 1$ and t sufficiently large. Therefore we have an integral representation.

$$f(z) = \int_D f(\xi) K(z, \xi) \delta^t(\xi) dV(\xi), \tag{1}$$

where $K(z, \xi)$ is a kernel of type $n+t+1$,[14] that is a measurable function on $D \times D$ such that $|K(z, \xi)| \leqslant C|\tilde{\Phi}(z, \xi)|^{-(n+1+t)}$, where $\tilde{\Phi}(z, \xi)$ is so called Henkin-Ramirez function for D. From now on we work with a fixed Henkin-Ramirez function $\tilde{\Phi}$ and a fixed kernel K of type $n + t + 1$. We are going to use the following results from [14].

Lemma 1. ([14], Corollary 5.3.) *If $r > 0$, $0 < p \leqslant 1$, $s > -1$, $p(s + n + 1) > n$ and $f \in H(D)$, then we have*

$$\left(\int_D |f(\xi)| |\tilde{\Phi}(z, \xi)|^r \delta^s(\xi) dV(\xi) \right)^p \leqslant C \int_D |f(\xi)|^p |\tilde{\Phi}(z, \xi)|^{rp} \delta^{p(s+n+1)-(n+1)}(\xi) dV(\xi).$$

Lemma 2. ([14], Corollary 3.9.) *Assume $K(z, \xi)$ is a kernel of type β, and $\sigma > 0$ satisfies $\sigma + n - \beta < 0$. Then we have*

$$\int_D K(z, \xi) \delta^{\sigma-1}(z) dV(z) \leqslant C \delta^{\sigma+n-\beta}(\xi).$$

Note we use same δ function as a function of $z_1, \dots, z_m$ variables in product domain D^m as products of m such functions, this will be clear from context below. And we use V^m as a Lebegues measure on D^m A natural problem is to estimate $\operatorname{dist}_{A^\infty_s(D)}(f, A^q_s(D))$ where $0 < q < \infty$, $sq > n$ and $f \in A^\infty_s(D)$. We give sharp estimates below, treating cases $0 < q \leqslant 1$ and $q > 1$ separately. Note the same problem is valid for same spaces but on product domains.

Note moreover all we discussed is also valid in m-products of D domains and analytic spaces on them. This includes the last embedding, the integral representation and the estimates of Kernel. (m products of kernels). This can be obtained by using "one dimensional" result m -times by each variable separately. Below we formulate a general theorem in it is "m- product version". For $m = 1$ case this theorem can be seen before in [9]. To define all objects in product spaces and analytic spaces on products of pseudoconvex domains themselves we use standard procedures which we see in polydsk (see, for example, [16]). Note also various problems of analytic function theory on products of pseudoconvex domains were under attention, see, for example, [21] and [22] and references there.

THEOREM 1. *Let $m \in N$, $0 < q \leqslant 1$, $sq > n$, $f \in A_s^\infty(D^m)$ and $t > s$ is sufficiently large. Then $\omega_1 = \omega_2$ where*

$$\omega_1 = \operatorname{dist}_{A_s^\infty(D^m)}(f, A_s^q(D^m)),$$

$$\omega_2 = \inf\Bigg\{\varepsilon > 0 :$$

$$\int_{D^m}\left(\int_{\Omega_{\varepsilon,s}} |K^m(z_1,\dots,z_m,\xi)|\delta^{t-s}(\xi)dV^m(\xi)\right)^q \delta^{sq-n-1}(z)dV^m(z) < \infty\Bigg\},$$

where $K(z,\xi)$ is the above kernel of type $n+t+1$, $K_m(z_1,\dots,z_m,\xi)$ is a m-product of such kernels and

$$\Omega_{\varepsilon,s} = \{z \in D^m : |f(z)|\delta^s(z) \geqslant \varepsilon\}.$$

PROOF. We assume $m = 1$.The proof of general case is the same. Let us prove that $\omega_1 \leqslant \omega_2$. We fix $\varepsilon > 0$ such that the above integral is finite and use (1):

$$f(z) = \int_{D\setminus\Omega_{\varepsilon,s}} f(\xi)K(z,\xi)dV(\xi) + \int_{\Omega_{\varepsilon,s}} f(\xi)K(z,\xi)dV(\xi) = f_1(z) + f_2(z).$$

We estimate f_1:

$$|f_1(z)| \leqslant C\varepsilon\int_D |K(z,\xi)|\delta^{t-s}dV(\xi) \leqslant C\varepsilon\int_D \frac{\delta^{t-s}(\xi)dV(\xi)}{|\tilde\Phi(z,\xi)|^{n+t+1}} \leqslant C\varepsilon\delta^{-s}(z),$$

where the last estimate is contained in [14] (see p. 375). Next,

$$\|f_2\|_{A_s^q}^q = \int_D |f_2(z)|^q\delta^{sq-n-1}(z)dV(z) \leqslant$$

$$\leqslant C\int_D\left(\int_{\Omega_{\varepsilon,s}} |f(\xi)|K(z,\xi)\delta^t(\xi)dV(\xi)\right)^q \delta^{sq-n-1}(z)dV(z) \leqslant C'\|f\|_{A_s^\infty}^q.$$

Now we have

$$\operatorname{dist}_{A_s^\infty(D)}(f, A_s^q(D)) \leqslant \|f - f_2\|_{A_s^\infty(D)} = \|f_1\|_{A_s^\infty(D)} \leqslant C\varepsilon.$$

Now assume that $\omega_1 < \omega_2$. Then there are $\varepsilon > \varepsilon_1 > 0$ and $f_{\varepsilon_1} \in A^q_s(D)$ such that $\|f - f_{\varepsilon_1}\|_{A^\infty_s} \leqslant \varepsilon_1$ and

$$I = \int_D \left(\int_{\Omega_{\varepsilon,s}} |K(z,\xi)| \delta^{t-s}(\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) = \infty.$$

As in the case of the upper half-plane one uses $\|f - f_{\varepsilon_1}\|_{A^\infty_s} \leqslant \varepsilon_1$ to obtain

$$(\varepsilon - \varepsilon_1) \chi_{\Omega_{\varepsilon,s}}(z) \delta^{-s}(z) \leqslant C |f_{\varepsilon_1}(z)|.$$

Now the following chain of estimates leads to a contradiction:

$$\begin{aligned} I &= \int_D \left(\int_D \chi_{\Omega_{\varepsilon,s}}(\xi) \delta^{t-s}(\xi) K(z,\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\leqslant C \int_D \left(\int_D |f_{\varepsilon_1}(\xi)| \delta^t(\xi) K(z,\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\leqslant C \int_D \left(\int_D |f_{\varepsilon_1}(\xi)| \delta^t(\xi) \frac{dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+t+1}} \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\leqslant C \int_D \int_D |f_{\varepsilon_1}(\xi)|^q \frac{\delta^{sq-n-1}(z) \delta^{q(t+n+1)-(n+1)}(\xi)}{|\tilde{\Phi}(z,\xi)|^{q(n+t+1)}} dV(z) dV(\xi) \\ &\leqslant C \int_D |f_{\varepsilon_1}(\xi)|^q \delta^{sq-n-1}(\xi) dV(\xi) < \infty, \end{aligned} \tag{2}$$

where we used Lemma 1 and Lemma 2 with $\beta = q(n+t+1)$, $\sigma = sq - n$. □

Next theorem deals with the case $1 < q < \infty$.

THEOREM 2. *Let $m \in N$, $q > 1$, $sq > n$, $t > s$, $t > \frac{s+n+1}{q}$ and $f \in A^\infty_s(D^m)$. Then $\omega_1 = \omega_2$ where*

$$\omega_1 = \operatorname{dist}_{A^\infty_s(D^m)}(f, A^q_s(D^m)),$$

$$\omega_2 = \inf \left\{ \varepsilon > 0 : \int_{D^m} \left(\int_{\Omega_{\varepsilon,s}} |K^m(z,\xi)| \delta^{t-s}(\xi) dV^m(\xi) \right)^q \delta^{sq-n-1}(z) dV^m(z) < \infty \right\}.$$

PROOF. We again restric ourselves to $m = 1$ case. See also [9]. The proof of general case is the same. A carefull inspection of the proof of the previous theorem shows that it extends to this $q > 1$ case also, provided one can prove the estimate:

$$\begin{aligned} J &= \int_D \left(\int_D |f_{\varepsilon_1}(\xi)| \delta^t(\xi) K(z,\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\leqslant C \int_D |f_{\varepsilon_1}(\xi)|^q \delta^{sq-n-1}(\xi) dV(\xi) < \infty \end{aligned}$$

where $q > 1$. Using Hölder's inequality and Lemma 2, with $\sigma = 1$ and $\beta = n+1+p\varepsilon$, we obtain

$$\begin{aligned} I(z) &= \left(\int_D |f_{\varepsilon_1}(\xi)|\delta^t(\xi)K(z,\xi)dV(\xi)\right)^q \\ &\leqslant \int_D \frac{|f_{\varepsilon_1}(\xi)|^q\delta^{tq}(\xi)dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+1+tq-\varepsilon q}} \cdot \left(\int_D \frac{dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+1+p\varepsilon}}\right)^{q/p} \\ &\leqslant C\int_D \frac{|f_{\varepsilon_1}(\xi)|^q\delta^{tq}(\xi)dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+1+tq-\varepsilon q}}\,\delta^{-q\varepsilon}(z), \end{aligned}$$

and this gives

$$\begin{aligned} J &\leqslant C\int_D\int_D \frac{|f_{\varepsilon_1}(\xi)|^q\delta^{tq}(\xi)\delta^{-q\varepsilon+sq-n-1}(z)}{|\tilde{\Phi}(z,\xi)|^{n+1+tq-\varepsilon q}}dV(z)\,dV(\xi) \\ &\leqslant C\int_D |f_{\varepsilon_1}(\xi)|^q\delta^{sq-n-1}(\xi)dV(\xi) < \infty, \end{aligned}$$

where we again used Lemma 2, with $\beta = n+1+tq-\varepsilon q$ and $\sigma = q(s-\varepsilon)-n > 0$. □

Remark 1. *We note that some results of this section on distances can be extended to so-called minimal bounded homogeneous domains.* $\Omega \subset \mathbb{C}^n$ *and* A^2_α *Hilbert spaces on them. Indeed, our proofs are based on Bergman representation formula, and Forelli-Rudin type estimates for integrals*

$$\int_\Omega K^\alpha(w,w)|K(z,w)|^\beta dV(w), \quad \alpha > 0,\ \beta > 0\ \ z \in \Omega,$$

where $K(z,w)$ *is a Bergman reproducing kernel for the weighted Bergman space* $A^2_\alpha(\Omega)$. *All relevant estimates for this case can be found in recent papers [19] and [20].*

3. Extremal problems in analytic spaces in bounded domains with C^2 boundary.

The intention of this section which is heavily based on previous one to consider the same extremal problem but in more general setting. All results of this section are based on series of recent theorems on embeddings of analytic function spaces in bounded domains with C^2 boundary(see [1], [2], [3], [4] and references there). Note our general results provide only one side estimates for dist function. Note also we will not provide proofs but only short comment after each assertion. Since all proofs are modifications of proofs of theorems of previous section. Let $D \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary and let $H(D) = \{f \text{ holomorphic in } D\}$. For

$z \in D$ let $\delta_D(z)$ denote the distance from z to ∂D. We define for every $\alpha > 0$, $p < \infty$ the measure $dv_\alpha = (\delta_D(z)^{\alpha-1})dV$ where dV (or dV_α) denotes the volume element and the weighted Bergman space

$$A^p_\alpha(D) = \left\{ f \in H(D) : \|f\|^p_{p,\alpha} = \int_D |f(w)|^p dV_\alpha(w) < \infty \right\};$$

$$A^{-\sigma}(D) = \{ f \in H(D) : \|f\|_{-\sigma} = \sup(\delta_D(w)^\sigma |f(w)|) < \infty : w \in D \},\ \sigma > 0.$$

$A^p_0 = H^p$ for all positive p. If D has C^2-boundary then the following inequality holds (see [1], [2])

$$|f(z)| \leqslant (\delta_D(z)^{-\frac{n+\alpha}{p}})\|f\|_{p,\alpha};\ \forall f \in A^p_\alpha(D); \forall z \in D$$

and

$$A^p_\alpha(D) \subset A^{-\frac{n+\alpha}{p}}(D);\ \textit{see } [1].$$

Also we have $H^\infty(D) \subset A^{-\sigma}(D)$, $\sigma > 0$, see [1].

In summary we have if D has C^2 boundary then we have the following chains of embeddings

$$H^\infty(D) \subset A^{-\sigma}(D) \subset A^p_\alpha(D) \subset A^{-\frac{n+\alpha}{p}}(D) \subset N_\beta(D)$$

for all $0 < p, \alpha < \infty$, $0 < \sigma < \frac{\alpha}{p}$; $\beta > 0$; (see [1]) where

$$N_\beta(D) = \left\{ f \in H(D) : \int_D (\log^+ |f(w)|) dV_\beta(w) < \infty \right\},\ \beta > 0$$

is a Bergman-Nevanlinna class in D (see for all this [1] and references there). Each embedding as we noticed above pose a problem of finding $dist_Y(f, X)$, $f \in Y \backslash X$, $X \subset Y$.

In [2] the authors showed for Hardy spaces $H^p(D) \subset A^q_\beta(D)$; for $0 < p < q < \infty$; $\beta > 0$ with $\frac{n}{p} = \frac{n+\beta}{q}$; for all bounded domains in $\mathbb{C}^n$ with C^2 boundary. This poses a problem $dist_{A^q_\beta}(f, H^p)$; $f \in A^q_\beta$ for any bounded domain in $\mathbb{C}^n$ with C^2 boundary.

We formulate a general result for A^p_α space.

Let

$$B(A^p_\alpha, f) = \inf \left\{ \varepsilon > 0 : \sup_{|z|<1} \left[\mathcal{X}_{\Omega_{\tau,\varepsilon}}(z)(\delta_D(z))^{-\frac{n+\alpha}{p}+s} \right] < \infty \right\},$$

where $s \geqslant \frac{n+\alpha}{p}$, $\tau = -\frac{n+\alpha}{p}$ and

$$\Omega_{\tau,\varepsilon} = \{ z \in D : |f(z)|\delta^\tau_D(z) \geqslant \varepsilon \};\ \tau = -\frac{n+\alpha}{p};\ \alpha > 0.$$

THEOREM 3. *Let D be any bounded domain in $\mathbb{C}^n$. Let also D has C^2 boundary. Then for all f, $f \in A^{-\frac{n+\alpha}{p}}$*

$$B(A^p_\alpha, f) \leqslant c(dist_{A^{-\frac{n+\alpha}{p}}(D)}(f, A^p_\alpha(D)));\ 0 < p < \infty;\ \alpha > 0.$$

The proof follows directly from arguments of proof of theorem 1 and will be omitted. Note similarly we can get one side estimates for first and third embeddings in chain of embeddings we provided above Similar results are valid for pairs (B, B^p), where B is a Bloch space,B^p is an analytic Besov space. Since $B^p \subset B$ as we mentioned above for all $p \in (0, \infty)$. Similarly using other embeddings we mentioned above we can formulate a result like this using the embedding $H^\infty \subset A^{-\sigma}(D)$, where $\sigma \in (0, \frac{\alpha}{p})$ on estimates of $dist_{A^{-\sigma}}(f, H^\infty(D))$ or $dist_{A^{-l}}(f, H^p(D))$; $0 < l < n(\frac{1}{p} - \frac{1}{q})$on bounded D domain with C^2 boundary in $\mathbb{C}^n$.

The proof of theorem 3 can be easily recovered from arguments we provided above in theorem 1 (see, for example, for one-dimensional case [16] and references there).

The following theorem is based on already mentioned embedding for any bounded D domain with C^2 boundary.

Let $\Omega_{\varepsilon,\sigma} = \{z \in \Omega : |f(z)|\delta(z)^\sigma \geqslant \varepsilon\}$, then we have

THEOREM 4. *Let D be bounded domain with C^2 boundary. Let $f \in A^{-\sigma}(D)$. Then for $\sigma \in (0, \frac{\alpha}{p})$*

$$dist_{A^{-\sigma}(D)}(f, H^\infty(D)) \geqslant \inf\{\varepsilon > 0 : \sup_{z \in D}[\mathcal{X}_{\Omega_{\varepsilon,\sigma}}(z)]\delta(z)^\sigma < \infty\}$$

The proof follows easily from discussion we had above based on arguments of proof of theorem 1 we omit details. Finally based on embeddings (see [1], [2], [3]) $H^\infty \subset H^q$, and $A^p_\alpha \subset H^q$, $p < q < \infty$, $\alpha > 0$, $\frac{n+\alpha}{p} = \frac{n}{q}$ similar results based on proof of theorem 1 can be obtained for any bounded domain with C^2 boundary. We leave this to interested readers.

4. On extremal problems in analytic spaces in admissible domains and in bounded strictly pseudoconvex domains with smooth boundary

The main goal of this section to turn to recent results from [12], [11], [6] on general admissible domains to get new estimates for dist function in this setting. All results are one side estimates and we don't discuss proofs since all of them are based on a proof of theorems 1, 2 from section 1. We say a smoothly bounded domain in $\mathbb{C}^n$ is an admissible domain (see [7]) if D is one of the following.

1) a strictly pseudoconvex domain

2) a pseudoconvex domain of finite type in $\mathbb{C}^2$

3) a convex domain of finite type in $\mathbb{C}^n$

Such type domains were under intensive study [13], [11], [12], [6]. It is natural to give estimates for dist function also in this case in this more general setting. But first we generalize all results of first section (in strongly pseudoconvex domains with smooth boundary)to so -called mixed norm spaces. Note we again omit details of proofs since it is a modification of proof of theorem 1.

Let $D = \{z : \rho(z) < 0\}$ be a bounded strictly pseudoconvex domain of $\mathbb{C}^N$ with C^∞ boundary. We assume that the strictly plurisubharmonic function ρ is of class C^∞ in a neighbourhood of $\bar{D}$, that $-1 \leqslant \rho(z) < 0$, $z \in D$, $|\partial\rho| \geqslant c_0 > 0$ for $|\rho| \leqslant r_0$. Let $A^{p,q}_{\delta,k}(D) = \{f$ holomorphic in D such that $\|f\|_{p,q,\delta,k} < \infty\}$ where

$$\|f\|_{p,q,\delta,k} = \left(\sum_{|\alpha|\leqslant k}\int_0^{r_0}\left(\int_{\partial D_r}|D^\alpha f|^p d\sigma_r\right)^{\frac{q}{p}}(r^{\delta\frac{q}{p}-1})dr\right)^{\frac{1}{q}}$$

here we denote by $D_r = \{z \in \mathbb{C}^n : \rho(z) < -r\}$, ∂D_r it is boundary $d\sigma_r$ the normalized surface measure on ∂D_r and by dz the normalized volume element on D (see [5], [10]).

For $p = q$, $k = 0$:

$$\|f\|_{p,\delta} = \left(\int_D |f(z)|^p(-\rho(z))^{\delta-1}dv(z)\right)^{\frac{1}{p}}$$

and hence we have a standard Bergman class $A^p_\delta(D)$ see [5], [10].

$$A^p_{q,s} = \left\{f \in H(D) : \left(\sum_{|\alpha|\leqslant s}\int_D|\partial^\alpha f|^p\delta^{q-1}(z)dv(z)\right)^{\frac{1}{p}} < \infty\right\},$$
$$q > 0,\ s \in \mathbb{N},\ 0 < p < \infty.$$

Theorem 5. *Let $m > 0$, $0 < q \leqslant 1$; $sq > n$, $q \geqslant p$, $f \in A^\infty_s(D)$, $t > t_0$, where t_0 is a large enough positive number. Then for any strictly bounded pseudoconvex domain D with smooth boundary we have $\omega_2 \leqslant c\omega_1$ where*

$$\omega_1 = dist_{A^\infty_s(D)}(f, A^{q,p}_{sq-n}(D)) = dist_{A^\infty_s}(f, A^{q,p}_{sq-n+mq,m});$$

and

$$\omega_2 = \inf\left\{\varepsilon > 0 : \int_D\left(\int_{\Omega_{\varepsilon,s}}|K(z,\zeta)|\delta^{t-s}(\zeta)dV(\zeta)\right)^q\delta^{sq-n-1}dV(z) < \infty\right\}.$$

Theorem 6. *Let D be any strictly bounded pseudoconvex domain with smooth boundary . Let $m > 0$, $q > 1$; $sq > n$, $q \geqslant p$, $q \leqslant \infty$, $t > t_0$, where t_0 is large enough positive number ,$f \in A^\infty_s(D)$. Then $\omega_2 \leqslant c\omega_1$ where*

$$\omega_1 = dist_{A^\infty_s}(f, A^{q,p}_{sq-n+mq,m});$$

and

$$\omega_2 = \inf\left\{\varepsilon > 0 : \int_D \left(\int_{\Omega_{\varepsilon,s}} |K(z,\zeta)|\delta^{t-s}(\zeta)dV(\zeta)\right)^q \delta^{sq-n-1}dV(z) < \infty\right\}.$$

For proofs of these assertions we refer the reader to our paper [6] and embeddings from [10], [5]. Combining these two we'll get proofs immediately.

Note for some values of parameters a sharp results for these pair of spaces were obtained previously by us see [16]. We now define several new analytic spaces in D domain. We list then various (known) embeddings which lead to distance estimates. Then provide new one side estimates.

For a bounded strictly pseudoconvex domain D we have also $BMOA(D) \subset H^2(D)$, where we define for any admissible domain $H^p(D)$ and $BMOA(D)$ in the following standard way (see [8], [12])

$$H^p(D) = \left\{f \in H(D) : \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{\partial D_\varepsilon} |f|^p d\sigma_\varepsilon\right)^{\frac{1}{p}} < \infty\right\},$$

where again $D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\},\ \varepsilon > 0$; and $D = \{z \in \mathbb{C}^n : \rho(z) < 0\},\ 0 < p \leqslant \infty$, $d\sigma_\varepsilon$ is surface measure on ∂D_ε. For any admissible domain we define also

$$BMOA(D) = \Big\{f \in H^1(D) : \|f\|^2_{BMOA} =$$

$$= \sup \int_{\partial D} |f(w) - f(z)|^2 \times P(z,w)d\sigma(w); z \in D\Big\} < \infty,$$

where $P(z,w) = (S(z,z)^{-1})|S(z,w)|^2; z \in D, w \in \partial D$ is a Poisson-Szego Kernel, $S(z,w)$ - Szego Kernel. see [5], [13], [8].

Let $p > n$, D is any bounded domain with C^2 boundary in $\mathbb{C}^n$. Then the Besov space is a space with norm see [11], [10]

$$\|f\|_{B^p(D)} = \left(\int_D |\nabla^{n+1} f(z)|^p [\delta(z)]^{p(n+1)} d\lambda(z)\right)^{\frac{1}{p}} < \infty, \tag{3}$$

where

$$|\nabla^{n+1}(f(z))| = \sum_{|\alpha|\leqslant n+1} \left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)\right|;$$

and $d\lambda(z) = K(z,z)dV(z)$ - biholomorphically invariant measure and $K(z,w)$ is the Bergman kernel, as above dV is a volume measure of D [10], [11].

For $p = \infty$, $B^\infty = Bl(D)$ this is a classical Bloch space (see [5], [13], [8]).

We add more embeddings from [13], [8], [11], [5] for admissible D domains.

We say $f \in VMOA$ if $f \in BMOA$ and

$$\lim_{z\to\partial D} \int_{\partial D} |f(w) - f(z)|^2 P(z,w)d\sigma(w) = 0.$$

We have $B^p(D) \subset VMOA(D)$ for $1 < p < \infty$. Also we have

$$\|f\|_{Bl} \leqslant \sup_{z\in D}\left[|\nabla^{n+1}(f(z))|(\delta(z)^{n+1})(\log\frac{c}{\delta(z)})\right] \ n \geqslant 0$$

and $H^\infty \subset Bl$, $H^\infty \subset B^p$; where

$$\|f\|_{Bl} = \sup_{z\in D}|\nabla^k f(z)|(\delta^k(z)) < \infty,\ 1 \leqslant p \leqslant \infty, k > 0.$$

The log Besov space is the following class of analytic functions [13], [8], [11].

$$LB^p(D) = \{f \in H(D) : \left[\int_D |\nabla^{n+1} f(z)|^p(\delta(z)^{(n+1)p})(\log\frac{c}{\delta(z)})^{\frac{1}{p}} d\lambda(z)\right]^{\frac{1}{p}} < \infty\}$$

$p \in (0,\infty)$. We have $LB^p \subset B^p(D) \subset B^q(D) \subset Bl(D)$: $0 < p < q < \infty$ (see [11]).

We also have for bounded strictly pseudoconvex domains $BMOA \subset L^p(\partial D) \cap H(D) : 1 \leqslant p < \infty$ (see [12]). We define Nevanlinna classes for all bounded domains with C^2 boundary in C^n (see [13], [8], [11]).

Let $D_\varepsilon = \{\rho < (-\varepsilon)\}$; D be bounded domain with C^2 boundary; $D = \{\rho < 0\}$; $d\rho \neq 0$ on ∂D. D_ε are subdomain of D $0 < \varepsilon < \varepsilon_0$ with C^1 boundary σ, σ_ε are surface measures on ∂D and ∂D_ε respectively.

For $0 < p < \infty$ Hardy class is as above

$$H^p(D) = \left\{ f \in H(D) : \sup_{0<\varepsilon<\varepsilon_0}\left(\int_{\partial D_\varepsilon} |f|^p d\sigma_\varepsilon\right)^{\frac{1}{p}} < \infty \right\}$$

if we put $\log^+|f| = \max(\log|f|, 0)$ instead of $|f|^p$ then we set analytic Nevanlinna class $N(D)$. Note $N(D) \supset H^p(D)$ for all $1 \leqslant p \leqslant \infty$. Also it is known for all bounded D domains with C^2 boundary (see [7], [8], [9]).

$$\sup_{z\in D}|f(z)|dist(z,\partial D)^{\frac{n}{p}} \leqslant c_1\|f\|_{H^p(D)};$$

$$\sup_{z\in D}|f(z)|dist(z,\partial D)^{\frac{n+1}{p}} \leqslant c_2\|f\|_{A^p(D)};$$

$p \in (0,\infty)$

We also have an embedding of atomic Hardy spaces into standard Hardy classes for admissible domains $H^p_A(D) \subset H^p(D)$; $0 < p \leqslant 1$, where

$$H^p_A(D) = \{\sum_{j=1}^m \lambda_j A_j : A_j \text{ is holomorphic p-atom} \sum_{j=1}^\infty |\lambda_j|^p < \infty\}$$

(see about these spaces and the definition of p-atom (holomorphic) in [9] for all admissible domains).

Based on several from these embeddings we formulate another (not sharp) three theorems on distances below. Short proofs of all of them are fully based on arguments of proof of theorem 1 and we leave them to interested readers.

In all assertions below our domain is admissible,in an assertion related with analytic Nevanlinna spaces it can be also simply bounded with C^2 boundary. A sharp version of our first assertion is probably valid based on projection theorems for Besov spaces from [11].

THEOREM 7. *Let $p > n$; $f \in Bl$. Let $S_{\varepsilon,f,n} = \{z \in D : |\nabla^{n+1} f(z)|(\delta(z))^{n+1} \geqslant \varepsilon\}$, $n \in \mathbb{N}$.*

Let $\tilde{\omega}_1 = dist_{Bl}(f, B^p)$ and also let

$$\tilde{\omega}_2 = \inf\{\varepsilon > 0 : \int_D \mathcal{X}_{S_{\varepsilon,f}}(z) d\lambda(z) < \infty\}.$$

Then $\tilde{\omega}_1 \geqslant \tilde{\omega}_2$.

Note similar assertion is valid if we replace $B^p(D)$ by $LB^p(D)$ or any analytic function space X, so that $X \subset Bl(D)$ for any admissible D domain.

Next we formulate an assertion which in particular concerns analytic Nevanlinna spaces in bounded domains in C^n with C^2 boundary.

THEOREM 8. *1) Let $p > n$ and let also $f \in H^p$. Then we have $K_1 \geqslant cK_2$, where $K_1 = dist_{H^p}(f, B^p)$ and*

$$K_2 = \inf\{\varepsilon > 0 : \int_0^r \mathcal{X}_{K_{\varepsilon,f}(\rho)} \rho^{2(n+1)p-(n+1)} d\rho < \infty\}$$

where

$$K_{\varepsilon,f} = \{\rho \in (0, \varepsilon_0) : \int_{\partial D_\rho} |f(\zeta)|^p d\sigma_\rho(\zeta) \geqslant \varepsilon\}$$

2) Let $f \in H^1(D)$. Then we have $s_1 \geqslant c(s_2)$ where

$$s_1 = dist_{H^1}(f, H^1 at)$$

$$s_2 = \inf\{\varepsilon > 0 : \sup_{0<r<\varepsilon_0} \lambda_{K_{\varepsilon,f}}(r) < \infty\}$$

and where

$$K_{\varepsilon,f} = \{\rho \in (0, \varepsilon_0) : \int_{\partial D_\rho} |f(\zeta)| d\sigma_\rho(\zeta) \geqslant \varepsilon\}$$

3) Let $p > 1$, let also $f \in N(D)$, then we have $\tau_1 \geqslant c\tau_2$, where

$$\tau_1 = dist_{N(D)}(f, H^p(D))$$

and where

$$M_{\varepsilon,f} = \{\rho \in (0, \varepsilon_0) : \int_{\partial D_\rho} (\log^+ |f(\zeta)|)^p d\sigma_\rho(\zeta) \geqslant \varepsilon\}$$

$$\tau_2 = \inf\{\varepsilon > 0 : \sup_{0<r<\varepsilon_0} (\lambda_{M_{\varepsilon,f}}(r)) < \infty\}$$

Finally we formulate another assertion which concerns BMOA type spaces in admissible domains in C^n.

THEOREM 9. *Let D be any admissible domain in C^n*
1) Let $f \in H^1$ then we have $\tilde{K}_1 \geqslant c\tilde{K}_2$, where

$$\tilde{K}_1 = dist_{H^1}(f, BMOA)$$

$$\tilde{K}_2 = \inf\{\varepsilon > 0 : \sup_{0<r<\varepsilon_0} \lambda_{K_{\varepsilon,f}}(r) < \infty\}$$

where

$$K_{\varepsilon,f} = \{\rho \in (0, \varepsilon_0) : \int_{\partial D_\rho} |f(\zeta)| d\sigma_\rho(\zeta) \geqslant \varepsilon\}$$

2) Let $f \in A^\infty_{n/p}$ then we have for $p \in (0, \infty)$ $\tilde{R}_1 \geqslant c\tilde{R}_2$, where

$$\tilde{R}_1 = dist_{A^\infty_{n/p}}(f, H^p)$$

$$\tilde{R}_2 = \inf\{\varepsilon > 0 : \sup_{0<r<\varepsilon_0} \mathcal{X}_{K_{\varepsilon,f}}(r) < \infty\}$$

5. Conclusion

Conditions of similar type for distance function based on embeddings $A^p_{p(s_2-s_1)} \subset H^p_{s_1}$ with some restriction on p, s_1, s_2 can be obtained similarly. These are conditions of type $\sup\limits_{0<r<\varepsilon_0} \lambda_{M_{\varepsilon,f}}(r) < \infty$ where $M_{\varepsilon,f}$ should be appropriately defined. We leave this to readers. Proofs of all these assertions are modifications of proof of theorem 1 and we omit details.

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