ТЕОРИЯ ВЕРОЯТНОСТЕЙ И ЕЕ ПРИМЕНЕНИЯ

Том XVI 1971 *Выпуск 3*

LOCAL FUNCTIONALS AND GENERALIZED RANDOM FIELDS WITH INDEPENDENT VALUES *

M. M. RAO

1. Introduction. Let $\mathcal{K}(R^n)$ be the space of infinitely differentiable real functions on the Euclidean n-space R^n, which vanish outside of compact sets. If $f_n \in \mathcal{K}$ $(=\mathcal{K}(R^n)$ hereafter), then, the sequence $\{f_n\}$ is said to converge to zero provided the supports of all the f_n lie in a fixed compact set and $f_n(x)$, together with their derivatives, tend uniformly to zero. With this, $\mathcal{K}$ becomes a locally convex linear topological space. (For a theory of these spaces, see [16].) If $M(\cdot): \mathcal{K} \to$ scalars, is a map, it is termed *local* by Gel'fand and Vilenkin (cf. [10], Ch. III, Sec. 4.1, Footnote 2), provided M satisfies: (a) $M(f_1+f_2) = M(f_1) + M(f_2)$ whenever $|f_1|\cdot|f_2| = 0$, for any f_1, f_2 in $\mathcal{K}$. (b) M is continuous, i. e. $f_n \in \mathcal{K}$, $f_n \to f$ in $\mathcal{K}$ implies $M(f_n) \to M(f)$. (Since bounded sets are relatively compact in $\mathcal{K}$, the continuity of M implies its boundedness on bounded sets of $\mathcal{K}$.) (c) For each $g \in \mathcal{K}$, if $M_g(f) = M(f+g) - M(g)$, then $M_g(\cdot)$ verifies (a). It should be noted that in [10], conditions (b) and (c) were not postulated in the definition of local functionals. In this generality the problem is more involved. The latter conditions are satisfied in most applications, and, in any case, they will always be assumed in this paper.

Local functionals are important in the theory of generalized random fields with independent values. The latter concept is defined thus. A linear map $F: \mathcal{K} \to \tilde{C}$, the space of scalar random variables on a fixed probability space $(\Omega, \Sigma, \mathbf{P})$, is said to be a *generalized random field* (*g.r.f.*) if F is continuous in the sense that $f_n \in \mathcal{K}$, $f_n \to 0$ in $\mathcal{K}$ implies $F(f_n) \to 0$ in probability, ([9], [10]). See also [7]. If $\tilde{C} \subset L^2(\Omega, \Sigma, \mathbf{P})$ and, in the above, $F(f_n) \to 0$ in L^2, then the concept was introduced in this form in [11]. (When comparable, these two concepts can be shown to coincide.) If a g.r.f. has the property that for any f_1, f_2 in $\mathcal{K}$, with $|f_1|\,|f_2| = 0$, $F(f_1)$ and $F(f_2)$ are mutually independent, then F is said *to have independent values* ([9], [10]). The work of [10] indicates the intimate relationship between the local functionals and the characteristic functionals (cf. f.) of g.r.f.'s on $\mathcal{K}$ with independent values. In that connection, special forms of $M(\cdot)$ were used by these authors who raised the problem of characterizing local functions.

* This research was supported under the NSF Grants GP—7679, GP-8777, and GP-15632.

The purpose of this paper is to present some characterizations of local functionals, on $\mathcal{K}$, and then to consider the ch.f.'s of g.r.f.'s (Section 3). Necessary and sufficient conditions for certain classes of functionals to be ch.f.'s of g.r.f.'s with independent values are given. For this class of ch. f.'s an explicit form is presented (Section 4) and it generalizes the classical Lévy — Khintchine representation formula for processes with independent increments. Some miscellaneous results, complementing the above characterizations, are included in the last section. The results of this paper extend and complement the fundamental work of Gel'fand [9] in many ways.

A different aspect of the g.r.f.'s with independent values (related to the stable laws) was considered by Urbanik [17], and another related study centering around the Lévy — Khintchine formulas was recently given in [10]. For other connected work, see [15]. A brief account of the main results of the present paper (with somewhat stronger hypotheses) was included in [14] without detailed proofs. [Condition (c) above was omitted by oversight in the definition of local functionals there.]

2. Local functionals. In this section some characterizations of local functionals on $\mathcal{K}$ will be considered. The classes studied are as follows.

Definition. 2.1. If $M(\cdot) : \mathcal{K} \to$ scalars, is a local functional, then it is said to be of *finite order* ($\leqslant m < \infty$), if $M(\cdot)$ is continuous in the topology of $\mathcal{K}^m(R^n)$, the space of m times continuously differentiable real functions with compact support.

For the theory of the spaces $\mathcal{K}$ and $\mathcal{K}^m(R^n)$, see [16].

Definition 2.2. A set $N \subset R^n$ is termed the *support* of a local functional $M(\cdot)$ on $\mathcal{K}$ if (a) N is closed, (b) for any open $V \subset R^n$ with $V \cap N \neq \emptyset$, there is a g in $\mathcal{K}$ with support in $V \cap N$ such that $M(g) \neq 0$, and (c) for any $f \in \mathcal{K}$ with support in $R^n - N$, $M(f) = 0$.

First a characterization of local functionals of finite order will be considered and later (in Section 5) a discussion of those functionals with compact support will be presented. It is seen that, if a local functional is uniformly continuous on $\mathcal{K}$ (instead of simple continuity as above), and of compact support, the theory of such functionals reduces to that of finite order. At the suggestion of the referee, the proofs of this section are outlined and the results on $C_\infty(G)$ (cf. [14]) are omitted. They will appear elsewhere.

Theorem 2.3. *Let $M(\cdot) : \mathcal{K} \to$ scalars, be a local functional of finite order, say m. Then it can be represented as:*

$$M(f) = \int_{R^n} \Phi(f(t), Df(t), \ldots, D^\alpha f(t), t)\, d\mu(t), \quad f \in \mathcal{K}, \tag{2.1}$$

where $D = \dfrac{\partial^\alpha}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_n}}$, $\alpha_i \geqslant 0, |\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant m$, *$\mu$ is a Borel measure in R^n such that $\mu(A) < \infty$ for compact $A \subset R^n$, and where the function $\Phi : R^\nu \times R^n \to$ scalars, has the following properties.* (i) *$\Phi(\cdot, \ldots, t)$ is continuous for each t,* (ii) *$\Phi(0, \ldots, 0, t) = 0$ for each t,* (iii) *$\Phi(x, \ldots, z, \cdot)$ is a Borel function on R^n for each real $x, \ldots, z$, and* (iv) *for each com-*

pact $A \subset R^n$, *a* $\delta > 0$ *and* $f \in \mathcal{K}$ *with* $\mathrm{supp}(f) \subset A$, $|D^\beta f(x)| < \delta$, $0 \leqslant |\beta| \leqslant m$ *one has* $|\Phi(f(t), Df(t), \ldots, D^\alpha f(t), t)| \leqslant C(A, \delta) < \infty$, *almost all* (t), *where* $C(A, \delta)$ *is a constant, depending on* A *and* δ. *Here* ν *is the number of partial derivatives of order* $\leqslant m$. *Conversely a function* Φ *on* $R^\nu \times R^n$ *and a Borel measure* μ *in* R^n *satisfying the above conditions define a local functional of order* m *on* $\mathcal{K}$ *through the formula* (2.1).

The proof of this result is a simple consequence of the next two lemmas.

Lemma 2.4. *Let* $\mathcal{K}(N)$ *be the subspace of* $\mathcal{K}$ *all of whose functions vanish outside of a compact rectangle* $N \subset R^n$. *If* $M : \mathcal{K}(N) \to$ *scalars, is a local functional of order* m $(< \infty)$, *then there exists a Borel measure* μ_0 *with* $\mu_0(N) < \infty$, *and a function* Φ_N *on* $R^\nu \times N \to$ *scalars, with properties* (i) — (iv) *of the theorem such that* (2.1) *holds with* Φ_N *for* Φ *there.*

Proof. Since $M(\cdot)$ is of order m, for each $f \in \mathcal{K}(N)$ it depends only on $D^\alpha f$, $0 \leqslant |\alpha| \leqslant m$. So consider the vector

$$\Psi = \{\psi_\alpha : \psi_\alpha(x) = D^\alpha f(x), x \in N, 0 \leqslant |\alpha| \leqslant m\}, \tag{2.2}$$

of length ν where ν is the number of partial derivatives of order not exceeding m. For each f, there is a Ψ and its correspondence $f \leftrightarrow \Psi$ is one-to-one. Let Δ^ν be the space of all vectors Ψ of ν components given by (2.2) for $f \in \mathcal{K}(N)$.

Let S be the space of all continuous functions on N and $S^\nu = S \times \ldots \times S$. If $g = (g_1, \ldots, g_\nu) \in S^\nu$ where $g_i \in S$, let

$$\|g\| = \max\{\sup[|g_i(x)| : x \in N] : 1 \leqslant i \leqslant \nu\}. \tag{2.3}$$

Then $\|\cdot\|$ is a norm under which S^ν is a Banach space and $\Delta^\nu \subset S^\nu$ is a linear subspace. Next define a functional $M_1(\cdot)$ on Δ^ν by

$$M_1(\Psi) = M(f), f \in \mathcal{K}(N), \tag{2.4}$$

where f and Ψ are determined by (2.2) uniquely. Consequently $M_1(\cdot)$ on Δ^ν is well-defined and it follows that M_1 is also a local functional on Δ^ν. By continuity, M_1 can be extended to the closure of Δ^ν under the metric (2.3) (or one not stronger than (2.3)) so as to remain a local functional. Since a metric space is normal, the functional M_1 can be extended to all S^ν, say M_2, by the generalized Tietze extension theorem. What is more, by a slight modification of the construction of this extension given in [1] and [4], M_2 can and will be taken to be a local functional. The result will follow if M_2 is characterized and then by restricting M_2 to Δ^ν to give M_1 and hence M.

The Cartesian product space S^ν can be identified with the space $C(N; R^\nu)$, the continuous function space on N with values in R^ν (cf., e.g., [5], p. 89), with the same norm (2.3). If $\nu = 1$, a local functional on such a space was characterized in [2], under the name «additive functional», whose hypothesis is satisfied by that of this paper. Since $\nu < \infty$, an extension of [2] to $C(N; R^\nu)$ presents no new difficulties. Hence, $M_2(\cdot)$ on $C(N, R^\nu)$ can be represented as follows: There exists a finite regular Borel measure μ_0 on the Borel sets of N and a function $\tilde{\Phi}_N$ on $R^\nu \times N$ to scalars, such that (i) $\tilde{\Phi}_N(\cdot, \ldots, \cdot, t)$ is continuous on R^ν for μ_0 almost all t, (ii) $\tilde{\Phi}_N(x, \ldots, z, \cdot)$ is Borel measurable, (iii) $\widetilde{\Phi}_N(0, \ldots, 0, t) = 0$ for μ_0 almost all t, and (iv) for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|\widetilde{\Phi}_N(x_1, \ldots, x_\nu, t)| < \varepsilon$, a.a. (t), if $\max\limits_{1 \leqslant i \leqslant \nu} |x_i| < \delta$, in terms of which the following equation holds,

$$M_2(g) = \int\limits_N \widetilde{\Phi}_N(g_1(t), \ldots, g_\nu(t), t)\, d\mu_0(t), \tag{2.5}$$

for all $g = (g_1, \ldots, g_\nu) \in S^\nu$. Since S^ν is separable and N is also separable, one can find a fixed null set $A_0 \subset N$ such that if $t \notin A_0$ then $\widetilde{\Phi}(\cdot, \ldots, \cdot, t)$ is continuous. Hence defining $\widetilde{\Phi}_N$ on A_0 by giving an arbitrary value, $\widetilde{\Phi}_N(\cdot, \ldots, \cdot, t)$ may be chosen continuous in (2.5), for each $t \in N$. With this in (2.6) let Φ_N be the function $\widetilde{\Phi}_N$ when M_2 is restricted to Δ^ν, so that $M_2 | \Delta^\nu = M_1$. This and (2.4) yield (2.1) at once. This completes the proof.

Lemma 2.5. *If M on $\mathcal{K}$ is a local functional of finite order, say m, then there exists a regular Borel measure μ in R^n such that $\mu(A) < \infty$ for each compact $A \subset R^n$ and a function $\Phi : R^\nu \times R^n \to$ scalars, satisfying the conditions* (i) — (iv) *of the theorem, in terms of which the formula* (2.1) *holds.*

P r o o f. Since R^n is countable at infinity, it can be written as $R^n = \bigcup\limits_{n=1}^{\infty} N_n$ where each N_i is compact and is contained in the interior of N_{i+1}. Let $\mathcal{K}(N_i)$ be the corresponding space (of Schwartz) on N_i. Then $\mathcal{K}(N_i) \subset \mathcal{K}(N_{i+1})$ and $\mathcal{K} = \bigcup\limits_{i=1}^{\infty} \mathcal{K}(N_i)$, is the strict inductive limit of $\mathcal{K}(N_i)$. But for each i, since $M_i = M | \mathcal{K}(N_i)$, the restriction of M to $\mathcal{K}(N_i)$, is local on $\mathcal{K}(N_i)$, it follows, by the preceding lemma, that

$$M_i(f) = \int\limits_{N_i} \Phi_{N_i}(f(t), Df(t), \ldots, D^\alpha f(t), t)\, d\mu_i(t), f \in \mathcal{K}(N_i). \tag{2.6}$$

The general case is now obtained by piecing together these (Φ_{N_i}, μ_i), where μ_i are the corresponding regular Borel measures on N_i.

Let $\widetilde{N}_1 = N_1$, and if $\widetilde{N}_i$ is defined, let $\widetilde{N}_{i+1} = N_{i+1} - N_i$. It is clear that $\widetilde{N}_i$ are disjoint Borel sets, $\widetilde{N}_i \subset N_i$, all i, and $R_n = \bigcup\limits_{i=1}^{\infty} \widetilde{N}_i$. Here all the μ_i may be assumed, for convience, positive since otherwise μ_i may be replaced by their variation measures $v(\mu_i)$ (i. e. $\mu_i \ll v(\mu_i)$, so by the Radon — Nikodým theorem $d\mu_i = \frac{d\mu_i}{dv(\mu_i)} \cdot dv(\mu_i) = h_i dv(\mu_i)$ and the measurable function h_i can be absorbed into Φ_{N_i} to start with). Next define the measure μ on the Borel sets of R^n starting with the equation:

$$\mu(\cdot) = \sum_{i=1}^{\infty} \mu_i(\widetilde{N}_i \cap \cdot). \tag{2.7}$$

This is well-defined and $\mu | \widetilde{N}_i = \mu_i$, so that $\mu(A) < \infty$ for each compact A since A is intersected by at most a finite number of $\widetilde{N}_i$'s, and $\mu_i(\widetilde{N}_i) <$

$< \infty$. Finally let

$$\Phi (x, y, \ldots, z, t) = \Phi_i (x, y, \ldots, z, t), \text{ for } t \in N_i. \tag{2.8}$$

Then Φ is well-defined, $\Phi | R^\nu \times N_i = \Phi_i$ and satisfies the conditions (i)— — (iv) since $t \in R^n$ if and only if $t \in \tilde{N}_i$ for a unique i. The relations (2.6) — (2.8) yield

$$M (f) = \int_{R^n} \Phi (f (t), Df (t), \ldots, D^\alpha f (t), t)\, d\mu (t), f \in \mathcal{K}. \tag{2.9}$$

The integral is well-defined since each f has compact support, and thus the integral reduces to one on a compact set, which exists by (2.6). This implies the lemma, and with it the theorem follows.

Definition 2.6. If $h \in R^n$ and τ_h is the translation operator on $\mathcal{K}$, so that $\tau_h f (x) = f (x + h)$ for all $f \in \mathcal{K}$, and $M(\cdot)$ is a local functional on $\mathcal{K}$ then it is said to be *translation invariant* if $M (\tau_h f) = M (f)$ for all $f \in \mathcal{K}$, and $h \in R^n$.

For translation invariant local functionals on $\mathcal{K}$, a more precise result holds.

Theorem 2.7. *Let* $M (\cdot) : \mathcal{K} \to$ *scalars, be a translation-invariant local functional of finite order, say* m. *Then there exists a continuous function* $\Phi : R^\nu \to$ *scalars, satisfying the conditions* (i), (ii) *and* (iv) *of Theorem* 2.3 *(whithout mention of* $t \in R^n$ *there and* (iii) *being automatic, since* Φ *does not depend on* t *and is continuous now), such that*

$$M (f) = \int_{R^n} \Phi (f (t), Df (t), \ldots, D^\alpha f (t))\, d\mu (t), \quad f \in \mathcal{K}, \tag{2.10}$$

where $| \alpha | \leq m$ *and* μ *is the Lebesgue measure in* R^n. *Conversely, a function* Φ *of the above description, and the Lebesgue measure* μ, *together define a translation invariant local functional* $M (\cdot)$ *on* $\mathcal{K}$ *through formula* (2.10).

Proof. It $M (\cdot) : \mathcal{K} \to$ scalars, is the given local functional, then let $M_2 (\cdot) : S^\nu \to$ scalars, be the extended functional as in the proof of Lemma 2.4. But if $M(\cdot)$ is translation invariant, then, as noted before, $M_2 (\cdot)$ *can be chosen to be* also translation invariant. Then a measure μ_h is induced by a translation invariant additive set function as:

$$\mu_h (A) = \lim_{\lambda \to 0} M_2 (P_{h, \lambda}) = \lim_{\lambda \to 0} M_2 (\tau_a P_{h, \lambda}) = \lim_{\lambda \to 0} M_2 (\hat{P}_{h, \lambda}) = \mu_h (A + a)$$

where $\hat{P} = \tau_a P_{h,\lambda}$ is the (vector) peak function of [2], which is translated by a units along its bases, and $A + a = \{x + a : x \in A\}$. The work of [2] shows that the limit exists and μ_h is a measure. It follows that μ_h and hence μ of (2.3) are both translation invariant. Since μ is a Randon measure on R^n (i. e. every compact set has finite μ-measure) it follows from known results that μ is a Lebesgue measure on R^n (cf., e.g. [12]). It thus remains to show that Φ of (2.1) does not depend on t.

Since $M_2 (\tau_h g) = M_2 (g)$, $g \in C_\infty (R^n, R^\nu)$, the space of continuous functions on R^n with compact supports and with values in R^ν, and μ is translation invariant, this equation can be expressed, by a change in variables in (2.9), as

$$\int_{R^n} \Phi(g_1(t), \ldots, g_\nu(t), t-h)\, d\mu(t) = \int_{R^n} \Phi(g_1(t), \ldots, g_\nu(t), t)\, d\mu(t) \qquad (2.11)$$

for all $g \in C_\infty(R^n, R^\nu)$. Since Φ was originally defined in such a way that $\Phi(\alpha\chi_A, t)$ was defined for all bounded Borel sets $A \in R^n$, the integrands in (2.11) can be identified for almost all t. Thus identifying and setting $h = t$ one gets

$$\Phi(x, y, \ldots, z, 0) = \Phi(x, y, \ldots, z, t),\ t \in R^n. \qquad (2.12)$$

Then letting $\Phi(x, y, \ldots, z) = \Phi(x, y, \ldots, z, 0)$, it follows from (2.12) that Φ can be expressed as a function on R^ν only, and that it has the stated properties of the theorem. Since the converse implication is clear, the result follows. (Note that the mention of Borel sets is unnecessary to deduce (2.12) from (2.11).)

3. Characteristic functionals. If $F(\cdot): \mathcal{K} \to \tilde{R}$ is a generalized random field, where $\tilde{R}$ is the set of all real random variables, on $(\Omega, \Sigma, \mathbf{P})$, then F is said to be a g.r.f. with *independent values* whenever $f, g \in \mathcal{K}$ and $|f| \cdot |g| = 0$ implies $F(f)$ and $F(g)$ are mutually independent random variables. This concept, due to Gel'fand [9], is a generalization of that of the ordinary (= nongeneralized) processes with independent increments.

Let $L: \mathcal{K} \to$ scalars, be a map defined by $L(f) = \mathbf{E}(e^{iF(f)})$ where $\mathbf{E}$ is the mathematical expectation on $(\Omega, \Sigma, \mathbf{P})$, so that $L(\cdot)$ is the characteristic functional (cf. f.) of the g.r.f. F. According to ([10], Ch. II, Sec. 4) a continuous functional $L(\cdot)$ on $\mathcal{K}$ is a ch.f. of a g.r.f. with independent values if and only if, it is positive definite, and $L(f_1 + f_2) = L(f_1)L(f_2)$ whenever $f_1 \cdot f_2 = 0$ for any f_1, f_2 in $\mathcal{K}$. Moreover a g.r.f. F is said to be (*strictly*) *stationary* whenever $L(\tau_h f) = L(f)$, $f \in \mathcal{K}$, $h \in R^n$. In this section some structure theorems for the ch. f.'s $L(\cdot)$ on $\mathcal{K}$ will be given extending and complementing the basic work of Gel'fand ([9], [10]).

First a special class of ch.f.'s $L(\cdot)$ on $\mathcal{K}$ will be considered. These relate the ch.f's of certain g.r.f.'s with independent values and the local functionals considered in the preceding section. Using this form, certain general ch.f.'s and their structure will be analyzed.

Suppose $L(\cdot)$ on $\mathcal{K}$ is a continuous functional which never vanishes so that $\log L(\cdot)$ can be defined. If moreover $L(f_1 + f_2) = L(f_1)L(f_2)$ for $f_1 f_2 = 0$, then $M(\cdot) = \log L(\cdot)$ defines a local functional on $\mathcal{K}$, and $L(\cdot) = \exp M(\cdot)$. A sufficient condition on a ch.f. $L(\cdot)$ to admit of the above representation is this.

Proposition 3.1. *Let $L(\cdot): \mathcal{K} \to$ scalars, be a ch.f. of some g.r.f. F with independent values. Suppose* (i) *$L(\chi_A)$ is also defined (by extension) for all indicator functions χ_A of bounded Borel sets, $A \subset R^n$, and* (ii) *$L(\chi_{A_n}) \to 1$ as $\{A_n\}$ decrease to the void set. Then there exists a local functional M on $\mathcal{K}$ such that $L(f) = \exp M(f)$, all $f \in \mathcal{K}$.*

Proof. Let $\varphi_A(s) = L(s\chi_A)$ for any real s and bounded Borel set $A \subset R^n$. Then $\varphi_A(\cdot)$ is the characteristic function of a real random variable for each A. If A_1, A_2 are disjoint bounded Borel sets, then, from the multiplicative property of $L(\cdot)$ and condition (ii) of the hypothesis, it follows that

$$\varphi_{A_1 \cup A_2}(s) = \varphi_{A_1}(s)\,\varphi_{A_2}(s), \text{ and } \lim_n \varphi_{A_n}(s) = 1, \tag{3.1}$$

uniformly for s in any finite interval.

Now if A is any compact rectangle in R^n, it can be written as $A = \bigcup_{i=1}^{m} A_i^{(m)}$, A_i^m are disjoint (half-open) rectangles in A for each m. Then from the first of (3.1) one has

$$\varphi_A(s) = \prod_{i=1}^{m} \varphi_{A_i^{(m)}}(s). \tag{3.2}$$

Now, given $\varepsilon > 0$, one can choose $m(\varepsilon)$ large enough and the diameters of A_i^m uniformly small, such that $|\varphi_{A_i^m}(s) - 1| < \varepsilon$ uniformly for any finite s-interval, while A is fixed. Then by ([3], p. 132, Theorem 4.1), $\varphi_A(\cdot)$ satisfying (3.2) is infinitely divisible and hence never vanishes. Since $A \subset R^n$ is an arbitrary (bounded) rectangle, and hence the result also holds for Borel sets, it follows that $L(f) \neq 0$ for any piecewise continuous f of bounded support. Consequently, the functional $M(\cdot)$ given by $M(f) = \log L(f)$, $f \in \mathcal{K}$, is well-defined and the continuity and multiplicative properties of $L(\cdot)$ imply the continuity and additivity of $M(\cdot)$. For condition (c), let $M_g(f) = \log L_g(f)$, with $L_g(f) = L(f+g)/L(g)$. If $f_1, f_2 \in \mathcal{K}$, $f_1 \cdot f_2 = 0$, a simple computation shows $L(f_1 + f_2 + g)/L(g) = L(f_1 + g) \cdot L(f_2 + g)/L^2(g)$, and hence $M_g(f_1 + f_2) = M_g(f_1) + M_g(f_2)$. So $M(\cdot)$ is a local functional on $\mathcal{K}$, completing the proof.

Corollary 3.2. *If $L(\cdot)$ is a ch.f. on $\mathcal{K}$, suppose either $L(\cdot)$ satisfies the conditions of the above proposition or, more generally, it is of the form $L(\cdot) = \exp M(\cdot)$ where $M(\cdot)$ is a local functional on $\mathcal{K}$. Then for each $f \in \mathcal{K}$, ψ_f, given by $\psi_f(s) = L(sf)$, defines a characteristic function of an infinitely divisible random variable.*

The conditions of the Proposition 3.1 on $L(\cdot)$ are not necessary. Consequently, the general class of functions given by

$$L(f) = \exp M(f),\ f \in \mathcal{K}, \tag{3.3}$$

where $M(\cdot)$ is a local functional on $\mathcal{K}$ of order m ($< \infty$) will be considered. Though the above argument does not seem to extend, it is likely that the ch.f. $L(\cdot)$ of a g.r.f. with independent values does not vanish and that (3.3) is the most general form of such $L(\cdot)$.

As in the preceding section, a functional $L(\cdot)$ given by (3.3) is said to be of *finite order*, say m, on $\mathcal{K}$ if the corresponding local functional is of order m. Since by [10], a functional $L(\cdot)$ on $\mathcal{K}$ is a ch.f. of some g.r.f. if and only if $L(\cdot)$ is continuous and positive definite with $L(0) = 1$, it will be of utmost importance, for further study, to find conditions on $M(\cdot)$ such that the $L(\cdot)$ of (3.3) is a ch.f. The following result goes in that direction, and extends a result in [9].

Theorem 3.3. *Suppose a functional* $L(\cdot)$ *on* $\mathscr{K}$ *is given by* (3.3) *where* $M(\cdot)$ *is an arbitrary local functional on* $\mathscr{K}$, *of order* m ($<\infty$). *Then* $L(\cdot)$ *is a ch.f. (necessarily of some g.r.f. with independent values) if and only if*

$$\exp\left(\int_A \Phi(x_1, \ldots, x_\nu, t)\, d\mu(t)\right), \quad x_i \in R, \tag{3.4}$$

is positive definite as a function of $x_1, \ldots, x_\nu$ *for all compact* $A \subset R^n$ *where* (Φ, μ) *is the pair determined by* M *through* (2.1), *and* ν *is the number of partial derivatives of order not exceeding* m.

P r o o f. Since $M(\cdot)$ is of order m, by Theorem 2.3, $L(\cdot)$ of (3.3) can be expressed as

$$L(f) = \exp\left(\int_{R^n} \Phi(f(t), Df(t), \ldots, D^\alpha f(t), t)\, d\mu(t)\right), \quad f \in \mathscr{K}, \tag{3.5}$$

where (Φ, μ) is a pair associated with $M(\cdot)$ through (2.1). As in Lemma 2.4, let g be the vector of ν components given by

$$g = \{g_\alpha : g_\alpha(x) = D^\alpha f(x),\ f \in \mathscr{K},\ |\alpha| \leqslant m\}. \tag{3.6}$$

Since the correspondence between g and f is one-to-one, define $L_1(\cdot)$ on Δ^ν by

$$\exp M_1(g) = L_1(g) = L(f) = \exp M(f), \tag{3.7}$$

where M_1 is the corresponding local functional to $M(\cdot)$ given by (2.4).

Suppose now that $L(\cdot)$ given by (3.5) is a ch.f., i. e., is positive definite and continuous on $\mathscr{K}$. Hence this implies, by (3.3) and (3.7), for any fixed but arbitrary set of complex numbers $\xi_1, \ldots, \xi_k$, the inequality ('bar' denotes complex conjugation below),

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \exp M_1(g_i - g_j)\, \xi_i \bar{\xi}_j \geqslant 0. \tag{3.8}$$

Let $M_2(\cdot)$ be a local functional which is an extension of M_1 onto $C_\infty(R^n, R^\nu)$. The extension procedure of [4] is such (M_2 is a certain positive linear combination of $M(\cdot)$) that (3.8) remains true when M_1 is replaced by M_2, and Δ^ν by $C_\infty(R^n, R^\nu)$. Also from the form of this functional (exponential form) it can clearly be extended to the space $\mathscr{Y} \supset C_\infty(R^n, R^\nu)$ of piece-wise continuous functions (as in [10], Ch. III, Sec. 4).

Now consider the vector $g_i = (g_{i1}, \ldots, g_{i\nu})$, $i = 1, \ldots, k$, where $g_{ij}(t) = x_{ij}$ for $t \in A$, A compact, $1 \leqslant j \leqslant \nu$, and $= 0$ otherwise. Then (3.8) takes the form, after rearrangement,

$$\sum_{i=1}^{k}\sum_{i'=1}^{k} \xi_i \bar{\xi}_{i'} \exp\left(\int_A \Phi(x_{i1} - x_{i'1}, \ldots, x_{i\nu} - x_{i'\nu}, t)\, d\mu(t)\right) \geqslant 0. \tag{3.9}$$

Since $\{\xi_i\}$ are arbitrary this implies that (3.4) is positive definite for each compact $A \subset R^n$. The properties of (Φ, μ) imply that (3.9) is always finite for all compact $A \subset R^n$. Thus the condition (3.4) is necessary.

For the converse, suppose (3.4) is positive definite. The proof that $L(\cdot)$ is positive definite and hence, because of (3.3), is a ch.f. follows closely that of [10], and is outlined. Again let $L_2(\cdot) = \exp M_2(\cdot)$ be the extended functional onto $C_\infty(R^n, R^\nu)$. Let $g_1, \ldots, g_k$ be a fixed but arbitrary set in $C_\infty(R^n, R^\nu)$. Their supports may be taken to lie in the compact rectangle $(b \geqslant 0)$ $N = \underset{i=1}{\overset{n}{\times}} [-b, b]^i \subset R^n$. Consider the expression, for $1 \leqslant i, j \leqslant k$,

$$a_{ij} = \exp\left(\int_N \Phi(g_{i1}(t) - g_{j1}(t), \ldots, g_{i\nu}(t) - g_{j\nu}(t), t)\, d\mu(t)\right). \tag{3.10}$$

Since $\Phi(\cdot, t)$ is continuous and the integral above exists on N ($\mu(N) < \infty$!), this integral may be approximated by step functions $g_{ip}^r \to g_{ip}$ pointwise and boundedly, as $r \to \infty$, where $g_{ip}^r = \sum_{q=1}^{t_r} a_{ipq} \chi_{A_q^r}$, with A_q^r as disjoint Baire sets in N for each r, $a_{ipq} \in R$, $1 \leqslant p \leqslant \nu$, so that, for $1 \leqslant i, j \leqslant k$,

$$a_{ij} = \lim_{r\to\infty} \prod_{q=1}^{t_r} \exp\left(\int_{A_q^r} \Phi(a_{i1q}^r - a_{j1q}^r, \ldots, a_{i\nu q}^r - a_{j\nu q}^r, t)\, d\mu(t)\right) =$$

$$= \lim_{r\to\infty} \prod_{q=1} a_{ij}^{qr}, \text{ say.} \tag{3.11}$$

Thus to prove that L_1 is positive definite, it suffices to show that the matrix (a_{ij}) is positive definite. For this, by a classical theorem of Schur (cf. [10]), it is enough to show that, for each q and r, the matrix (a_{ij}^{qr}) is positive definite. But by hypothesis this is true if in (3.11) each A_q^r is a compact set. Since (3.4) is positive definite for each compact set, it implies the result for all bounded Borel sets and thus (a_{ij}^{qr}) is positive definite. So $L_1(\cdot)$ on $C_\infty(R^n, R^\nu)$ and hence $L(\cdot)$ on $\mathcal{K}$ satisfy the same property. Since $L(0) = 1$ by (3.3), and is continuous, $L(\cdot)$ is a ch.f., and hence necessarily of a g.r.f. with independent values by [10]. This completes the proof of the theorem.

Corollary 3.4. *If $L(\cdot)$ on $\mathcal{K}$, given by* (3.3), *is of order m ($< \infty$) and is translation invariant,* (*i. e.* $L(\tau_h f) = L(f)$, $f \in \mathcal{K}$, $h \in R^n$) *then $L(\cdot)$ is a ch.f.* (*necessarily of a g.r.f. with strictly stationary and independent values*) *if and only if*

$$\exp(s\Phi(x_1, \ldots, x_\nu)) \tag{3.12}$$

is positive definite for all real $x_1, \ldots, x_\nu$ and any $s > 0$, where Φ on R^ν is the function given in Theorem 2.7, *and ν is the total number of partial derivatives of orders not exceeding m.*

This follows from the theorem since the translation invariance of L is equivalent to that of $M(\cdot)$ and then by Theorem 2.7, Φ does not depend on the last variable and μ is the Lebesgue measure, so that $s = \mu(A) > 0$, and (3.4) is equivalent to (3.12). If $\nu = 1$, the above corollary implies a result obtained in ([9], [10]), and the necessity answers affirmatively a question in [10].

The above theorem raises the problem of characterizing those local functionals $M(\cdot)$ for which (3.4) is positive definite. Only then it can be used in the theory of g.r.f.'s with advantage. This is now given in the next result.

Theorem 3.5. *Let* (Φ, μ) *be a pair satisfying the conditions of Theorem 2.1. Then in order that the function*

$$\exp\left(\int_A \Phi(\cdot, \dots, \cdot, t)\, d\mu(t)\right) \tag{3.13}$$

be positive definite in R^ν *for each compact* $A \subset R^\nu$, *it is sufficient that for* μ*-almost all* $t \in R^n$ *and any fixed but arbitrary set of complex numbers* $\xi_1, \dots, \xi_k$ *with* $\xi_1 + \dots + \xi_k = 0$, $\Phi(-x, t) = \overline{\Phi(x, t)}$ *and*

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \xi_i \bar{\xi}_j \Phi(x_{i1} - x_{j1}, \dots, x_{i\nu} - x_{j\nu}, t) \geqslant 0 \tag{3.14}$$

hold for all $x_i = (x_{i1}, \dots, x_{i\nu}) \in R^\nu$, $i = 1, \dots, k$. *If moreover, the measure* μ *is non-atomic, then the condition is also necessary.*

R e m a r k. This result extends a corresponding one in ([10], Ch. III, Sec. 4, Theorem 4) given when $\nu = 1$, μ the Lebesgue measure and Φ not involving t. However, the important sufficiency proof given there needs an additional argument. [The argument in [10] is insufficient for the case $\Phi(x, t) = ix$, cf. Math. Reviews, March 1968, 679), because the expression (11) in the proof there has no finite minimum!] The needed modifications together with the details of proof will be supplied here.

P r o o f. To prove the first part of the result, if $\xi_1, \dots, \xi_k$ are any given complex numbers and $A \subset R^n$ is compact, it should be shown that for all $x_i \in R^\nu$, $i = 1, \dots, k$, and with (3.14), one has

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \xi_i \bar{\xi}_j \exp \tilde{f}(x_i - x_j, A) > 0, \tag{3.15}$$

where $\tilde{f}(x, A) = \int_A \Phi(x, t)\, d\mu(t)$. By Schur's theorem, used in the proof of Theorem 3.3, it is enough to establish the positive definiteness of the matrix (a_{ij}^m) where $a_{ij}^m = 1 + \frac{s}{m}\tilde{f}(x_i - x_j, A)$ for any positive s and large enough m (because $a_{ij}^m \to \exp(s\tilde{f}(x_i - x_j, A))$ as $m \to \infty$ for any $s > 0$). Thus it should be shown that ($\tilde{f}(-x, A) = \overline{\tilde{f}(x, A)}$ being true by hypothesis),

$$\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}^m \xi_i \bar{\xi}_j = \left|\sum_{i=1}^{n} \xi_i\right|^2 + s_0 \sum_{i=1}^{k}\sum_{j=1}^{k} \xi_i \bar{\xi}_i \tilde{f}(x_i - x_j, A) \geqslant 0 \tag{3.16}$$

holds true. Now if $\sum_{i=1}^{k} \xi_i = 0$ and (3.14) is true simultaneously then (3.16) is clearly true. So consider the case that $\sum_{i=1}^{k} \xi_i \neq 0$. Dividing through by the non-zero number $\sum_{i=1}^{k} \xi_i$, if necessary, it may and will be assumed, for convenience, that $\sum_{i=1}^{k} \xi_i = 1$ in establishing (3.16) in this case.

Let $0 < \beta < \infty$ be a fixed but arbitrary number, and let $U_\beta = \{x \in R^\nu : |x| < \beta\}$. Then consider the continuous function (in x_i's) for any fixed compact $A \subset R^n$:

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \tilde{f}(x_i - x_j, A), \quad \sum_{i=1}^{k} \xi_i = 1, \; x_i, x_j \in U_\beta. \tag{3.17}$$

The continuity of $\tilde{f}(\cdot, A)$ on R^ν implies for any fixed $\xi_1, \ldots, \xi_k$ in (3.17), the function has a finite minimum as x_i's vary on the compact set $\bar{U}_\beta$, the closure of U_β. Let this be c_β and so $0 \leqslant |c_\beta| < \infty$. It follows that (3.16) has the following lower bound in this case: $\left(\text{since } \sum_{i=1}^{k} \xi_i = 1\right)$

$$\left|\sum_{i=1}^{k} \xi_i\right|^2 + s_0 \sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \tilde{f}(x_i - x_j, A) \geqslant 1 + s_0 c_\beta. \tag{3.18}$$

But $s_0 = s/m \downarrow 0$ as $m \to \infty$. So choosing $s_0 c_\beta \geqslant -1$ (i.e., if $c_\beta \geqslant 0$ any s_0 will do here and if $c_\beta < 0$, let $0 < s_0 \leqslant |c_\beta|^{-1}$), it follows that (3.18) is non-negative. This means the matrix (a_{ij}^m) is positive definite for all $x \in U_\beta$ for large enough m (i. e., so that $\frac{s}{m} = s_0 \leqslant |c_\beta|^{-1}$). Translated back, it is shown that for each compact $A \subset R^n$, $\exp(\tilde{f}(x; A))$ is positive definite (and continuous in R^ν) for x in a ball of finite radius. Consequently by a result of M. G. Krein (cf. [13], p. 209) the function $\exp(\tilde{f}(\cdot, A))$ coincides with a characteristic function on U_β. But this function being an exponential, is regular and it has just been shown to coincide with a characteristic function on U_β for any $\beta < \infty$. It then follows, as a consequence of a result of J. Marcinkiewicz (cf. [13], pp. 212—213) that $\exp(\tilde{f}(\cdot, A))$ determines uniquely a characteristic function on R^ν itself. Since this is clearly (3.13) itself (simply let $\beta \to \infty$ so that $U_\beta \to R^\nu$ in the above), it follows that (3.15) always holds for all $x \in R^\nu$. Since $A \subset R^n$ is arbitrary (3.15) implies that (3.13) is positive definite. This completes the sufficiency.

For the converse, let μ be non-atomic and (3.13) be positive definite. Using the notation of the preceding part, and noting that $e^\theta = 1 + \theta + \theta^2\delta$ for some $|\delta| < 1$ for small enough θ, one can expand $\exp(\tilde{f}(x, A))$, on noting that $\tilde{f}(\cdot, A)$ is a continuous function and $|\tilde{f}(x, A)| \to 0$ as $\mu(A) \to 0$

uniformly in $x \in \bar{U}_\beta$ for any $\beta < \infty$, to get the following:

$$0 \leqslant \sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \exp(\tilde{f}(x_i - x_j, A)) = \left(\left|\sum_{i=1}^{k} \xi_i\right|\right)^2 +$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \tilde{f}(x_i - x_j, A) + \frac{1}{2} \delta \sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \tilde{f}(x_i - x_j, A)^2 =$$

$$= 0 + \sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_i \tilde{f}(x_i - x_j, A) + \frac{1}{2} \delta \sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \tilde{f}(x_i - x_j, A)^2, \qquad (3.19)$$

since $\sum_{i=1}^{k} \xi_i = 0$. Given any $\varepsilon > 0$, and any fixed but arbitrary β, for all $x_i \in U_\beta$, there exists an η $(= \eta(\varepsilon, \beta))$ such that $\mu(A) < \eta$ implies $|\tilde{f}(x_i - x_j, A)| < \varepsilon$. Thus, if (3.14) were false, there exists a set A, and ε, β, η, for which the above inequalities are true and such that in (3.19)

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \xi_i \bar{\xi}_j \tilde{f}(x_i - x_j, A) + \varepsilon^2 < 0.$$

Since this contradicts the left inequality of (3.19), which is true by hypothesis, (3.14) must hold as stated. The positive definiteness of (3.13) implies immediately that $\tilde{f}(-x, A) = \overline{\tilde{f}(x, A)}$ and the arbitrariness of A then implies $\Phi(-x, t) = \overline{\Phi(x, t)}$ for almost all $t \in R^n$. This proves the necessity and with it the theorem. (Clearly the nonato micity of μ is not fully used here.)

The condition (3.14) can be stated equivalently in another form using integrals for sums there. Thus for any compact $A \subset R^n$, $\exp(\tilde{f}(\cdot, A))$ is positive definite if and only if

$$\int_{R^\nu} \int_{R^\nu} \tilde{f}(x - y, A) h(x) \overline{h(y)} dx dy \geqslant 0 \qquad (3.20)$$

for any $h \in \mathscr{K}(R^\nu)$ such that $\int_{R^\nu} h(x) dx = 0$. Since for any $h \in \mathscr{K}(R^\nu)$, $\int_{R^\nu} Dh(x) dx = 0$, where D is the differential operator of order one, the condition (3.20) can be stated more conveniently, but equivalently, as

$$\int_{R^\nu} \int_{R^\nu} \tilde{f}(x - y, A) Dh(x) \overline{Dh(y)} dx dy \geqslant 0, \quad h \in \mathscr{K}(R^\nu). \qquad (3.21)$$

In the terminology of [10], this condition means that for each compact $A \subset R^n$, $\tilde{f}(\cdot, A)$ determines a conditionally positive definite generalized function of order one on $\mathscr{K}(R^\nu)$. Because of its importance the theorem is formally stated as follows:

Theorem 3.5′. *If* $\Phi : R^\nu \times R^n \to$ *scalaras, has the properties,* (a) $\Phi(\cdot, t)$ *is continuous on* R^ν *and* $\Phi(0, t) = 0$, *for each* $t \in R^n$, *and* (*b*), $\Phi(x, \cdot)$ *is Borel measurable, and if* μ *is a Radon measure in* R^n, *then* $\exp\left(\int_A \Phi(\cdot, t) d\mu(t)\right)$

is a positive definite function for each compact $A \subset R^n$, if $\Phi(\cdot, t)$ determines in $\mathcal{K}(R^\nu)$ a conditionally positive definite generalized function, of order one, and $\Phi(-x, t) = \overline{\Phi(x, t)}$, for almost all $t \in R^n$. The converse holds whenever μ is also non-atomic.

R e m a r k. If in Theorems 3.5 $L(\cdot)$ is also translation invariant then the function Φ does not involve the last variable and the measure μ is the Lebesgue measure. This case reduces to that of [9], since Corollary 3.4 is applicable now.

4. **Generalized Lévy — Khintchine representation formula.** The results of the preceding sections enable a generalization of the Lévy — Khintchine representation formula for the ch.f.'s of the g.r.f.'s on $\mathcal{K}$ with independent values. This work extends the results of Gel'fand's (cf. [9], [10]) to the ch.f. of the form (3.3) and of finite order. Recall that a continuous functional $L(\cdot)$ on $\mathcal{K}(R^n)$ is of order m ($< \infty$) if it is continuous in the topology of $\mathcal{K}^m(R^n)$, the space of m-times continuously differentiable real functions in R^n with compact supports (cf. [16]). Another extension of the formula, in a different direction, was recently considered in [8].

The main result is here contained in

Theorem 4.1. *Let $L(\cdot) : \mathcal{K} \to$ scalars, be a functional of order m ($< \infty$), and suppose $L(f)$ is given by (3.3) for all $f \in \mathcal{K}$. Then $L(\cdot)$ is a characteristic functional of a g.r.f. on $\mathcal{K}$ with independent values if and only if it can be represented as:*

$$L(f) = \exp\left\{\int_{R^n} \Phi(f(t), Df(t), \ldots, D^\alpha f(t), t)\, d\mu(t)\right\}, \quad f \in \mathcal{K}, \tag{4.1}$$

where $|\alpha| \leqslant m$, μ is a Radon measure in R^n (also non-atomic for the 'only if' part), and Φ on $R^\nu \times R^n$ is given by

$$\Phi(x, t) = \int_{|y|>0} [e^{i(x, y)} - \alpha(y)(1 + i(x, y))]\, \sigma(dy; t) +$$

$$+ a_0(t) + \sum_{|k|=1}^{2} a_k(t) \frac{(ix)^k}{k!}. \tag{4.2}$$

Here $\sigma(\cdot; t)$ is a positive tempered measure in R^ν for each $t \in R^n$, $\sigma(A, \cdot)$ is a Borel function in R^n for each Borel $A \subset R^\nu$, (x, y) is the scalar product in R^ν, $\alpha(\cdot)$ is an entire analytic function of exponential type (i. e., it belongs to the space Z, cf. [10]) *such that $\alpha(y) - 1$ has a zero of order three at $y = 0$, and the $a(\cdot)$'s are some Borel functions in R^n determined by Φ (i. e., by $L(\cdot)$) all of which satisfy the following two conditions, for almost all $t \in R^n$:*

$$\int_{0<|y|<1} |y|^2 \sigma(dy; t) + \int_{|y|>1} \sigma(dy; t) < \infty, \tag{4.3}$$

$$\int_{|y|>0} (1 - \alpha(y))\, \sigma(dy; t) + a_0(t) = 0, \qquad \sum_{|r|=|s|=1} a_{r+s}(t)\, \xi_r \bar{\xi}_s \geqslant 0$$

where the ξ's are complex numbers. (The standard notation of [16] is used here; namely, $|k| = k_1 + \ldots + k_\nu$, $|y|^2 = (y, y)$, $k! = k_1! \ldots k_\nu!$ and $x^k =$

$= x_1^{k} \dots x_\nu^{k_\nu}$.) *Finally, if $L(\cdot)$ is also translation invariant then Φ of (4.1) does not involve the last variable t, and then the σ and a_i's do not involve t.*

P r o o f. Since $L(\cdot)$ never vanishes, set $M(\cdot) = \log L(\cdot)$. It is clear that $M : \mathscr{K} \to$ scalars, is well-defined and is of order m, (see equation (3.3) above). To prove the sufficiency, let $L(\cdot)$ be the ch. f. of a g.r.f. on $\mathscr{K}$ with independent values. Then $L(\cdot)$ is multiplicative and by (3.3) $M(\cdot)$ defined above is a local functional of order m. Hence by Theorem 2.3, there exists a pair (Φ, μ) such that Φ satisfies the conditions (i) — (iv) there and μ is a Radon measure, and one has

$$L(f) = \exp\Big(\int_{R^n} \Phi(f(t), Df(t), \dots, D^\alpha f(t), t)\, d\mu(t)\Big), \quad f \in \mathscr{K}. \tag{4.4}$$

If as before $\tilde{f}(x, A) = \int_A \Phi(x, t)\, d\mu(t)$, utilizing the fact that $L(\cdot)$ is a ch.f., then, by Theorem 3.3, $\exp(\tilde{f}(\cdot, A))$ is a positive definite function in R^ν for each compact $A \subset R^n$. Consequently by Theorem 3.5′, this implies that $\tilde{f}(\cdot, A)$ determines a generalized function $(\tilde{f}(\cdot, A), \cdot)$ on $\mathscr{K}$ which is conditionally positive definite of order one, treating A as a parameter.

As a consequence of an important theorem of N. Ya. Vilenkin and A. M. Yaglom (cf. [10], Ch. II, Sec. 4.4, and [18], Sec. 3) the functional $(\tilde{f}(\cdot, A), \cdot)$, of order 1, on the space Z can be expressed as follows:

$$\int_{R^\nu} \tilde{f}(x, A) f(x)\, dx = (\tilde{f}(\cdot, A), f) = \int_{R_0^\nu} \Big[\hat{f}(t) - \alpha(t) \times$$

$$\times \sum_{|\beta|=0}^{1} \frac{D^\beta \hat{f}(0)}{\beta!} t^\beta\Big] \nu_0(dt, A) + \sum_{|\beta|=0}^{2} \tilde{a}_\beta(A) \frac{D^\beta \hat{f}(0)}{\beta!}, \quad f \in \mathscr{K},$$

where $\alpha(\cdot)$ is an entire analytic function of exponential type (i. e. $\alpha \in Z$) such that $\alpha(t) - 1$ has a zero of order 3 at $t = 0$, $\hat{f}$ is the Fourier transform of f, and $\nu_0(\cdot, A)$ is a positive tempered measure, for each compact $A \subset R^n$, on $R_0^\nu = R^\nu - \{0\}$. Moreover, $\tilde{a}_k(A) = \int_{R^\nu} \tilde{f}(x, A)\alpha(x) x^k dx$, $0 \leqslant |k| \leqslant 1$, and for $|k| = 2$, $\tilde{a}_k(A) = \nu_k(0, A)$ where the measures $\nu_k(\cdot, A)$ are uniquely defined by $(\tilde{f}(\cdot, A), f)$ through the equation $(\tilde{f}(\cdot, A), z^k f) = \int_{R^\nu} f(x)\, d\nu_k(x, A)$, and $\tilde{a}_k$'s are positive definite for $|k| = 2$. In other words, $\nu_0(\cdot, A)$ and $a_k(A)$'s satisfy (4.3) if t is replaced by A there and ν_0 for σ. From this, the actual expressions given in the theorem can be obtained as follows.

Substituting the expression for $\tilde{f}(\cdot, A)$ here and rearranging (by Fubini's theorem) one gets

$$\tilde{a}_k(A) = \int_A \int_{R^\nu} \Phi(x, t)\alpha(x) x^k dx d\mu(t) = \int_A a_k(t)\, d\mu(t), \text{ say,}$$

so that $a_k(\cdot)$, defined by the integral on R^ν, is a Borel function in R^n. Since, by the representation, $\nu_k(A, \cdot)$ is also countably additive and μ-continuous it can be expressed as

$$\tilde{a}_k(A) = \int_A \tilde{\nu}_k(0; t)\, d\mu(t).$$

Letting $a_k(t) = \tilde{\nu}_k(0, t)$, $|k| = 2$, it follows that $a_k(\cdot)$, $0 \leqslant |k| \leqslant 2$ satisfy the conditions of the theorem. Since $\nu_0(C, A) = \int_A \tilde{\nu}_0(C, t)\, d\mu(t)$, for Borel $C \subset R^\nu$, compact $A \subset R^n$, a similar computation shows that with $\tilde{\nu}_0(\cdot, t), = \sigma(\cdot, t)$, the right side of (4.5) yeilds the right side of (4.2). Sin conditions (4.3) hold for ν_0, it follows, as above, that (4.5) holds for $\sigma(\cdot, t)$ for μ-almost all $t \in R^n$. This completes the proof that every ch.f. of the theorem admits a representation (4.1) — (4.3).

To prove the necessity, let $\Phi : R^\nu \times R^n \to$ scalars, be given by (4.2), with (4.3) being true relative to a Randon measure μ in R^n. Then the converce part of the Vilenkin — Yaglom theorem yields the result that, for almost all $t \in R^n$, the function $\Phi(\cdot, t)$ (and hence $\tilde{f}(\cdot, A)$), determines a conditionally positive definite generalized function on $\mathcal{K}$. But then, when μ is also non-atomic, the converse part of Theorem 3.5 is applicable and shows that $\exp(\tilde{f}(\cdot, A))$ is a positive definite function in R^ν for each compact $A \subset R^n$. Then Theorem 3.3 yields the result that $L(\cdot)$ of (4.1) is a ch.f., of order m, of a g.r.f. on $\mathcal{K}$ with independent values, as asserted.

If $L(\cdot)$ is also translation invariant, then Φ does not involve t and the result then is immediate from Theorem 3.5, and the a_i's, and $\sigma(\cdot)$ do not involve t and conversely. Thus the theorem is completely proved.

That this result extends the classical Lévy—Khintchine representation formula (cf., [3], p. 418, eq. (7.3)) will be illustrated now. For simplicity only the translation invariant case (with $\nu = n = 1$) will be considered. Let $\alpha(\lambda) \equiv 1$, and $\sigma(\cdot)$ be a finite measure determined by a monotone non-decreasing bounded function such that $\int_{|x|>0} |x|\, \sigma(dx) < \infty$. Let $a_0 = 0$, $a_1 = -\int_{|x|>0} \frac{x^3}{1+x^2}\, d\sigma(x) + \gamma$ and $a_2 = -a^2$. Then a simple computation shows that (4.1) reduces to the following when μ concentrates at a point t, so that $L(t) = \exp \Phi(t)$ where

$$\Phi(t) = it\gamma - \frac{a^2 t^2}{2} + \int_{|x|>0} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) d\sigma(x),$$

which is one form of the Lévy — Khintchine formula (cf. [3]).

5. Miscellaneous results and remarks. In this section a few special results related to the work of Sections 2 and 3 will be briefly treated. If the hypothesis on local functionals is strengthened then the hypothesis of finiteness of the order can be dropped. This will hopefully clarify (rather than generalize) the structure of local functionals. A functional $M : \mathcal{K} \to$ scalars, is *strongly local* if conditions (a) and (c) of a local functional definition hold, and (b) is strengthened to: (b′) $M(\cdot)$ is uniformly continuous on $\mathcal{K}$ (i. e., for any

$\varepsilon > 0$, there is a neighborhood V of the origin in $\mathcal{K}$, such that for any f, g in V, $|M(f) - M(g)| < \varepsilon$).

The following result illustrates the nature of this hypothesis.

Proposition 5.1. *If M on $\mathcal{K}$ is a strongly local functional with compact support, then M is of finite order.*

Proof. Recall that the neighborhood basis for $\mathcal{K}$ is given by the sets $V(\{m\}, \{\varepsilon\}, \{\Omega\})$ where $\{\varepsilon\} = \{\varepsilon_1, \varepsilon_2, \ldots; \varepsilon_j \downarrow 0\}, \{m\} = \{m_1, m_2, \ldots; m_j \uparrow \uparrow \infty\}$, and $\{\Omega\} = \{\Omega_1, \Omega_2, \ldots, : \Omega_j = [x : |x| < j]\}$, in terms of which

$$V(\{m\}, \{\varepsilon\}, \{\Omega\}) = \{f : |D^\alpha f(x)| < \varepsilon_j, |\alpha| \leqslant m_j, x \notin \Omega_j, j = 1, 2, \ldots\}, \quad (5.1)$$

(cf. [16]). Now by hypothesis $M(\cdot)$ has compact support $N \subset R^n$. Then $M : \mathcal{K}(N) \to$ scalars, and is strongly local. Consequently, given $\varepsilon > 0$, there is a neighborhood $V(m, \delta, N) = V$, of zero in $\mathcal{K}(N)$ such that for all $f \in V$ one has $|M(f)| < \varepsilon$. This means for any $\lambda > 0$, one has $|(D^\alpha f)| < \lambda\delta$, $|\alpha| \leqslant m$, $x \notin N$, for some m (taking $\Omega_m = N$). Then $\frac{f}{\lambda} \in V$ so that $\left|M\left(\frac{f}{\lambda}\right)\right| < \varepsilon$, by the uniform continuity of $M(\cdot)$. Since λ is arbitrary, this implies $|M(f)| \to 0$ as $|D^\alpha f(x)| \to 0$ uniformly for $|\alpha| \leqslant m$. Hence $M(\cdot)$ is of order $m (< \infty)$, completing the proof.∎

Remark. Without the uniformity condition, this result is false in general since then V depends on λ also in the above.

The above proposition implies the work of Sections 2 and 3 also holds for the strongly local functionals of compact support. The following companion result can be given for general strongly local functionals.

Proposition 5.2. *If $M : \mathcal{K} \to$ scalars, is a strongly local functional, then there exists a sequence $\{M_r(\cdot)\}$ of strongly local functionals, each $M_r(\cdot)$ having a compact support, satisfying $\lim_{r\to\infty} M_r(f) = M(f)$, $f \in \mathcal{K}$, in terms of which the following representation holds:*

$$M(f) = \lim_{r\to\infty} \int_{R^n} \Phi_r(f(t), Df(t), \ldots, D^{\alpha_r} f(t), t)\, d\mu_r(t), \quad f \in \mathcal{K}, \quad (5.2)$$

where $|\alpha_r| \leqslant m_r (< \infty)$, the (Φ_r, μ_r) being the pair associated with M_r (where $\Phi_r(\cdot, t)$ is also uniformly continuous now) as in Theorem 2.3, and the $\{m_r\}$ is a (not necessarily bounded) sequence of positive integers.

Proof. Let $\{A_k\}$ be an open countable covering of R^n which (by refinement if necessary) may be assumed locally finite and A_k relatively compact. Then there exists a C^∞-partition of unity $\{\varphi_i\}_{i\in I}$ subordinate to the covering $\{A_k\}$, (cf. [16]). This means (i) each $\varphi_i \geqslant 0$, $\varphi_i \in \mathcal{K}$, (ii) $\Sigma\varphi_i(x) = 1$, $x \in R^n$, (iii) support of $\varphi_i \subset A_i$ and (iv) any compact set $B \subset R^n$ intersects only finitely many supports of φ_i. Now if $f_m = \{\Sigma\varphi_i f, i \in I_m\}$ where $I_m \uparrow \subset I$ is a finite set. Then $f_m \in \mathcal{K}$ and $f_m \to f$ in $\mathcal{K}$. The first assertion is obvious and the second one follows from a small computation yielding:

$$D^\alpha f_m = \sum_{i=0}^{\alpha} \binom{\alpha}{i} D^i f \sum_{r\in I_m} (D^{\alpha-i}\varphi_r) \quad (5.3)$$

Here the differential symbolism and the product rule (cf. [16]) are used. The first factor vanishes outside of a compact set and only finitely many supports of φ_r intersect it. So as $m \to \infty$ $(I_m \to I)$ the second factor of (5.3) tends to zero, unless $\alpha = i$ (by interchanging the sum and derivative) in which case it tends to one. Thus $D^\alpha f_m \to D^\alpha f$ for each α, so that $f_m \to f$ in $\mathcal{K}$.

The continuity of $M(\cdot)$ implies $\lim_{m\to\infty} M(f_m) = M(f)$. So let $M_r(f) = M\left(\sum_{i\in I_r} \varphi_i f\right)$, $f \in \mathcal{K}$. Then $M_r(\cdot)$ is a strongly local functional and the support of M_r is contained in that of $\sum_{i\in I_r} \varphi_i$, which is compact for each r. This proves the first assertion. The second part is now a simple consequence of Theorem 2.3 and the first part. This completes the proof.

Using this proposition many of the results of Sections 2 and 3 can be reformulated for arbitrary strongly local functionals. However, this class excludes some interesting simple cases. Thus it will be of interest to extend the general theory of local functionals of finite order, of the earlier sections, to arbitrary order and it seems to require a separate study.

Carnegie-Mellon University and the Institute for Advanced Study (U.S.A.)

Поступила в редакцию 2.9.68

REFERENCES

[1] R. Arens, Extension of functions on fully normal spaces, Pacific J. Math., 2 (1953), 11—22.

[2] R. V. Chacon, N. Friedman, Additive functionals, Arch. Rat. Mech. Anal., 18 (1965), 230—240.

[3] J. L. Doob, Stochastic Processes, Wiley, New York, 1953, (русск. перевод: Дж. Л. Дуб, Вероятностные процессы, М., ИЛ, 1956).

[4] J. Dugundji, An extension of Tietze's theorem, Pacific J. Math. **1** (1951), 353—367.

[5] N. Dunford, J. T. Schwartz, Linear Operators, Part I: General Theory, Interscience, New York, 1958, (русск. перевод: Н. Данфорд, Дж. Т. Шварц, Линейные операторы, Общая теория, М., ИЛ, 1962).

[6] N. Friedman, M. Katz, A representation theorem for additive functionals, Arch. Rat. Mech. Anal., **21** (1966), 49—57.

[7] X. Fernique, Processus linéarie, processus généralisés, Ann. Inst. Fourier, Grenoble, **17** (1967), 1—92.

[8] X. Fernique, Lois indéfiniment divisibles sur l'espace distributions, Inventiones Math., **3** (1967), 282—292.

[9] И. М. Гельфанд, Обобщенные случайные процессы, ДАН СССР, **100** (1955), 853—856.

[10] И. М. Гельфанд, Н. Я. Виленкин, Обобщенные функции, вып. 4 Некоторые применения гармонического анализа. Оснащенные гильбертовы пространства, М., Физматгиз, 1961.

[11] K. Ito, Stationary random distributions, Mem. Coll. Sci. Univ. Kyoto, 28A (1954), 209—223.

[12] M. Jerison, W. Rudin, Translation-invariant functionals, Proc. Amer. Math. Soc., **13** (1962), 417—423.

[13] M. Lòeve, Probability Theory (3rd ed.), D. Van Nostrand, Princeton, 1963.

[14] M. M. Rao, Local functionals and generalized random fields, Bull. Amer. Math. Soc., 74, (1968), 288—293.
[15] M. M. Rao, Representation theory of multidimensional generalized random fields, Proc. Second International Symp. Multivariate Analysis, Academic Press, New York (1969), 411—436.
[16] L. Schwartz, Théorie des Distributions, Tome 1 (2nd ed.), Hermann, Paris, 1957.
[17] K. Urbanik, Generalized stochastic processes with independent values, Proc. Fourth Berkley Symp. Math. Statist. Probability, Vol. 2 (1961), 369—380.
[18] A. M. Яглом, Некоторые классы случайных полей в n-мерном пространстве, связанные со стационарными случайными процессами. Теория вероят. и ее примен., II, 3 (1957), 292—338.

ЛОКАЛЬНЫЕ ФУНКЦИОНАЛЫ И ОБОБЩЕННЫЕ СЛУЧАЙНЫЕ ПОЛЯ С НЕЗАВИСИМЫМИ ЗНАЧЕНИЯМИ

М. М. РАО (США)

(Резюме)

В работе обобщаются и уточняются результаты И. М. Гельфанда [9], касающиеся обобщенных случайных процессов. Доказаны соответствующие теоремы для функционалов конечного порядка и для функционалов, связанных с локально компактными хаусдорфовыми пространствами. Уточнен ряд результатов о характеристических функционалах.