N. M. Bogoliubov, Form factors, plane partitions and random walks, *Zap. Nauchn. Sem. POMI*, 2008, Volume 360, 5–30

Записки научных
семинаров ПОМИ
Том 360, 2008 г.

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FORM FACTORS, PLANE PARTITIONS AND RANDOM WALKS

Abstract. An exactly solvable boson model, the so-called "phase model," is considered. A relation between certain transition matrix elements of this model and boxed plane partitions, three-dimensional Young diagrams placed into a box of finite size, is established. It is shown that the natural model describing the behavior of friendly walkers, ones that can share the same lattice sites, is the "phase model." An expression for the number of all admissible nests of lattice paths made by a fixed number of friendly walkers for a certain number of steps is obtained.

1. Introduction

A fascinating connection between statistical physics and combinatorics has led to the publication of a number of papers in various areas including solvable lattice models and quantum chains with different boundary conditions [1–5].

One of the main problems of statistical physics is the investigation of transitions from a given initial state to a given final one. For integrable models this problem is effectively solved by the algebraic Bethe ansatz of the quantum inverse method [6–8]. A relation between certain transition matrix elements of the exactly solvable boson model and boxed plane partitions, three-dimensional Young diagrams placed into a box of finite size (see Fig. 1), was established in the papers [9, 10].

The enumeration of plane partitions is a classical chapter in combinatorics [11, 12] and in the theory of symmetric functions [13]. The statistics of plane partitions with respect to natural probabilistic measures was studied in [14, 15]. The correlation functions for random plane partitions were calculated in [16].

The phase model related to plane partitions is defined as the $q \to \infty$ limit of the integrable q-boson model introduced in [17, 18]. The correlation functions of the model were calculated in [20]. The interpretation of the state vectors of the model with arbitrary parameter q in terms of the

The work was partially supported by the RFBR grant 07-01-00358 .

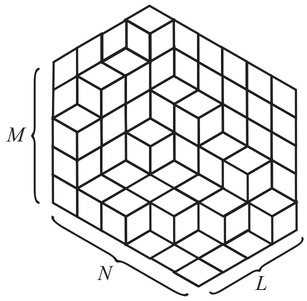

Fig. 1. A plane partition inside the $L \times N \times M$ box.

algebra of symmetric functions was done in [20], and for the phase model in [9] and [21].

The random walks problem first introduced by Fisher [22] attracts much attention in mathematical physics. Different connections to other research fields, such as Young diagrams and the theory of random matrices, have been revealed one after another [23–32]. Two essentially different types of walkers may be distinguished, namely vicious and friendly ones. Walkers are called vicious since arriving at the same lattice site they annihilate not only each other but all the rest as well. On the contrary, any number of friendly walkers may share lattice sites. In the paper [33], the so-called random turns model [22] for friendly walkers [27, 29] on a one-dimensional periodical lattice was considered. For the random turns model at each tick of the clock dt only a single randomly chosen walker moves one step to the left or one step to the right while the rest are staying. It was shown that the generating function for the number of all admissible nests of lattice paths that made N friendly walkers is expressed as the N-particle correlation function of the phase model over the vacuum state. Spatio-temporal trajectories of the walkers are two-dimensional directed paths that cannot turn back and mutually intersect. Typical nests of paths of vicious and friendly walkers are represented in Fig. 2.

In this paper, we shall give the detailed analysis of the main results obtained in the papers [9, 10, 33] and calculate the multi-particle correlation functions of the phase model in the sector with a fixed number of particles.

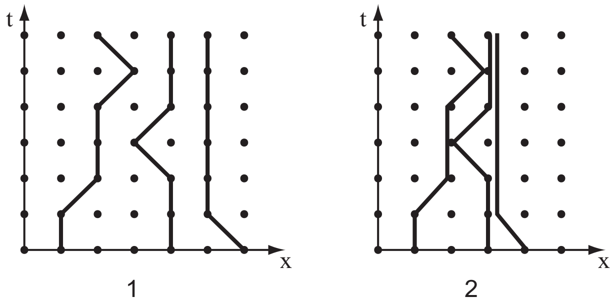

Fig. 2. Typical nests of paths of vicious (1) and friendly (2) walkers in the random turns model.

2. Phase model

Consider the system on a periodic lattice of circumference $M+1$ defined by the Hamiltonian

$$H = H_{\mathrm{hop}} + \widehat{N} = -\frac{1}{2}\sum_{n,m=0}^{M} \Delta_{nm}\phi_n^{\dagger}\phi_m + \sum_{n=0}^{M} N_n. \tag{1}$$

The entries of the hopping matrix Δ are equal to

$$\Delta_{nm} = \delta_{n+1,m} + \delta_{n-1,m}. \tag{2}$$

The periodic boundary conditions mean that $\Delta_{n+M+1,m} = \Delta_{n,m+M+1} = \Delta_{nm}$. The operators of the "exponential phase" [34] ϕ_n, $\phi_n^{\dagger}$ and the number operator N_n satisfy the commutation relations

$$[N_i, \phi_j] = -\phi_i\delta_{ij}, \quad [N_i, \phi_j^{\dagger}] = \phi_i^{\dagger}\delta_{ij}, \quad [\phi_i, \phi_j^{\dagger}] = \pi_i\delta_{ij}, \tag{3}$$

where π_j is the local vacuum projector: $\phi_j\pi_j = \pi_j\,\phi_j^{\dagger} = 0$. The total number operator $\widehat{N}$

$$\widehat{N} = \sum_{n=0}^{M} N_n \tag{4}$$

commutes with the hopping term of the Hamiltonian

$$[H_{\mathrm{hop}}, \widehat{N}] = 0. \tag{5}$$

A local Fock space $\mathcal{F}_j$ with the index j corresponds to a lattice site j. The relations

$$\begin{aligned} \phi_j|0\rangle_j &= 0, \\ \phi_j|n_j\rangle_j &= |n_j-1\rangle_j, \quad \phi_j^\dagger|n_j\rangle_j = |n_j+1\rangle_j, \\ N_j|n_j\rangle_j &= n_j|n_j\rangle_j, \\ \pi_j|0\rangle_j &= |0\rangle_j \end{aligned} \tag{6}$$

hold for the local orthonormal Fock states $|n_j\rangle_j$ $({}_k\langle n_m|n_i\rangle_j = \delta_{im}\delta_{kj})$.

The operators ϕ_j, $\phi_j^\dagger$, and N_j are represented in the following form:

$$\phi_j = \sum_{n=0}^{\infty} (|n\rangle\langle n+1|)_j\,, \quad \phi_j^\dagger = \sum_{n=0}^{\infty} (|n+1\rangle\langle n|)_j\,, \quad N_j = \sum_{n=0}^{\infty} (|n\rangle\langle n|)_j\,.$$

The introduced phase operators are expressed in terms of the canonical Bose operators $b_j^\dagger$, b_j, $\left[b_i, b_j^\dagger\right] = \delta_{ij}$, $N_j = b_j^\dagger b_j$:

$$\phi_j = (N_j+1)^{-1/2}\, b_j, \quad \phi_j^\dagger = b_j^\dagger\, (N_j+1)^{-1/2}\,.$$

The phase model is associated with the following L-operator [19]:

$$L_n(u) = \begin{pmatrix} u^{-1} & \phi_n^\dagger \\ \phi_n & u \end{pmatrix}, \tag{7}$$

where $u \in \mathcal{C}$ is a parameter.

This L-operator satisfies the intertwining relation

$$R(u,v)\,(L(n|u)\otimes L(n|v)) = (L(n|v)\otimes L(n|u))\,R(u,v) \tag{8}$$

in which $R(u,v)$ is the R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{9}$$

where

$$f(v,u) = \frac{u^2}{u^2-v^2}, \quad g(v,u) = \frac{uv}{u^2-v^2}. \tag{10}$$

The introduced R-matrix may be considered as the special limit of the standard XXZ R-matrix

$$R(u,v) = \lim_{\gamma\to\infty} e^{-2\gamma} \left(1\otimes e^{-\gamma\sigma^z}\right) R_{\mathrm{XXZ}}(u,v) \left(1\otimes e^{\gamma\sigma^z}\right),$$

where the XXZ R-matrix is of the form

$$R_{\mathrm{XXZ}}(u,v) = \begin{pmatrix} \widetilde{f}(v,u) & 0 & 0 & 0 \\ 0 & \widetilde{g}(v,u) & 1 & 0 \\ 0 & 1 & \widetilde{g}(v,u) & 0 \\ 0 & 0 & 0 & \widetilde{f}(v,u) \end{pmatrix},$$

$$\widetilde{f}(v,u) = \frac{u^2e^{2\gamma} - v^2e^{-2\gamma}}{u^2-v^2}, \quad \widetilde{g}(v,u) = \frac{uv}{u^2-v^2}\left(e^{2\gamma} - e^{-2\gamma}\right).$$

The monodromy matrix is introduced as

$$T(u) = L(M|u)\dots L(1|u)L(0|u) \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{11}$$

The commutation relations of the matrix elements of the monodromy matrix are given by the same R-matrix (9):

$$R(u,v)\left(T(u)\otimes T(v)\right) = \left(T(v)\otimes T(u)\right)R(u,v). \tag{12}$$

The most important relations are

$$\begin{aligned} C(u)B(v) &= g(u,v)\left\{A(u)D(v) - A(v)D(u)\right\}, \\ A(u)B(v) &= f(u,v)B(v)A(u) + g(v,u)B(u)A(v), \\ D(u)B(v) &= f(v,u)B(v)D(u) + g(u,v)B(u)D(v), \\ [B(u),B(v)] &= [C(u),C(v)] = 0. \end{aligned} \tag{13}$$

The transfer matrix $\tau(u)$ is the matrix trace of the monodromy matrix

$$\tau(u) = \operatorname{tr} T(u) = A(u) + D(u). \tag{14}$$

The relation (12) means that $[\tau(u),\tau(v)] = 0$ for arbitrary values of parameters u, v and the transfer matrix is considered to be the generating function of integrals of motion.

The Hamiltonian of the phase model (1) is expressed in terms of the transfer matrix in the following way:

$$-2H = \frac{\partial u^{M+1}\tau(u)}{\partial u^2}\Big|_{u=0} + \frac{\partial u^{-M-1}\tau(u)}{\partial u^{-2}}\Big|_{u=\infty} - 2\widehat{N}, \tag{15}$$

where $\widehat{N}$ is the total number operator (4).

Matrix elements of the monodromy matrix (11) act in the Fock space spanned on the local state vectors $|n_j\rangle_j$:

$$|n\rangle = \prod_{j=0}^{M} |n_j\rangle_j = \prod_{j=0}^{M} (\phi_j^\dagger)^{n_j} |0\rangle, \quad \sum n_j = n, \tag{16}$$

where

$$|\Omega\rangle = \prod_{j=0}^{M} |0\rangle_j \tag{17}$$

is the vacuum vector. The dual vacuum $\langle\Omega| = |\Omega\rangle^\dagger$, $\langle\Omega|\Omega\rangle = 1$.

The vacuum vector (17) is the generating state of the model. It is annihilated by the operator $C(u)$,

$$C(u)|\Omega\rangle = 0, \tag{18}$$

and is an eigenvector of the operators $A(u)$ and $D(u)$:

$$A(u)|\Omega\rangle = \alpha(u)|\Omega\rangle, \quad D(u)|\Omega\rangle = \delta(u)|\Omega\rangle. \tag{19}$$

For the phase model (1), $\alpha(u)$ and $\delta(u)$ are equal to

$$\alpha(u) = u^{-(M+1)}, \quad \delta(u) = u^{M+1}. \tag{20}$$

The state vectors are taken to be of the form

$$|\Psi_N(u_1, u_2, \dots, u_N)\rangle = \prod_{j=1}^{N} B(u_j)|\Omega\rangle, \tag{21}$$

and the state conjugated to (21) is

$$\langle\Psi_N(u_1, u_2, \dots, u_N)| = \langle\Omega| \prod_{j=1}^{N} C(u_j). \tag{22}$$

The representation of the state vectors (21) in the coordinate form is obtained by systematic application of formula (A.2.4) in [8] and are expressed in the form

$$|\Psi_N(u_1,u_2,\dots,u_N)\rangle = \prod_{k=1}^{N} B(u_k)|0\rangle$$
$$= \sum_{\substack{0\le n_0,n_1,\dots,n_M\le N\\ n_0+n_1+\dots+n_M=N}} f_{\{n\}}(u_1,u_2,\dots,u_N)\prod_{j=0}^{M}|n_j\rangle_j. \quad (23)$$

From the commutativity of the $B(u)$ operators (see (13)) it follows that the amplitudes $f_{\{n\}}(u_1,u_2,\dots,u_N)$ are symmetric functions and are expressed via symmetric Schur functions $S_{(\lambda_1,\dots,\lambda_N)}$:

$$f_{\{n\}}(u_1,u_2,\dots,u_N) = (u_1u_2\cdot\ldots\cdot u_N)^{-M}\, S_\lambda(u_1^2,u_2^2,\dots,u_N^2). \quad (24)$$

The Schur function that corresponds to the partition $\lambda=(\lambda_1,\dots,\lambda_N)$ is equal to

$$S_{(\lambda_1,\dots,\lambda_N)}(u_1,u_2,\dots,u_N) = \frac{\det\left(u_j^{N-i+\lambda_i}\right)}{\prod\limits_{1\le i<j\le N}(u_i-u_j)}. \quad (25)$$

There is a one-to-one correspondence between the configuration of occupation numbers $\{n\}$ and the partition $\lambda=(M^{n_M},(M-1)^{n_{M-1}},..,1^{n_1},0^{n_0})$: the number S appears in this partition n_S times.

The parts λ_k of the partition λ may be considered as the coordinates of the particles (the coordinate λ_k corresponds to the kth particle). The state of the system is spanned by the orthonormal basis $|\lambda\rangle = |\lambda_1,\lambda_2,\dots,\lambda_m\rangle$: $\langle\mu|\lambda\rangle = \delta_{\{\mu\},\{\lambda\}}$. The N-particle state vector (23) is given then by the equation

$$|\Psi_N(u_1,u_2,\dots,u_N)\rangle$$
$$= (u_1u_2\cdot\ldots\cdot u_N)^{-M}\sum_{\lambda\subseteq\{M^N\}} S_{\{\lambda\}}(u_1^2,u_2^2,\dots,u_N^2)|\lambda\rangle, \quad (26)$$

where the sum is taken over all partitions λ into at most N parts each of which is less than or equal to M.

According to the algebraic Bethe ansatz method, the state vector (21) is an eigenvector of the transfer matrix (14),

$$\tau(v)|\Psi_N(u_1,u_2,\dots,u_N)\rangle = \Theta_N(v;u_1,\dots,u_N)|\Psi_N(u_1,u_2,\dots,u_N)\rangle, \quad (27)$$

if the parameters $u_1,\dots,u_N$ satisfy the Bethe equations [17, 18]:

$$u_n^{-2(M+N+1)} = (-1)^{N-1}U, \quad U = \prod_{j=1}^{N} u_j^{-2}. \quad (28)$$

The eigenvalue of the transfer matrix (14) is equal to

$$\Theta_N(v;\{u\}) = v^{-(M+1)}\left\{(-1)^N U^{-1} + v^{2(N+M+1)}\right\}\prod_{j=1}^{N}\frac{1}{v-u_j^2}. \quad (29)$$

After parametrization $u^2 = \exp(-ip)$, with p playing the role of momenta, the Bethe equations (28) will take the form

$$e^{ip_n(M+N+1)} = (-1)^{N-1}e^{iP}, \quad P = \sum_{j=1}^{N} p_j, \quad (30)$$

where P is the total momenta of the system. The solution of these equations is

$$p_n = \frac{2\pi I_n + P}{M+N+1}, \quad (31)$$

where I_n are integers or half-integers (depending on N being odd or even) lying in the interval $M+N+1 > I_l \geq 0$. For the unique definition of the eigenstates (23) it is sufficient to restrict oneself to the set of N different numbers I_l satisfying the condition

$$M+N+1 > I_1 > I_2 > \cdots > I_N \geq 0. \quad (32)$$

The Bethe eigenvectors (27), $|\Psi_N(u_1,u_2,\dots,u_N)\rangle$, with u_j the solutions of (28), are the eigenstates of the Hamiltonian (1), $H|\Psi_N\rangle = E_N|\Psi_N\rangle$, with the eigenenergies that can be found from Eqs. (15) and (29):

$$E_N = -\sum_{k=1}^{N}\left(\frac{u_k^2+u_k^{-2}}{2} - 1\right) = 2\sum_{k=1}^{N}\sin^2\frac{p_k}{2}. \quad (33)$$

The hopping energy is equal, respectively, to

$$E_N^{\text{hop}} = -\sum_{k=1}^{N}\frac{u_k^2+u_k^{-2}}{2} = -\sum_{k=1}^{N}\cos p_k. \quad (34)$$

3. Scalar products and form factors

From the general theory of the scalar products of the state vectors of integrable systems [8] it follows that for the models associated with the R-matrix (9), and hence with the commutation relations (13), the scalar product of the state vectors (21) and (22) with arbitrary values of the parameters u and v

$$S(N,M|\{v\},\{u\}) = \langle\Omega|C(v_1)\dots C(v_N)B(u_1)\dots B(u_N)|\Omega\rangle \tag{35}$$

is expressed in the determinantal form

$$S(N,M|\{v\},\{u\}) = \Big\{\prod_{j>k} g(v_j,v_k)\prod_{l>m} g(u_m,u_l)\Big\}\det H, \tag{36}$$

where the matrix elements of the $N\times N$ matrix H are

$$\begin{aligned} H_{jk} &\equiv H(v_j,u_k) \qquad (37)\\ &= \Big\{\alpha(v_j)\delta(u_k)\Big(\frac{u_k}{v_j}\Big)^{N-1} - \alpha(u_k)\delta(v_j)\Big(\frac{u_k}{v_j}\Big)^{-N+1}\Big\}\times\frac{1}{\frac{u_k}{v_j}-\Big(\frac{u_k}{v_j}\Big)^{-1}}. \end{aligned}$$

These formulas are valid for arbitrary values of the parameters $\{u\}$ and $\{v\}$.

Substituting (20) into this formula we obtain the answer for the scalar product of the state vectors of the phase model [9]:

$$\begin{aligned} &\widetilde{S}(N,M|\{v\},\{u\}) = \langle 0|\widetilde{C}(v_N)\dots\widetilde{C}(v_1)\widetilde{B}(u_1)\dots\widetilde{B}(u_N)|0\rangle \qquad (38)\\ &= \Big\{\prod_{N\ge j>k\ge 1}\frac{v_j^2v_k^2}{v_k^2-v_j^2}\prod_{N\ge l>m\ge 1}\frac{u_l^{-2}u_m^{-2}}{u_m^{-2}-u_l^{-2}}\Big\}\det\left[\frac{1-\Big(\frac{u_k}{v_j}\Big)^{2(M+N)}}{1-\Big(\frac{u_k}{v_j}\Big)^2}\right]_{j,k=1}^N. \end{aligned}$$

For the considered phase model the answer for the scalar product (38) may be evaluated directly from the coordinate representation of the state vectors (26) and the Cauchy relation on the Schur functions:

$$\begin{aligned} &\sum_{\lambda\subseteq\{L^N\}} s_\lambda(x_1^2,\dots,x_N^2)s_\lambda(y_1^2,\dots,y_N^2)\\ &= \Big(\prod_{j=1}^N x_j^L y_j^L\Big)\Big\{\prod_{N\ge j>k\ge 1}\frac{y_j^2y_k^2}{v_j^2-v_k^2}\prod_{N\ge l>m\ge 1}\frac{x_l^2x_m^2}{u_l^2-u_m^2}\Big\}\det H_{jk}, \qquad (39) \end{aligned}$$

where the entries of the $N \times N$ matrix H_{jk} are equal to

$$H_{jk} = \frac{(x_k y_j)^{N+L} - (x_k y_j)^{-N-L}}{x_k y_j - (x_k y_j)^{-1}}, \qquad (40)$$

and the sum is over all partitions, λ, into at most N parts each of which is less than L.

Knowing the answer for the scalar product of the state vectors we may calculate different correlation functions. One of them is a form factor of the Kth power of the annihilation operator ϕ_0:

$$\widetilde{S}(K,N,M|\{v\},\{u\}) = \langle 0|\widetilde{C}(v_{N-K})\dots\widetilde{C}(v_1)\phi_0^K \widetilde{B}(u_1)\dots\widetilde{B}(u_N)|0\rangle. \qquad (41)$$

To calculate this form factor notice that the monodromy matrix may be represented, in particular, as

$$T(u) = \begin{pmatrix} A_1(u) & B_1(u) \\ C_1(u) & D_1(u) \end{pmatrix} \begin{pmatrix} u^{-1} & \phi_0^\dagger \\ \phi_0 & u \end{pmatrix},$$

$$\begin{pmatrix} A_1(u) & B_1(u) \\ C_1(u) & D_1(u) \end{pmatrix} \equiv L(M|u)\dots L(1|u).$$

From this representation it follows that

$$A(u) = A_1(u)u^{-1} + B_1(u)\phi_0, \quad B(u) = A_1(u)\phi_0^\dagger + B_1(u)u, \qquad (42)$$

$$C(u) = C_1(u)u^{-1} + D_1(u)\phi_0, \quad D(u) = C_1(u)\phi_0^\dagger + D_1(u)u, \qquad (43)$$

and the matrix elements of the monodromy matrix are related [20]:

$$\begin{aligned} \phi_0 B(u) = uA(u), \qquad & A(u)\phi_0^\dagger = u^{-1}B(u), \\ C(u)\phi_0^\dagger = u^{-1}D(u), \quad & \phi_0 D(u) = uC(u). \end{aligned} \qquad (44)$$

The form factor (41) may be rewritten in the form

$$\widetilde{S}(K,N,M|\{v\},\{u\}) = \langle 0|\phi_0^K \widetilde{C}(v_{N-K})\dots\widetilde{C}(v_1)\widetilde{B}(u_1)\dots\widetilde{B}(u_N)|0\rangle, \qquad (45)$$

since from (43) it follows that $[\phi_0, \widetilde{C}(v)] = 0$. The operator $\widetilde{C}(v)$ possesses the property $\lim_{v\to\infty} \widetilde{C}(v) = \phi_0$, and thus

$$\widetilde{S}(K,N,M|\{v\},\{u\}) = \lim_{v_N,\dots,v_{N-K+1}\to\infty} \widetilde{S}(N,M|\{v\},\{u\}).$$

Taking this limit in (38) we obtain

$$\widetilde{S}(K,N,M|\{v\},\{u\}) \tag{46}$$
$$=\left\{ \left(v_1^2 v_2^2 \dots v_{N-K}^2\right)^K \prod_{N-K\ge j>k\ge 1} \frac{v_j^2 v_k^2}{v_k^2 - v_j^2} \prod_{N\ge l>m\ge 1} \frac{u_l^{-2} u_m^{-2}}{u_m^{-2} - u_l^{-2}} \right\} \det Q,$$

where the matrix elements of the $N \times N$ matrix Q are equal to

$$Q_{jk} = \frac{1-\left(\frac{u_k}{v_j}\right)^{2(M+N)}}{1-\left(\frac{u_k}{v_j}\right)^2}, \qquad 1 \le j \le N-K;$$
$$Q_{jk} = u_k^{2(N-j)}, \qquad N-K+1 \le j \le N.$$

Using the equality $\phi_0 \widetilde{B}(u) = \widetilde{A}(u)$ (see (42)), where we put $\widetilde{A}(u) = u^{M+1}A(u)$, and the property $[\phi_0, \widetilde{C}(u)] = [\phi_0, \widetilde{A}(u)] = 0$ we may rewrite the form factor (45) in the form

$$\widetilde{S}(K,N,M|\{v\},\{u\})$$
$$= \langle 0|\widetilde{C}(v_{N-K})\dots\widetilde{C}(v_1)\widetilde{A}(u_N)\dots\widetilde{A}(u_{N-K+1})\widetilde{B}(u_1)\dots\widetilde{B}(u_{N-K})|0\rangle.$$

This equality expresses the form factor as the average of the $\widetilde{A}(u)$ operators.

Consider now the case when the parameters of the state vectors satisfy the Bethe equations (28).

The norms of the Bethe eigenvectors (27)

$$\mathcal{N}^2(u_1,u_2,\dots,u_N) = \langle \Psi_N(u_1,u_2,\dots,u_N)|\Psi_N(u_1,u_2,\dots,u_N)\rangle$$

are found from (36). Now we have $u_k = v_k$ satisfying the Bethe equations (28), and the matrix elements are equal to

$$H_{jj} = M+N,$$
$$H_{jk} = -\left(\frac{u_j}{u_k}\right)^2, \quad j \ne k.$$

It is not difficult to calculate the determinant

$$\det H = (M+1)(M+N+1)^{N-1},$$

and for the norm of any Bethe eigenvector we obtain

$$\mathcal{N}^2(u_1, u_{2,} \dots, u_N) = U^{-(N-1)} \prod_{j \neq k} \frac{1}{u_k^2 - u_j^2} (M+1)(M+N+1)^{N-1}. \quad (47)$$

Consider the form factor (41) on the solutions of the Bethe equations. Taking into account that the parameters u_k satisfy Eqs. (28) while the parameters v_k satisfy the Bethe equations

$$v_n^{-2(M+1+N-K)} = (-1)^{N-K-1} V, \quad V = \prod_{k=1}^{N-K} v_k^{-2}, \quad (48)$$

we will obtain the following representation for the form factor:

$$\langle \Psi_{N-K}(v_1, v_2, \dots, v_{N-K}) | \phi_0^K | \Psi_N(u_1, u_2, \dots, u_N) \rangle = V^{-(N-1)} \prod_{N-K \geq j > k \geq 1} \frac{1}{v_k^2 - v_j^2} \prod_{N \geq l > m \geq 1} \frac{1}{u_l^2 - u_m^2} \det Q, \quad (49)$$

where the matrix elements of the $N \times N$ matrix Q are equal to

$$Q_{jk} = \frac{1 - (-1)^K u_k^{-2} v_j^{-2(K-1)} U^{-1} V}{1 - u_k^2 v_j^{-2}}, \quad 1 \leq j \leq N-K;$$

$$Q_{jk} = u_k^{2(N-j)}, \quad N-K+1 \leq j \leq N.$$

The translational invariance of the model on the periodic chain allows us to write

$$\langle \Psi_{N-K}(v_1, v_2, \dots, v_{N-K}) | \phi_m^K | \Psi_N(u_1, u_2, \dots, u_N) \rangle = \left(\frac{U}{V} \right)^m \langle \Psi_{N-K}(v_1, v_2, \dots, v_{N-K}) | \phi_0^K | \Psi_N(u_1, u_2, \dots, u_N) \rangle.$$

The solutions of the Bethe equations (28) were studied in detail in [19], and it was demonstrated that they provide a complete and orthogonal basis of eigenvectors (21), (22). This means that the eigenvectors provide the resolution of the identity operator

$$I = \sum_{\{u\}} \frac{|\Psi_N(u_1, u_2, \dots, u_N)\rangle \langle \Psi_N(u_1, u_2, \dots, u_N)|}{\mathcal{N}^2(u_1, u_2, \dots, u_N)}, \quad (50)$$

where the summation is over all different solutions of the Bethe equations (28).

Knowing the answers for the form factors and taking into account the completeness and orthogonality of the Bethe vectors [19] one can calculate different correlation functions of the phase model. The temperature-dependent correlation function of the fields in the Kth order in the $N-K$ particle sector is defined as

$$< \phi_m^K e^{-tH} (\phi_0^\dagger)^K > \tag{51}$$

$$\equiv \frac{1}{Z} \sum_{\{v\}} \frac{\langle \Psi_{N-K}(v_1, v_2, \dots, v_{N-K}) | \phi_m^K e^{-tH_{ph}} (\phi_0^\dagger)^K | \Psi_{N-K}(v_1, v_2, \dots, v_{N-K}) \rangle}{\mathcal{N}^2(v_1, v_2, \dots, v_{N-K})},$$

where the summation is over all solutions of the Bethe equations (48), t is the inverse temperature $t = 1/T$ and

$$Z = \sum_{\{v\}} e^{-tE_{N-K}}$$

where E_{N-K} are the eigenenergies (33). Inserting the resolution of the identity operator into (51) and taking into account the translational invariance we obtain

$$< \phi_m^K e^{-tH} (\phi_0^\dagger)^K > \tag{52}$$

$$= \frac{1}{Z} \sum_{\{v\}} \sum_{\{u\}} \Big(\frac{U}{V}\Big)^m \frac{e^{-tE_{N-K}} \left| \langle \Psi_{N-K}(v_1, v_2, ..., v_{N-K}) | \phi_m^K | \Psi_N(u_1, u_2, ..., u_N) \rangle \right|^2}{\mathcal{N}^2(v_1, v_2, ..., v_{N-K}) \mathcal{N}^2(u_1, u_2, ..., u_N)}$$

$$= \frac{1}{Z(M+1)^2 (M+N+1)^{N-1} (M+N-K+1)^{N-K-1}} \sum_{\{v\}} \sum_{\{u\}} \Big(\frac{U}{V}\Big)^m e^{-tE_{N-K}} |\det Q|^2.$$

Here the summation is over all solutions of the Bethe equations (48) and (28), the matrix Q is defined in (49).

4. Boxed plane partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ of nonnegative integers λ_j called the parts of λ.

Plane partitions are a two-dimensional generalization of ordinary partitions. A plane partition π is an array π_{ij} of nonnegative integers satisfying $\pi_{ij} \geq \pi_{i+1j}$ and $\pi_{ij} \geq \pi_{ij+1}$ for all $i, j \geq 1$. Each column or raw of a plane partition is itself an ordinary partition. A symmetric plane partition is a plane partition for which $\pi_{ij} = \pi_{ji}$. The integers π_{ij} are called the parts of the plane partition, and $|\pi| = \sum \pi_{ij}$ is its volume.

Plane partitions are usually interpreted as solid stacks of unit cubes – three-dimensional Young diagrams. The height of a stack with coordinates ij is π_{ij}. The volume $|\pi|$ of the partition is equal to the total number of cubes of the diagram $|\pi| = \sum \pi_{ij}$. It is said that a plane partition is contained in the $L \times N \times M$ rectangular box if $\pi_{ij} \leq M$ for all i and j, and $\pi_{ij} = 0$ whenever $i > L$ or $j > N$; such a plane partition is called a boxed plane partition. Allowing for some of the partitions to be empty we may express a plane partition inside the $L \times N \times M$ box as a $N \times N$ matrix (without loss of generality we may put $N > L$) with the matrix elements of the last $K = N - L$ rows equal to zero: $\pi_{jm} \equiv 0$ for $j \geq L$.

The plane partition corresponding to the matrix

$$\pi = \begin{pmatrix} 5 & 3 & 3 & 2 & 2 & 1 \\ 4 & 2 & 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{53}$$

is expressed in Fig. 3.

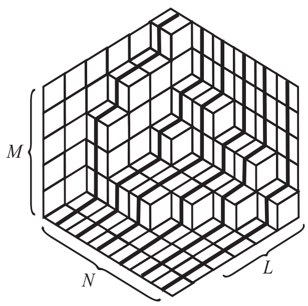

Fig. 3. A plane partition with gradient lines.

A plane partition inside the $L \times N \times M$ box may be represented as a collection of nests of lattice paths on a square lattice. Consider first a partition inside the $N \times N \times M$ box. The plane partition corresponds to a nest of N lattice paths on the $2N \times (1 + M)$ square lattice. The first N columns starting from the left have numbers $(-N, \dots, -1)$, and the last N ones, $(1, \dots, N)$. The mth path is running from the down left $(-N+m-1; 0)$ vertex to the top right $(m; M)$ one, and always moves east or north. The number of squares in the jth column of the lattice (the numbering starts from the right) under the mth path may be considered as the matrix element π_{jm} of the plane partition π. Only one path is allowed on vertical lattice edges (this rule is equivalent to the condition $\pi_{jm} \geq \pi_{jm+1}$) but any number of paths can share horizontal ones. The condition $\pi_{jm} \geq \pi_{j+1m}$ is equivalent to the rule that all paths are moving always east or north.

A plane partition inside the $L \times N \times M$ box $(N > L)$ may be represented as a nest of lattice paths described above on the same $2N \times (1+M)$ square lattice with the additional condition that the first lattice path $(m = 1)$ is going only along the 0-th row of the lattice up to the $(-L; 0)$ vertex.

The nest of the lattice paths in Fig. 4 corresponds to the plane partition in Fig. 3 and to the array (53), respectively.

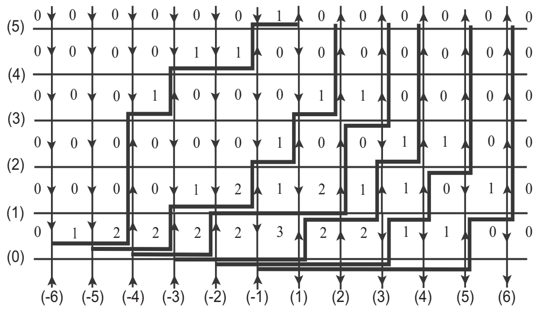

Fig. 4. A typical nest of admissible lattice paths.

According to a classical formula of MacMahon there are exactly

$$A(L, N, M) = \prod_{j=1}^{L} \prod_{k=1}^{N} \prod_{i=1}^{M} \frac{i+j+k-1}{i+j+k-2} = \prod_{j=1}^{L} \prod_{k=1}^{N} \frac{M+j+k-1}{j+k-1} \qquad (54)$$

plane partitions contained in the $L \times N \times M$ box. The generating function of plane partitions

$$Z_q(L, N, M) = \sum_{p.p} q^{|\pi|}, \qquad (55)$$

where the sum is over all partitions confined in the $L \times N \times M$ box, may be considered as the partition function of three-dimensional Young diagrams with the Boltzmann weight equal to the volume $|\pi|$ of a partition, and $q = e^{-\frac{1}{T}}$. The proof that

$$Z_q(L, N, M) = \prod_{j=1}^{L} \prod_{k=1}^{N} \prod_{i=1}^{M} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} = \prod_{j=1}^{L} \prod_{k=1}^{N} \frac{1 - q^{M+j+k-1}}{1 - q^{j+k-1}} \tag{56}$$

may be found in [12, 13]. It is clear that $A(L, N, M)$ is the $q \to 1$ limit of the generating function $Z_q(L, N, M)$.

The generating function of symmetric plane partitions ($\pi_{ij} = \pi_{ji}$) is equal to

$$Z_q^{sym}(N, N, M) = \sum_{sym} q^{|\pi|}$$
$$= \prod_{j=1}^{N} \frac{1 - q^{M+2j-1}}{1 - q^{2j-1}} \prod_{1 \le i < j \le N} \frac{1 - q^{2(M+i+j-1)}}{1 - q^{2(i+j-1)}}, \tag{57}$$

where the sum is over all symmetric partitions confined in the $N \times N \times M$ box. This enumeration formula was proposed by MacMahon and proved by Macdonald [13].

5. Form factors and enumeration of plane partitions

To make a connection of the form factor (45) with the problem of enumeration of boxed plane partitions we shall use a graphic representation of these objects.

One can visualize the matrix elements of the L-operator as a vertex with attached arrows (see Fig. 5). The matrix element $l_{12}(n|u) = \phi_n^\dagger$ corresponds to a vertex (b), $l_{21}(n|u) = \phi_n$ corresponds to a vertex (c), $l_{11}(n|u) = u^{-1}$ to a vertex (a), and $l_{22}(n|u) = u$ to a vertex (d), respectively.

The matrix elements of the monodromy matrix (11) are expressed then as sums over all possible configurations of arrows on the one-dimensional lattice with $M + 1$ sites with different boundary conditions (see Fig. 5). Namely, from the definition (11) of the operator $B(u)$,

$$B(u) = \sum_{k_M, \dots, k_1 = 1}^{2} l_{1k_M}(M|u) l_{k_M k_{M-1}}(M-1|u) \dots l_{k_1 2}(0|u),$$

it follows that this operator corresponds to the boundary conditions when the arrows on the top and bottom of the lattice are pointing outward (configuration (B)). Respectively, the operator $C(u)$ corresponds to the boundary conditions when the arrows on the top and bottom of the lattice are pointing inward (configuration (C)). The operators $A(u)$ and $D(u)$ correspond to the boundary conditions when the arrows on the top and bottom of the lattice are pointing in one direction, up and down (configurations (A) and (D)), respectively.

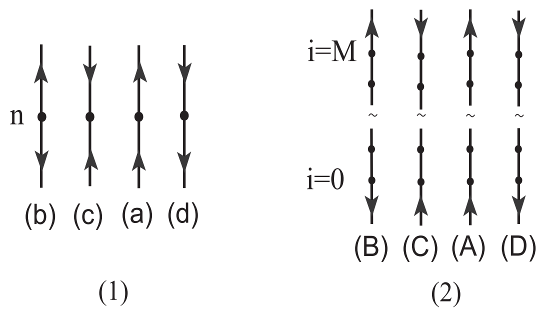

Fig. 5. Graphic representation of the matrix elements of the L-operator (1) and of the monodromy matrix (2).

Consider the two-dimensional square lattice with $2N \times (M+1)$ sites. The first N columns of the lattice with the numbers $(-N, \ldots, -1)$ are associated with the operators $C(v_j)$, and the last N ones with the numbers $(1, \ldots, N)$ are associated with the operators $B(u_j)$ (see Fig. 4). The rows of the lattice are associated with the local Fock spaces, the ith row with the ith space, respectively. Each horizontal edge of the lattice is labelled with the occupation number n_j of the correspondent Fock vector $|n_j\rangle_i$. The scalar product (35) is equal to the sum over all allowed configurations of the vertices (a), (b), (c), and (d) on the square lattice with the following boundary conditions: the external arrows on the first N columns are pointing inwards; on the last N ones they are pointing outwards; on the right and on the left boundaries all occupation numbers are equal to zero. Allowed configurations of the form factor (45) satisfy the additional condition that the internal arrows of the first left K columns are pointing downwards.

Allowed configurations may be represented in terms of lattice paths. These configurations are nests of noncrossing N lattice paths starting from the down left vertices $(-N; 0), (-N+1; 0), \ldots, (-1; 0)$ and ending at the top right ones $(1;M),(2;M),\ldots,(N;M)$. In the vertical direction paths follow the arrows pointing upwards, and only one path is allowed on a vertical lattice edge but any number of paths can share horizontal ones. The number of paths sharing a horizontal edge is equal to the corresponding occupation number of this edge. Allowed configurations of the form factor (45) satisfy the additional condition that the first lattice path is going only along the 0th row of the lattice up to the $(-N+K;0)$ vertex. One can see that the nests of lattice paths that correspond to plane partitions define allowed configurations of vertices in the scalar product (35) and in the form factor (45).

The form factor (45) may be represented in the form

$$\widetilde{S}(K,N,M|\{v\},\{u\}) = \sum_{p.p}\prod_{k=-1}^{-N} v_{-k}^{t_k^d - t_k^a - M}\prod_{j=1}^{N} u_j^{t_j^d - t_j^a + M}, \qquad (58)$$

where the sum is over the nests of lattice path that correspond to plane partitions in the $N\times N\times M$ box with the additional conditions discussed above, and t_j^a and t_j^d are the numbers of (a) and, respectively, (d) vertices in the jth column of the lattice. The volume $|\pi|$ of the plane partition π in the $N\times N\times M$ box is equal to ([9])

$$2|\pi| = \sum_{k=-1}^{-N} k\left(l_k^d - l_k^a - M\right) + \sum_{j=1}^{N}(j-1)\left(l_j^d - l_j^a + M\right). \qquad (59)$$

The substitution of the parametrization $v_j = q^{-\frac{j}{2}}$, $u_j = q^{\frac{j-1}{2}}$ into (58) gives for the form factor

$$\widetilde{S}(K,N,M|q) = \sum_{p.p} q^{|\pi|},$$

and thus it is the generating function (58) for plane partitions in the $L\times N\times M$ box, where $L=N-K$.

The determinant of the $N\times N$ matrix $\widehat{H}$ with the matrix elements

$$\widehat{H}_{jk} = \frac{1-s^{j+k-1}}{1-q^{j+k-1}}, \quad 1\le j\le N-K;$$
$$\widehat{H}_{jk} = q^{(k-1)(N-j)}, \quad N-K+1\le j\le N,$$

may be evaluated in the way similar to that of the paper [35], and it is equal to

$$\det \widehat{H} = (-1)^{\frac{N(N-1)+(N-K)(N-K-1)}{2}} q^{\frac{1}{6}[N(N-1)(N-2)+(N-K)(N-K+1)(N+2K-1)]}$$

$$\prod_{N-K\geq j>k\geq 1} \left(1-q^{j-k}\right) \prod_{N\geq l>m\geq 1} \left(1-q^{l-m}\right) \prod_{k=1}^{N} \prod_{j=1}^{N-K} \frac{1-sq^{j-k}}{1-q^{j+k-1}}. \quad (60)$$

For $K=0$ this determinant was evaluated by Kuperberg [35] in connection with the alternating sign matrices enumeration problem.

The substitution of the parametrization $v_j = q^{-\frac{j}{2}}$, $u_j = q^{\frac{j-1}{2}}$ and $s = q^{N+M}$ into (46) gives equality (56) with $L = N-K$ for the generating function (55).

From the representation (23), it follows that the projection of the state vector to the vector

$$|P\rangle = \sum_{\substack{0\leq n_0, n_1, \ldots, n_M \leq N \\ n_0+n_1+\ldots+n_M=N}} \prod_{j=0}^{M} |n_j\rangle_j \quad (61)$$

is

$$\langle P| \prod_{j=1}^{N} \widetilde{B}(u_j)|0\rangle = \sum_{\lambda\subseteq\{M^N\}} S_{\{\lambda\}}(u_1^2, u_2^2, \ldots, u_N^2), \quad (62)$$

where the sum is over all partitions λ into at most N parts each of which is less than or equal to M.

In [9], it was proved that

$$\langle P| \prod_{j=1}^{N} \widetilde{B}(q^{\frac{2j-1}{2}})|0\rangle = \sum_{sym} q^{|\pi|},$$

where the sum is taken over all symmetric plane partitions, and hence their generating function is equal to

$$Z_q^{sym}(N,N,M) = \sum_{\lambda\subseteq\{M^N\}} S_{\{\lambda\}}(q, q^3, \ldots, q^{2N-1}).$$

This sum of Schur functions may be expressed as a product (see [13, Ex. I.5.17]), and we obtain representation (57).

6. Random turns friendly walkers

The moves of a single walker on the lattice may be expressed by the matrix Δ (2). This choice of the matrix Δ means that the walker can step to the right or to the left from an arbitrary site m. After K steps all admissible lattice paths running from l are given by $\sum_{n_1,\dots,n_K} \Delta_{n_K n_{K-1}} \dots \Delta_{n_2 n_1} \Delta_{n_1 l}$. The number $|P_K(l \to j)|$ of all lattice paths of length K from l into j is equal to

$$|P_K(l \to j)| = \sum_{n_1,\dots,n_{K-1}} \Delta_{j n_{K-1}} \dots \Delta_{n_2 n_1} \Delta_{n_1 l}, \tag{63}$$

and if $|j-l| < K$, then $|P_K(l \to j)| = 0$.

The selective temporal evolution of the states obtained by the selective creation of the boson particle $\phi_l^\dagger |\Omega\rangle$, where $|\Omega\rangle$ is the vacuum vector (17), is defined by the one-particle correlation function

$$\langle \Omega | \phi_j e^{-tH} \phi_l^\dagger |\Omega\rangle = e^{-t} \langle \Omega | \phi_j e^{-tH_{\text{hop}}} \phi_l^\dagger |\Omega\rangle \equiv e^{-t} F(j,l|t), \tag{64}$$

where H and H_{hop} are the Hamiltonians defined by equality (1). The prefactor e^{-t} appeared because of the commutativity $[H_{\text{hop}}, \widehat{N}] = 0$ (5).

The function $F(j,l|t)$ may be expressed in terms of $|P_K(l \to j)|$ (63). Really, expanding $F(j,l|t)$ in powers of t one has

$$F(j,l|t) = \sum_K \frac{t^K}{2^K K!} \langle \Omega | \phi_j (2H_{\text{hop}})^K \phi_l^\dagger |\Omega\rangle. \tag{65}$$

Applying the commutation relation

$$[2H_{\text{hop}}, \phi_m^\dagger] = \sum_n \Delta_{mn} \phi_n^\dagger \pi_m,$$

one obtains

$$\begin{aligned} (2H_{\text{hop}})^K \phi_l^\dagger |\Omega\rangle &= (2H_{\text{hop}})^{K-1} [2H_{\text{hop}}, \phi_l^\dagger] |\Omega\rangle \\ &= (2H_{\text{hop}})^{K-1} \sum_{n_1} \Delta_{n_1 l} \phi_{n_1}^\dagger |\Omega\rangle \\ &= \sum_{n_1,\dots,n_K} \Delta_{n_K n_{K-1}} \dots \Delta_{n_2 n_1} \Delta_{n_1 l} \phi_{n_K}^\dagger |\Omega\rangle. \end{aligned} \tag{66}$$

This equality may be interpreted in the following way. The position of the walker on the lattice is labeled by the position of the boson particle on the lattice site with the occupation number of the correspondent Fock state equal to one, while the vacuum states correspond to empty sites of the lattice. The state $\phi_l^\dagger|\Omega\rangle$ describes a particle on the lth site of the lattice. Equality (66) enumerates all admissible trajectories of the walker starting from the site l. The operator ϕ_j annihilates the particle on the site j. The multiplication of equality (66) from the left by the state $\langle\Omega|\phi_j$ will fix the ending point of the trajectory because of the orthogonality of the local Fock states. Finally, one has

$$\begin{aligned}\langle\Omega|\phi_j(2H_{\rm hop})^K\phi_l^\dagger|\Omega\rangle &= |P_K(l\to j)|\\ &= \sum_{n_1,\dots,n_{K-1}} \Delta_{jn_{K-1}}\dots\Delta_{n_2n_1}\Delta_{n_1l},\end{aligned} \qquad (67)$$

where $|P_K(l\to j)|$ is the number of all lattice paths from l to j of length K (63), and, respectively, for the correlation function we have the following answer:

$$\langle\Omega|\phi_j e^{-tH}\phi_l^\dagger|\Omega\rangle = e^{-t}\sum_K \frac{t^K}{2^K K!}|P_K(l\to j)|. \qquad (68)$$

In the one-particle sector, the solutions of the Bethe equations (31) are equal to

$$p_k = \frac{2\pi k}{M+1}; \quad 0\le k\le M.$$

Inserting the resolution of the unity (50) into (65) we obtain

$$F(j,l|t) = \frac{1}{M+1}\sum_{k=0}^{M} e^{2t\cos\frac{2\pi}{M+1}k} e^{i\frac{2\pi}{M+1}k(j-l)}, \qquad (69)$$

and for the $|P_K(l\to j)|$ we have the well-known answer [27]:

$$|P_K(l\to j)| = \frac{1}{M+1}\sum_{k=0}^{M}\left(2\cos\frac{2\pi}{M+1}k\right)^K e^{i\frac{2\pi}{M+1}k(j-l)}.$$

Consider now the multi-particle correlation function

$$\begin{aligned}&\langle\Omega|\phi_{j_1}^{n_{j_1}}\phi_{j_2}^{n_{j_2}}\dots\phi_{j_s}^{n_{j_S}} e^{tH}(\phi_{l_1}^\dagger)^{n_{l_1}}(\phi_{l_2}^\dagger)^{n_{l_2}}\dots(\phi_{l_z}^\dagger)^{n_{l_z}}|\Omega\rangle\\ &= e^{-tN}\langle\Omega|\phi_{j_1}^{n_{j_1}}\phi_{j_2}^{n_{j_2}}\dots\phi_{j_s}^{n_{j_S}} e^{tH_{\rm hop}}(\phi_{l_1}^\dagger)^{n_{l_1}}(\phi_{l_2}^\dagger)^{n_{l_2}}\dots(\phi_{l_z}^\dagger)^{n_{l_z}}|\Omega\rangle\\ &= e^{-tN}F\left(n_{j_1},n_{j_2},\dots,n_{j_s};n_{l_1},n_{l_2},\dots n_{l_z}|t\right),\\ &n_{j_1}+n_{j_2}+\dots+n_{j_s} = n_{l_1}+n_{l_2}+\dots+n_{l_z} = N.\end{aligned} \qquad (70)$$

One easily finds that

$$[2H_{\rm hop},(\phi^{\dagger}_m)^l]|\Omega\rangle=\sum_{j=0}^{l-1}(\phi^{\dagger}_m)^j[2H_{\rm hop},\phi^{\dagger}_m](\phi^{\dagger}_m)^{l-1-j}|\Omega\rangle$$

$$=\sum_{j=0}^{l-1}(\phi^{\dagger}_m)^j\sum_n\Delta_{mn}\phi^{\dagger}_n\pi_m(\phi^{\dagger}_m)^{l-1-j}|\Omega\rangle=\sum_n\Delta_{mn}\phi^{\dagger}_n(\phi^{\dagger}_m)^{l-1}|\Omega\rangle,$$

where the property $\pi\phi^{\dagger}=0$ was used. Successively applying the above equation we see that the average

$$\langle\Omega|\phi_{j_1}^{n_{j_1}}\phi_{j_2}^{n_{j_2}}\ldots\phi_{j_s}^{n_{j_S}}(2H_{\rm hop})^K(\phi^{\dagger}_{l_1})^{n_{l_1}}(\phi^{\dagger}_{l_2})^{n_{l_2}}\ldots(\phi^{\dagger}_{l_z})^{n_{l_z}}|\Omega\rangle$$

is equal to the number $|P_K(n_{l_1},n_{l_2},\ldots n_{l_Z}\to n_{j_1},n_{j_2},\ldots,n_{j_S})|$ of mutually noncrossing lattice paths that were made in K steps by $N=n_{l_1}+n_{l_2}+\ldots+n_{l_z}$ friendly walkers initially located in the sites $M\geq l_1>l_2>\ldots>l_Z\geq 0$ and terminated in the sites $M\geq j_1>j_2>\ldots>j_S\geq 0$, $N=n_{j_1}+n_{j_2}+\ldots+n_{j_s}$. We see that the multi-particle correlation function (70) is expressed through the numbers of all admissible nests of lattice paths made by N walkers:

$$\langle\Omega|\phi_{j_1}^{n_{j_1}}\phi_{j_2}^{n_{j_2}}\ldots\phi_{j_s}^{n_{j_S}}e^{tH_{\rm hop}}(\phi^{\dagger}_{l_1})^{n_{l_1}}(\phi^{\dagger}_{l_2})^{n_{l_2}}\ldots(\phi^{\dagger}_{l_z})^{n_{l_z}}|\Omega\rangle \tag{71}$$

$$=e^{-tN}\sum_K\frac{t^K}{2^KK!}|P_K(n_{l_1},n_{l_2},\ldots n_{l_Z}\to n_{j_1},n_{j_2},\ldots,n_{j_S})|.$$

Inserting the resolution of unity (50) into the correlation function (70) and taking into account the coordinate representation of the N-particle state vector (26) on the solutions of the Bethe equations (13) we have

$$(M+1)(M+N+1)^{N-1}F\left(n_{j_1},n_{j_2},\ldots,n_{j_S};n_{l_1},n_{l_2},\ldots n_{l_Z}|t\right)$$

$$=\sum_{\{I\}}e^{tE_N^{\rm hop}}\det\left\{e^{i\frac{2\pi I_m+P}{M+N+1}(N-k+\lambda_k^j)}\right\}\det\left\{e^{-i\frac{2\pi I_m+P}{M+N+1}(N-k+\lambda_k^l)}\right\} \tag{72}$$

$$=\sum_{\{I\}}e^{tE_N^{\rm hop}}e^{i\frac{1}{(M+1)(M+N+1)}(|r|-|q|)\sum\limits_{k=1}^{N}I_k}\det\left\{e^{i\frac{2\pi I_m}{M+N+1}r_k}\right\}\det\left\{e^{-i\frac{2\pi I_m}{M+N+1}q_k}\right\},$$

where r_s and q_s are the parts of the strict partition $M+N \geq r_1 > r_2 > \ldots > r_N \geq 0$, $M+N \geq q_1 > q_2 > \ldots > q_N \geq 0$, obtained from the partitions $\lambda^j = (j_1^{n_{j_1}}, j_2^{n_{j_2}}, \ldots, j_S^{n_{j_S}})$, $\lambda^l = (l_1^{n_{l_1}}, l_2^{n_{l_2}}, \ldots, l_Z^{n_{l_Z}})$ by the shift on the partition $\delta = ((N-1), (N-2), \ldots, 0)$: $r = \lambda^j + \delta$, $q = \lambda^l + \delta$; and $|r| = \sum r_k$, $|q| = \sum q_k$ are the weights of the correspondent partitions. The numbers I_k are (32). The above expression may be simplified. Really,

$$(M+1)(M+N+1)^{N-1} F(n_{j_1}, n_{j_2}, \ldots, n_{j_S}; n_{l_1}, n_{l_2}, \ldots n_{l_Z} | t)$$

$$= \frac{1}{N!} \sum_{I_1, \ldots, I_N = 0}^{M+N} e^{tE_N^{\text{hop}}} e^{i \frac{1}{(M+1)(M+N+1)} (|r|-|q|) \sum\limits_{k=1}^{N} I_k} \det \left\{ e^{i \frac{2\pi I_m}{M+N+1} r_k} \right\}$$

$$\times \det \left\{ e^{-i \frac{2\pi I_m}{M+N+1} q_k} \right\}$$

$$= \frac{1}{N!} \sum_{I_1, \ldots, I_N = 0}^{M+N} e^{tE_N^{\text{hop}}} e^{i \frac{1}{(M+1)(M+N+1)} (|r|-|q|) \sum_{k=1}^{N} I_k}$$

$$\times \sum_P (-1)^{\epsilon(P)} e^{i \frac{2\pi}{M+N+1} I_{P_1} r_1} \ldots e^{i \frac{2\pi}{M+N+1} I_{P_N} r_N} \det \left\{ e^{-i \frac{2\pi I_m}{M+N+1} q_k} \right\},$$

where the sum over P means the summation over the permutations of indices, and $\epsilon(P)$ is the parity of the permutation. Using the antisymmetry of the determinants we can symmetrize the sum over P:

$$= \sum_{I_1, \ldots, I_N = 0}^{M+N} e^{tE_N^{\text{hop}}} e^{i \frac{1}{(M+1)(M+N+1)} (|r|-|q|) \sum_{k=1}^{N} I_k}$$

$$\times e^{i \frac{2\pi}{M+N+1} I_1 r_1} \ldots e^{i \frac{2\pi}{M+N+1} I_N r_N} \det \left\{ e^{-i \frac{2\pi I_m}{M+N+1} q_k} \right\}$$

$$= \sum_{I_1, \ldots, I_N = 0}^{M+N} e^{tE_N^{\text{hop}}} e^{i \frac{1}{(M+1)(M+N+1)} (|r|-|q|) \sum_{k=1}^{N} I_k}$$

$$\times e^{i \frac{2\pi}{M+N+1} I_1 r_1} \ldots e^{i \frac{2\pi}{M+N+1} I_N r_N}$$

$$\times \sum_P (-1)^{\epsilon(P)} e^{i \frac{2\pi}{M+N+1} I_1 q_{P_1}} \ldots e^{i \frac{2\pi}{M+N+1} I_N q_{P_N}}$$

$$= \sum_{I_1, \ldots, I_N = 0}^{M+N} e^{tE_N^{\text{hop}}} e^{i \frac{1}{(M+1)(M+N+1)} (|r|-|q|) \sum_{k=1}^{N} I_k}$$

$$\times \sum_P (-1)^{\epsilon(P)} e^{i \frac{2\pi}{M+N+1} I_1 (r_1 - q_{P_1})} \ldots e^{i \frac{2\pi}{M+N+1} I_N (r_N - q_{P_N})},$$

and finally we have

$$F\left(n_{j_1},n_{j_2},\dots,n_{j_S};n_{l_1},n_{l_2},\dots n_{l_Z}|t\right)=\frac{1}{(M+1)(M+N+1)^{N-1}} \quad (73)$$

$$\times\sum_{I_1,\dots,I_N=0}^{M+N} e^{tE_N^{\text{hop}}} e^{i\frac{1}{(M+1)(M+N+1)}(|r|-|q|)\sum\limits_{k=1}^{N}I_k} \det\left\{e^{i\frac{2\pi}{M+N+1}I_k(r_k-q_s)}\right\}_{k,s=1,\dots,N}.$$

From the above formula it follows that the number of all admissible nests of lattice paths that made N friendly walkers by K steps from the sites $l_1>l_2>\dots>l_Z$ to the sites $j_1>j_2>\dots>j_S$ is equal to

$$|P_K^F(n_{l_1},n_{l_2},\dots n_{l_Z}\to n_{j_1},n_{j_2},\dots,n_{j_S})|$$
$$=\frac{1}{(M+1)(M+N+1)^{N-1}}\sum_{I_1,\dots,I_N=0}^{M+N}\left\{2\sum_{l=1}^{N}\cos\left(\frac{2\pi I_l+P}{M+N}\right)\right\}^K e^{i\frac{(|r|-|q|)P}{M+N}}$$
$$\times\det\left\{e^{i\frac{2\pi I_k}{M+N}(r_k-q_s)}\right\}_{k,s=1,\dots,N}. \quad (74)$$

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Поступило 21 ноября 2008 г.