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ON A PROBLEM OF BEURLING

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This article is an extended version of a lecture presented to a conference at Brown University on 11 June 2017 in celebration of John Wermer's 90th birthday. Here, we discuss a complete solution to the weighted approximation problem for polynomials on an arbitrary bounded simply connected domain Ω in the complex plane. The problem had been studied extensively by Keldysh [13] prior to 1941 in the context of weighted L^2-approximation, and more recently by Beurling [5], where the emphasis is on uniform approximation.

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§1. Introduction

On the occasion of this tribute to John Wermer in celebration of his 90th birthday I am happy to recall a time long ago when I began work as his doctoral student, a time when Mergeljan's theorem on uniform polynomial approximation on compact planar sets was still the subject of much discussion. Although the statement of the theorem is quite simple, the original proof

Key words: weighted polynomial approximation, quasi-analyticity, asymptotically holomorphic functions, harmonic measure.

was rather complicated, and it was in this atmosphere that I attended my first advanced seminar at which John Wermer presented a simplified proof of Mergeljan's theorem based on joint work with Irving Glicksberg (cf. [24]). It was here that the seeds of my life long interest in questions having to do with approximation by both polynomials and rational functions were planted. It is my intention to summarize some of what has been achieved on certain questions where the application of potential theoretic ideas has proven to be particularly fruitful.

By way of introduction X will be a compact subset of the complex plane, and $C(X)$ will denote the linear space of all continuous functions on X endowed with the uniform norm. By $P(X)$ we shall mean the closure in $C(X)$ of the polynomials in the complex variable z. And, $A(X)$ will stand for the closed subspace of $C(X)$ consisting of all functions that are continuous on X and analytic in its interior X°. Thus,

$$P(X) \subseteq A(X) \subseteq C(X).$$

The problem for uniform polynomial approximation is to determine for which X is $P(X) = C(X)$ or $P(X) = A(X)$, and here in both cases there is a purely topological answer (cf. [17]).

Theorem (Lavrentiev, 1936). *$P(X) = C(X)$ if and only if $X^\circ = \varnothing$, and $\mathbb{C} \setminus X$ is connected.*

Theorem (Mergeljan, 1951). *$P(X) = A(X)$ if and only if $\mathbb{C} \setminus X$ is connected.*

§2. The Bernstein problem

The modern theory of uniform polynomial approximation may rightly be said to have begun in 1885 with Weierstrass' theorem to the effect that if $I = [a, b]$ is any compact interval on the real line $\mathbb{R}$, then $P(I) = C(I)$. Of course, $P(\mathbb{R}) \neq C(\mathbb{R})$ since the polynomials are always unbounded near ∞. In order to deal with that difficulty Bernstein [2] was forced in 1924 to consider approximation with respect to a weight $w(x) \geqslant 0$ defined on $\mathbb{R}$, and decreasing sufficiently rapidly at ∞ to ensure that

$$\lim_{|x|\to\infty} |x|^n w(x) = 0 \quad \text{for all} \quad n = 0, 1, 2 \dots .$$

In particular, $w(x)$ decreases faster at ∞ than any polynomial can grow. By definition

$$C_w = \{f \in C(\mathbb{R}) : \lim_{|x|\to\infty} f(x)w(x) = 0\},$$

and is given the norm $\|f\|_w = \sup_x |f(x)|w(x)$.

Bernstein's problem (1924). *For which weights w are the polynomials dense in C_w?*

Complete solutions have been given by several authors, including Ahiezer, Pollard, and Mergeljan (cf. [1, 19] and [15]). Of particular interest here is the following result of Mergeljan: Given any point $z \in \mathbb{C}$ consider the auxiliary weight w^* defined in terms of the function

$$M^*(z) = \sup_Q |Q(z)|,$$

where the supremum is taken over all polynomials Q such that

$$|Q(x)|(1+|x|)^{-1}w(x) \leqslant 1$$

for all real x; that is, $M^*(z)$ is the norm of the linear mapping $Q \to Q(z)$ in the $C_{\frac{w}{1+|x|}}$- norm. Here,

$$w^*(x) = \frac{1+|x|}{M^*(x)} \geqslant w(x),$$

and

$$w^*(x) \geqslant 1$$

for all $x \in \mathbb{R}$. Mergeljan [19, §4] has shown that *each* of the following is necessary and sufficient for the polynomials to be dense in C_w:

(1) $$\int_{-\infty}^{\infty} \frac{\log w^*(x)}{1+x^2}\, dx = +\infty,$$

(2) $$M^*(z) = +\infty \quad \text{for all } z, \quad \operatorname{Im} z \neq 0.$$

§3. A Problem of Beurling

Let us forego for now questions having to do with approximation on the real line, and consider instead weighted polynomial approximation on an arbitrary bounded simply connected domain Ω in the complex plane $\mathbb{C}$. By definition Ω_∞ will denote the unbounded component of $\mathbb{C}\setminus\bar{\Omega}$. We shall assume that $\mathbb{C}\setminus\bar{\Omega}$ is connected, and that

$$\partial\Omega \setminus \partial\Omega_\infty \neq \varnothing.$$

Of course, without the latter requirement

$$P(\bar{\Omega}) = A(\bar{\Omega})$$

by Mergeljan's theorem from 1951. For example, let Ω be a simply connected region obtained from a Jordan domain by introducing cuts or slits in the form of simple (but not necessarily smooth) arcs extending from the interior outward to the boundary. In this setting let $w(z) > 0$ be a bounded continuous function on Ω with the property that $w(z) \to 0$ as $z \to \partial\Omega$; that is, as z approaches each point in the full $\partial\Omega$. By analogy with the Bernstein problem Beurling [5, p. 413] replaces the real line $\mathbb{R}$ with a domain Ω, and he introduces two Banach spaces:

$$C_w(\Omega) = \{f : fw \in C(\bar{\Omega}) \text{ and } fw = 0 \text{ on } \partial\Omega\},$$

with norm defined by $\|f\|_w = \sup_\Omega |f|w$, and an associated subspace

$$A_w(\Omega) = \{f \in C_w(\Omega) : f \text{ is analytic in } \Omega\}.$$

Beurling's problem (1989). *For which weights w are the polynomials dense in $A_w(\Omega)$?*

Surprisingly, Beurling's problem turns out to be equivalent to another question concerning L^2-approximation by polynomials studied by Keldysh in 1941 (cf. [13, 14] and [18, § 7]).

§4. Approximation in the mean by polynomials

In this setting Ω will continue to be a bounded simply connected region in $\mathbb{C}$, and dA will denote two-dimensional Lebesgue (or area) measure. By definition, Ω is a *Carathéodory domain* if $\partial\Omega = \partial\Omega_\infty$. Although the closure of a Carathéodory domain need not have a connected complement (cf. [18, §1]), such domains are nevertheless the natural analogues in the case of L^p-approximation by polynomials to sets with connected complements in the case of uniform approximation. For Carathéodory domains two spaces, each defined for $1 \leqslant p < \infty$, are of particular interest:

(i) $H^p(\Omega, dA)$, *the closure of the polynomials in* $L^p(\Omega, dA)$.

(ii) $L^p_a(\Omega, dA)$, *the set of functions in* $L^p(\Omega, dA)$ *that are analytic in* Ω.

Evidentally,

$$H^p(\Omega) \subseteq L^p_a(\Omega) \quad \text{for all } p.$$

Moreover, in 1934 A. I. Markushevich and O. J. Farrell (cf. [18]) established independently the fact that if Ω is a Carathéodory domain then

$$H^p(\Omega) = L^p_a(\Omega)$$

whenever $1 \leqslant p < \infty$. Years later in 1966 S. O. Sinanjan [21] showed that

$$H^p(X) = L^p_a(X)$$

whenever X is a closed Carathéodory set; that is, whenever $\partial X = \partial X_\infty$. Here, of course, $L^p_a(X)$ stands for the functions in $L^p(X)$ that are analytic in X°.

Since approximation in the mean is completely understood in the case of a Carathéodory domain, it will henceforth be assumed that $\partial\Omega \neq \partial\Omega_\infty$. As previously indicated, $w(z) > 0$ will be a bounded continuous function on Ω with the property that $w(z) \to 0$ as $z \to \partial\Omega$. The spaces $H^p(\Omega, w\,dA)$ and $L^p_a(\Omega, w\,dA)$ are defined by analogy with (i) and (ii) above, and clearly

$$H^p(\Omega, w\,dA) \subseteq L^p_a(\Omega, w\,dA)$$

for all $p, 1 \leqslant p < \infty$.

Keldysh's problem (1941). *For which weights w is*

$$H^2(\Omega, w\,dA) = L^2_a(\Omega, w\,dA)$$

Due to the fact that the problems of Beurling and Keldysh are essentially equivalent, we shall first outline a complete solution in the case of L^2-approximation where the theory of Sobolev spaces and its associated potential theory are available.

§5. Weighted L^2-approximation

It will be assumed that Ω is a bounded simply connected domain, that the weight $w = w(G)$ depends only on Green's function G, and in particular that

$$w(z) = e^{-h(\log 1/|\varphi(z)|)},$$

where $\varphi : \Omega \to D$ conformally onto the open unit disk D. In addition we require that $yh(y) \uparrow +\infty$ as $y \downarrow 0$. With this understanding we have (cf. [8]):

Theorem 1. $H^2(\Omega, w\,dA) = L^2_a(\Omega, w\,dA)$ *whenever* $\int_0 \log h(y)dy = +\infty$; *that is, whenever*

$$\int\limits_0 \log\log\frac{1}{w(y)}\,dy = +\infty.$$

Proof of Theorem 1 (outline). Let $g \in L^2(\Omega, w\,dA)$ be any function such that $\int Pgw\,dA = 0$ for all polynomials P, and put

$$f(z) = \int\limits_\Omega \frac{gw(\zeta)}{(\zeta - z)}\,dA_\zeta. \tag{5.1}$$

Evidentally, $f \equiv 0$ in Ω_∞. Moreover, since w is bounded and

$$gw \in L^2(\Omega, w\,dA),$$

it follows from the Calderon–Zygmund theory of singular integrals that f belongs to the Sobolev space $W_1^2(\mathbb{R}^2)$, and is therefore *quasi-continuous* throughout the entire plane. Hence, $f = 0$ quasi-everyhere (with respect to capacity) on $\partial\Omega_\infty$. On the other hand, the restrictions placed on w guarentee that f is continuous on the entire boundary $\partial\Omega$, and therefore $f \equiv 0$ on $\partial\Omega_\infty$. Our goal is to prove that $f = 0$ on $\partial\Omega \setminus \partial\Omega_\infty$ as well, or equivalently that

$$f \in \overset{0}{W}{}_1^2(\Omega).$$

We may conclude, therefore, by an argument of Havin [12] that there exists a sequence $\eta_j \in C_0^\infty(\Omega)$ such that $\eta_j \to f$ in the norm of $W_1^2(\Omega)$. For any $\sigma \in L_a^2(\Omega, w\,dA) \cap L^2(\Omega, dA)$ we have

$$\int\limits_\Omega \sigma\, \bar{\partial}\eta_j\, dA = -\int\limits_\Omega \eta_j\, \bar{\partial}\sigma\, dA = 0, \tag{5.2}$$

and since

$$\int\limits_\Omega \sigma\, \bar{\partial}\eta_j\, dA \to \int\limits_\Omega \sigma g w\, dA$$

it can be inferred that

$$\int\limits_\Omega \sigma g w\, dA = 0$$

and therefore that $\sigma \in H^2(\Omega, w\,dA)$. Let us note here that the right hand integral in (5.2) vanishes because η_j is supported on a set where $\bar{\partial}\sigma = 0$. Since we can take $\sigma = \varphi^n(\varphi')$ for all $n = 0, 1, 2, \ldots$ and since, by virtue of the fact that w depends only on Green's function, these functions are dense in $L_a^2(\Omega, w\,dA)$ we easily deduce that $H^2(\Omega, w\,dA) = L_a^2(\Omega, w\,dA)$. □

We are therefore left with the task of demonstrating that the Cauchy integral in (5.1) vanishes identically on $\partial\Omega \setminus \partial\Omega_\infty$. Before we can accomplish that task, however, we will need some additional information regarding the manner in which the theory of Sobolev spaces is related to our problem.

§6. Sobolev spaces and asymptotically holomorphic functions

Throughout this discussion $\varphi : \Omega \to D$ will denote a conformal map of Ω onto the open unit disk D, and $\psi = \varphi^{-1}$. It will be essential to understand what effect a conformal change of coordinates has on $W_1^2(\Omega)$. If, for example, $f \in W_1^2(\Omega)$ and T is defined by setting $T(f) = f(\psi)$, then

$$T : \overset{0}{W}{}_1^2(\Omega) \to \overset{0}{W}{}_1^2(D).$$

In this case, it is even true that

$$T : W_1^2(\Omega) \to W_1^2(D),$$

but that depends on the nature (or smoothness) of the target domain D and is not true in general (cf. [8, p. 766]). Note also that if $F \in W_1^2(D)$ then F has boundary values

$$F(e^{i\theta}) \in L^2(\partial D, d\theta).$$

Let k be the *Legendre transform* of the function h in the statement of Theorem 1. By definition

$$k(x) = \inf_{y>0} (h(y) + xy), \ x > 0.$$

As a lower envelope of linear functions k is *concave* and, moreover,

(P$_1$) $yh(y) \uparrow +\infty$ as $y \downarrow 0$,

(P$_2$) $x^{-1/2}k(x) \uparrow +\infty$ as $x \uparrow +\infty$.

If h or k is smooth, which can always be arranged, then properties (P$_1$) and (P$_2$) are equivalent (cf. [8, p. 768]). The following Lemma expresses a connection between the growths of h and k, and was discovered independently by Beurling [4] and E. M. Dyn'kin [10].

Lemma (Beurling, Dyn'kin, 1972). *If h and k are related to one another as Legendre transforms, then*

$$\int_0^\delta \log h(y)\, dy \quad \textit{and} \quad \int_1^\infty \frac{k(x)}{x^2}\, dx$$

converge or diverge simultaneously.

The following result is due essentially to Dyn'kin [10], who proved it under the assumption that F is a continuous function on ∂D. Known as the Dyn'kin extension theorem, it was subsequently modified by Vol'berg [22] to cover functions $F \in L^1(\partial D, d\theta)$.

Theorem (Dyn'kin, 1972). *If $F(e^{i\theta}) \in L^2(\partial D, d\theta)$ with a Fourier expansion*

$$F(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

satisfying the two conditions

(i) $|a_{-n}| \leqslant e^{-k(n)}$, for $n = 1, 2, 3, \ldots$

(ii) $\sum_{n=1}^{\infty} \frac{k(n)}{n^2} = +\infty$

where $k(x)$ *is concave and* $x^{-1/2}k(x) \uparrow +\infty$ *as* $x \uparrow +\infty$*, then* F *can be replaced by a new Sobolev function* $\widetilde{F}$ *in such a way that*

(a) $\widetilde{F}$ *is continuous in the interior of* D*, and*

(b) $\widetilde{F}$ *has the same boundary values as* F*.*

In order to clarify the manner in which the new function $\widetilde{F}$ is to be obtained, we first transfer the original weight w from Ω to D by setting $W = w(\psi)$. Under the given hypotheses Dyn'kin constructs a function $\rho \in L^\infty(D)$ with the property that

$$\widetilde{F}(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{1}{\pi} \int_D \frac{\rho(\zeta) W(\zeta)}{(\zeta - z)} \, dA_\zeta$$

has exactly the boundary values $F(e^{i\theta})$. The first term on the right is continuous in D and the second is everywhere continuous, since it is the Cauchy integral of a bounded function. The main difficulty consists in showing that $\widetilde{F}$ has the same boundary values as F. Since $\bar{\partial}\widetilde{F} = -\rho W$, the inequality

$$|\bar{\partial}\widetilde{F}(z)| \leqslant C\, W(z)$$

is satisfied for all $z \in D$ and some absolute constant C. By choosing w, and hence W, appropriately we can arrange that

$$|\bar{\partial}\widetilde{F}(z)| \to 0 \quad \text{rapidly as} \quad z \to \partial D.$$

The rate at which $|\bar{\partial}\widetilde{F}(z)|$ decays at ∂D can be taken as a measure (or asymptotic estimate) of the extent to which $\widetilde{F}(e^{i\theta})$ deviates from the boundary values of an actual analytic function. If the drop off is sufficiently rapid and the consequent deviation is below a certain critical level, then $\widetilde{F}(e^{i\theta})$ will inherit many of the properties usually associated with analyticity. When this occurs such a function is said to be *asymptotically holomorphic*. Here, if the hypotheses of Theorem 1 are satisfied, then the function F obtained from the Cauchy integral in (5.1) also satisfies the conditions of Dyn'kin's theorem (cf. [8, p. 770]). We must prove that

$$\int_0^{2\pi} \log|F(e^{i\theta})| \, d\theta = -\infty \quad \text{implies} \quad F(e^{i\theta}) \equiv 0. \tag{6.1}$$

This, in essence, is an extension of a 1982 theorem of Vol'berg (cf. [22] and [23]).

To verify (6.1) in the context of Theorem 1 consider the set

$$E = \{z \in D : |\widetilde{F}(z)| \leqslant W(z)\}$$

and note that the complementary set $U = D \setminus E$ is open. We may assume that

(i) E is the countable union of smoothly bounded Jordan regions, and

(ii) only finitely many such regions meet any compact subset of D.

We may also assume that $\partial U \supset \partial D$, since otherwise $\widetilde{F} = 0$ on a subarc of ∂D and therefore, by a theorem of Levinson [16], $\widetilde{F} \equiv 0$ on ∂D. It can also be arranged in such a way that $|\widetilde{F}(z)| \leqslant 2W(z)$ on ∂U, and that $|\widetilde{F}(z)| > W(z)$ inside U.

§7. The uniqueness property

As noted above the proof of Theorem 1 rests on establishing that the function f given by the Cauchy integral (5.1) vanishes identically on $\partial\Omega \setminus \partial\Omega_\infty$, or along with (6.1) showing that

$$\int_0^{2\pi} \log|F(e^{i\theta})|\, d\theta = -\infty.$$

A good overall reference for much of what will be required here is Koosis' book [15, vol. 1] on the *Logarithmic Integral.* A full discussion of the details can be found in [8] (cf. also [9]).

Let $\lambda_0 = e^{i\theta_0}$ be a point on ∂D, and for each $\epsilon \leqslant 1/2$ form the sectorial box

$$B_\epsilon = \left\{ re^{i\theta} : 1 - \epsilon \leqslant r \leqslant 1,\ |\theta - \theta_0| \leqslant \frac{1}{2} \log \frac{1}{(1-\epsilon)} \right\}.$$

The mapping $\chi(z) = -i \log z$ carries B_ϵ onto a square, and the point

$$\lambda_\epsilon = \sqrt{1-\epsilon}\, e^{i\theta_0}$$

corresponds to the center of the image. If $\lambda_\epsilon \in U$ let $d\omega_\epsilon$ be harmonic measure for $B_\epsilon \cap U$ at the point λ_ϵ. It can be assumed that $B_\epsilon \cap U$ is connected, and there are two cases to consider.

$$(A)\quad \int\limits_{\partial U\cap B_\epsilon} \frac{d\omega_\epsilon}{(1-|z|)} \geqslant C > 0 \quad \text{for all } \epsilon \text{ (at some } \lambda_0 \in \partial D).$$

$$(B)\quad \int\limits_{\partial U\cap B_\epsilon} \frac{d\omega_\epsilon}{(1-|z|)} \to 0 \quad \text{for some sequence of } \epsilon \to 0 \text{ (at every } \lambda_0 \in \partial D).$$

Case (A). For each $\epsilon < 1$ consider the function

$$\widetilde{F}_\epsilon(z) = Q(z) + \frac{1}{\pi}\int\limits_{D_\epsilon} \frac{\rho(\zeta)W(\zeta)}{(\zeta - z)} dA_\zeta,$$

where $D_\epsilon = \{z : |z| < 1-\epsilon\}$ and Q is the analytic part of $\widetilde{F}$. Evidentally, $\widetilde{F}_\epsilon$ is analytic for $(1-\epsilon) < |z| < 1$ and, moreover,

(1) $|\widetilde{F}(z) - \widetilde{F}_\epsilon(z)| \leqslant e^{-ch(\epsilon)}$

(2) $|\widetilde{F}_\epsilon(z)| \leqslant K$

where c and K are constants independent of ϵ (cf. [8, p. 772]). Under the map $\chi(z) = -i\log z$ we obtain functions H and H_ϵ corresponding to $\widetilde{F}$ and $\widetilde{F}_\epsilon$, respectively so that

(3) H and H_ϵ are defined and periodic on $\mathbb{R}$;

(4) H_ϵ has an analytic extension to the interior of a horizontal strip S_ϵ of height $\log \frac{1}{(1-\epsilon)}$ and base lying along the real axis.

We now choose a rectangle R_ϵ with base $[a, b] \subset \mathbb{R}$, containing a full period of H, and top lying along the opposite side of S_ϵ. We may assume that both a and b correspond to $\lambda_0 \in \partial D$ under the map χ, and we form the Fourier transform

$$\widehat{H}(z) = \int\limits_a^b H(x)e^{izx}\, dx.$$

Replacing H by H_ϵ and appealing to the inequality (1), we integrate as in Beurling [3] over $\Gamma_\epsilon = \partial R_\epsilon \setminus [a,b]$ to obtain the estimate

$$\big|\widehat{H}(t)\big| \leqslant e^{-ch(\epsilon)} + \left| \int_{\Gamma_\epsilon} H_\epsilon(z) e^{itz}\, dz \right|.$$

Here we require that $yh(y) \uparrow +\infty$ as $y \downarrow 0$, and we form the corresponding Legendre transform

$$\widetilde{k}(x) = \inf_{y>0} y(h(y)+x), \quad x > 0.$$

Together with property (A), it can be shown (cf. [8, p. 773]) that

$$|\widehat{H}(t)| \leqslant K e^{-c\widetilde{k}(t)} \quad \text{for all } t \geqslant 1$$

and suitable constants K and c, from which it follows that

$$-\int_1^\infty \log|\widehat{H}(t)|\,\frac{dt}{t^2} \geqslant C \int_1^\infty \frac{\widetilde{k}(t)}{t^2}\, dt = +\infty.$$

The last integral diverges as a consequence of Lemma 1, and the assumption that

$$\int_0 \log h(y)\, dy = +\infty.$$

The implication here is that $\widehat{H}(z) = 0$, and that therefore $H = 0$ *a.e.*-dx on $\mathbb{R}$. Evidentally, then $F = \widetilde{F} = 0$ *a.e.*-$d\theta$ on ∂D, from which it follows that

$$F \in \overset{0}{W}{}_1^2(D) \quad \text{and} \quad f \in \overset{0}{W}{}_1^2(\Omega),$$

and therefore Case (A) is complete.

Case (B). Fix a point $z_0 \in U$. For each ϵ sufficiently small set

$$D_\epsilon = \{z : |z| < 1-\epsilon\},$$

let U_ϵ be the component of $D_\epsilon \setminus E$ containing z_0, and let $d\omega_{U_\epsilon}$ be harmonic measure for z_0 relative to U_ϵ. Since we are in Case (B), it follows that $\partial U_\epsilon \cap D$ is so sparse that $d\omega_{U_\epsilon}$ is *uniformly boundedly equivalent* to arc length $d\theta$ along ∂D_ϵ (cf. [7, pp. 38–40]). Next, consider the function $\Phi = \widetilde{F}e^R$ introduced by Vol'berg in [22], where

$$R(z) = -\frac{1}{\pi} \int_U \frac{1}{\widetilde{F}(\zeta)} \frac{\rho(\zeta)W(\zeta)}{(\zeta - z)}\, dA_\zeta. \tag{7.1}$$

Because $\bar\partial\Phi = 0$ *a.e.*-dA in U and the integrand in (7.1) is bounded, Φ itself possesses the following two properties:

(5) Φ is analytic in U, and

(6) $C_1|\widetilde{F}(z)| \leqslant |\Phi(z)| \leqslant C_2|\widetilde{F}(z)|$ with constants $C_1, C_2 > 0$.

Integrating $\log|\Phi|$ over ∂U_ϵ we obtain

$$\int\limits_{\partial U_\epsilon} \log|\Phi|\, d\omega_{U_\epsilon} = \int\limits_{\partial U_\epsilon} \log|\widetilde{F}|\, d\omega_{U_\epsilon} + \int\limits_{\partial U_\epsilon} Re(R)\, d\omega_{U_\epsilon},$$

where by reason of (7.1) the second term on the right is bounded. Moreover, it can be shown that

$$\int \log|\widetilde{F}|\, d\omega_{U_\epsilon} \approx \int\limits_{\partial D_\epsilon} \log|\widetilde{F}|\, d\theta \to -\infty$$

for some sequence of $\epsilon \to 0$ (cf. [8, pp. 774–777]). This implies that $\Phi \equiv 0$ in U and therefore $\widetilde{F}(e^{i\theta}) = F(e^{i\theta}) = 0$ *a.e.*-dθ on ∂D, since $\widetilde{F}(z) \to 0$ as $z \to \partial D$ through $D \setminus U$ by construction. This also completes Case (B).

§8. Weighted uniform approximation

We continue to assume that $w = e^{-h}$ is a weight depending only on Green's function and that

(i) $yh(y) \uparrow +\infty$ as $y \downarrow 0$, and

(ii) $\int\limits_0 \log h(y)\, dy = +\infty$.

For any such weight w on a region Ω there exists another weight $\widetilde{w}$ also satisfying properties (i) and (ii), and so that for all polynomials Q

$$\sup_\Omega |Q|\widetilde{w} \leqslant C\, \|Q\|_{L^1(\Omega,\, wdA)}. \tag{8.1}$$

Fix a point $\xi \in \Omega$, and let $\delta = \operatorname{dist}(\xi, \partial\Omega)$. By the *Koebe distortion theorem* (cf. [20, p. 9]), if $\varphi : \Omega \to D$ conformally then

$$1 - |\varphi(\xi)| \leqslant C\sqrt{\delta},$$

where C is an absolute constant. If $B = B_{\delta/2}(\xi)$ the area mean value theorem ensures that

$$Q(\xi) = \frac{1}{|B|}\int\limits_B Q\, dA = \frac{4}{\pi\delta^2}\int\limits_B Q\,\frac{w}{w}\, dA.$$

Setting $y = 1-|\varphi(\xi)|$, recalling that $w(y)$ decreases as $y \downarrow 0$ and that $y \leqslant C\sqrt{\delta}$, we obtain an inequality

$$|Q(\xi)|\, y^4 w(y/2) \leqslant C \int\limits_{\Omega} |Q|\, w\, dA,$$

which is valid for all polynomials Q, and all $\xi \in \Omega$. Taking

$$\widetilde{w}(z) = e^{-2h(\frac{1}{2}\log 1/|\varphi(z)|)}$$

and adjusting the constant C accordingly, (8.1) is satisfied.

In summary:

(a) given w we can find $\widetilde{w}$ so that (8.1) holds;

(b) conversely, given $\widetilde{w}$ we can argue backwards to find a w so that (8.1) again holds;

(c) in both instances w and $\widetilde{w}$ each enjoy properties (i) and (ii).

Before taking up the question of weighted uniform approximation, let us observe that the problem of weighted L^2-approximation can be posed and settled in the context of $L^p(\Omega, wdA)$ for any $p, 1 \leqslant p < \infty$, provided w continues to satisfy (i) and (ii). By analogy with Mergeljan's solution to the Bernstein problem in terms of the non-existence of certain bounded point evaluations (*bpe's*) off the real axis $\mathbb{R}$, we have the following theorem (cf. [6, p. 416]):

Theorem 2. *$H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$ if, and only if, $H^p(\Omega, wdA)$ has no bounded point evaluations on $\partial\Omega$. Moreover, if $H^p(\Omega, wdA) \neq L^p_a(\Omega, wdA)$ there exists a point $\xi_0 \in \partial\Omega$ and an open set U containing ξ_0 such that every function in $H^p(\Omega, wdA)$ admits an analytic continuation to U.*

Suppose now that $\widetilde{w}$ is a weight satisfying (i) and (ii) and that

$$H^p(\Omega, \widetilde{w}dA) \neq L^p_a$$

for some p. By Theorem 2 there is necessarily a *bpe* at some point $\xi_0 \in \partial\Omega$, and so by our remarks in the summary above there is another weight w having the same properties as $\widetilde{w}$ such that

$$|Q(\xi_0)| \leqslant C_1 \|Q\|_{L^p(\Omega, \widetilde{w}dA)} \leqslant C_2 \|Q\|_{L^1(\Omega, wdA)}$$

for all polynomials Q. We conclude, therefore:

Theorem 3. *For any* $p, 1 \leqslant p < \infty$, *the polynomials are dense in* $L_a^p(\Omega, wdA)$ (*i.e.* $H^p = L_a^p$) *whenever* w *is a weight satisfying* (i) and (ii).

Theorem 4. *If* (i) and (ii) *are satisfied, then the polynomials are dense in* $A_w(\Omega)$.

Proof of Theorem 4 (outline). . Fix $f \in A_w(\Omega)$, and for each $r < 1$ let $f_r = F(r\varphi)$, where as usual $F = f(\varphi^{-1})$. Evidentally, $f_r \in A_w(\Omega)$ and in view of the monotonicity of w,

$$\|f_r - f\|_w \to 0 \quad \text{as} \quad r \to 1.$$

Since f_r is also bounded, it can be approximated arbitrarily closely by a sequence of polynomials in $L^1(\Omega, \widetilde{w}dA)$ for any weight $\widetilde{w}$ dominating w in the sense of (8.1). The same sequence therefore converges to f_r in the norm of $C_w(\Omega)$, and the theorem follows. (Note, here in connection with (8.1), the roles of w and $\widetilde{w}$ have been reversed.) □

As we have repeatedly tried to emphasize, Beurling and Keldysh studied what at first appeared to be rather different approximation problems, but which in retrospect have turned out to be essentially equivalent. In the case of Beurling [5] the emphasis was on weighted uniform approximation, whereas Keldysh [14] was exclusively concerned with L^2-approximation. Having acknowledged the historical connection, it seems appropriate at this point to compare the contributions of each of the individuals in question, while noting that those earlier contributions are now largely subsumed in the theorems outlined above.

§9. The contributions of Beurling and Keldysh

Both authors characterize weighted completeness in terms of the growth of $\log\log(1/w)$. Here, Beurling [5, p. 413] obtains the sharper result with respect to the weight w, but at the expense of limiting the type of region Ω to which his argument applies. Based on the supposition that the conformal map $\psi : D \to \Omega$ extends continuously to $\bar{D}$, which implies (cf. [20, p. 20]) that

(1) $\partial\Omega$ is arcwise connected, and

(2) $\partial\Omega_\infty$ is a Jordan curve,

along with the additional, but unstated, assumption that $\partial\Omega_\infty$ carries positive harmonic measure, Beurling concludes that the polynomials are dense in

$A_w(\Omega)$ whenever

$$\int\limits_0 \log\log\frac{1}{w(y)}dy = +\infty.$$

This, of course, excludes consideration of the most general bounded simply connected domains. He does, however, remark that if w is monotonic and if the *inner boundary* $\partial\Omega \setminus \partial\Omega_\infty$ contains an isolated *smooth* arc, then the polynomials fail to be dense in $A_w(\Omega)$ whenever

$$\int\limits_0 \log\log\frac{1}{w(y)}\,dy < \infty. \tag{9.1}$$

That fact also follows from theorems of Levinson and Domar (cf. [15, vol. 1, pp. 374–379]).

Keldysh, on the other hand, considers weighted approximation on an *arbitrary* bounded simply connected domain Ω. In addition, he explicitly requires that $w(z)$ is a weight such that if $\widetilde{w} = w(\psi)$, then

$$H^2(D, \widetilde{w}\,dA) = L^2_a(D, \widetilde{w}\,dA),$$

which is evidentally the case anytime w depends only on Green's function. Under the stated conditions he is able to prove the following two theorems, where $w(z)$ depends on $\delta(z) = \operatorname{dist}(z, \partial\Omega)$ (cf. [13, 14] and [18, §7]):

(1) $H^2(\Omega, wdA) = L^2_a(\Omega, wdA)$ whenever $\displaystyle\liminf_{\delta(z)\to 0} \frac{\log\log\log\frac{1}{w(z)}}{\log\frac{1}{\delta(z)}} > 2.$

(2) There exists a domain Ω_0 with the property that

$$H^2(\Omega_0, wdA) \neq L^2_a(\Omega_0, wdA)$$

for any weight w satisfying

$$\limsup_{\delta(z)\to 0} \frac{\log\log\log\frac{1}{w(z)}}{\log\frac{1}{\delta(z)}} < 2. \tag{9.2}$$

In the second assertion (2) the domain Ω_0 is obtained by deleting from the open unit disk D a spiral which winds outward from the interior, and accumulates along the entire length of ∂D. Keldysh shows that if (9.2) is satisfied, then any sequence of polynomials converging in the $L^2(\Omega_0, w\,dA)$-norm is necessarily a normal family in D, and so every $f \in H^2(\Omega_0, w\,dA)$ admits an analytic continuation across the inner boundary of Ω_0 to the entire disk D.

In retrospect, that result might well be viewed as a precursor to Levinson's log log-theorem referred to in connection with (9.1) above.

§10. An example

In whatever context one considers the weighted approximation problem, either in the uniform norm or in any of the $L^p(dA)$ metrics, the argument for polynomial completeness is fundamentally the same. To identify a difficulty associated solely with the underlying topological structure of the domain Ω, and is therefore present in any of the cases under consideration, let us recall, in outline, the core of the argument in our proof of Theorem 1, which ran as follows: Given any function $g \in L^2(\Omega, w\,dA)$ such that $\int Pgw\,dA = 0$ for all polynomials P, form the Cauchy integral

$$f(z) = \int_{\Omega} \frac{gw(\zeta)}{(\zeta - z)} dA_{\zeta}.$$

Evidently, f is a Sobolev function, $f \equiv 0$ in Ω_∞, and by *continuity* f also vanishes identically on $\partial\Omega_\infty$. To prove Theorem 1 it is sufficient to verify that $f \equiv 0$ on $\partial\Omega \setminus \partial\Omega_\infty$ as well. Since the weight w depends only on Green's function, the original problem is conformally invariant and can be transferred to the open unit disk D by means of a conformal map $\psi : D \to \Omega$, yielding a new Sobolev function $F = f(\psi)$ whose boundary values $F(e^{i\theta})$ belong to $L^2(d\theta)$ and satisfy the conditions of Levinson's theorem from 1936 (cf. [8, p. 770]). In that case, if $F(e^{i\theta}) = 0$ a.e.-$d\theta$ on any sub-arc of ∂D, then $F(e^{i\theta}) = 0$ a.e.-$d\theta$ on ∂D by Vol'berg's theorem (cf. [22]), and therefore $f \equiv 0$ on $\partial\Omega$, from which the theorem follows.

The essential difficulty in establishing the completeness of the polynomials in a general context consists in proving that $F(e^{i\theta}) = 0$ a.e.-$d\theta$ on ∂D, even if it is not known, a priori, that $F(e^{i\theta}) = 0$ at more than a single point. And, that is undoubtedly the situation whenever Ω is a spiral domain of the type studied by Keldysh, since then $\partial\Omega_\infty$ represents a single prime end, and therefore corresponds to only a single point on ∂D under a conformal map. The same difficulty can also arise in the context of uniform weighted approximation, even if it is assumed that the conformal map $\psi : D \to \Omega$ extends continuously to $\bar{D}$ as stipulated by Beurling in [5, p. 412]. As stated, Beurling's theorem is correct, but the proof in [5] is incomplete. The deficiency becomes apparent in an example, first brought to my attention by D. E. Marshall, and is this:

Theorem 5. *There exists a bounded simply connected domain Ω with the property that the conformal map $\psi : D \to \Omega$ extends continuously to $\bar{D}$, and yet $\partial\Omega_\infty$ carries zero harmonic measure.*

Proof of Theorem 5. Let D be the open unit disk, and choose a collection of k points evenly spaced with respect to arc length along ∂D. At each such point attach a radial segment of length $1/2$, or 2^{-1}. At the midpoints of each of the k open arcs determined by the initial collection attach a radial segment of length 2^{-2}. This results in a collection of $2k$ disjoint open arcs lying along ∂D, to the midpoints of which we now attach radial segments of length 2^{-4}. Continuing in this way, *ad infinitum*, we obtain a bounded simply connected domain Ω such that $\partial\Omega$ is evidentally locally connected, and therefore the conformal map

$$\psi : D \to \Omega$$

has a continuous extension to $\bar{D}$ (cf. [20, p. 18]).

To complete the proof of the theorem let ω denote harmonic measure for Ω with the origin as base point. For each r, $1/2 \leqslant r \leqslant 1$, let

$$J_r = \{z \in \Omega : |z| = r\}$$

so that in each case J_r consists of finitely many arcs separating the origin from the set

$$E = \partial\Omega \cap (|z| = 1),$$

and having their end points in $\partial\Omega$. Denote by $\theta(r)$ the length of the longest arc, and hence the length of any one of the arcs, in J_r. By convention it is customary to set $\theta(r) = 0$ when $r < 1/2$. It follows from the theory of *extremal length* that

$$\omega(E) \leqslant C \exp\left(-\pi \int_0^1 \frac{dr}{r\theta(r)} \right), \tag{10.1}$$

where C is an absolute constant (cf. [11, pp. 480–484] and [15, vol. 2, pp. 100–105]). For each $n = 1, 2, 3, \ldots$ the portion of J_r corresponding to the interval

$$1 - \frac{1}{2^n} \leqslant r \leqslant 1 - \frac{1}{2^{n+1}} \tag{10.2}$$

will be denoted by I_n and consists of the union of $2^n k$ individual arcs, having corresponding lengths $2\pi r/2^n k$. Calculating the integral in (10.1) over I_n, and recalling that $\log(1-x) \sim -x$ as $x \downarrow 0$, we obtain the following estimate:

$$\frac{2^n k}{2\pi}\int_{I_n}\frac{dr}{r} = \frac{2^n k}{2\pi}\left\{\log\left(1-\frac{1}{2^{n+1}}\right) - \log\left(1-\frac{1}{2^n}\right)\right\} \approx \frac{k}{4\pi},$$

valid for all n sufficiently large. In particular, the integral in (10.1) is approximately equal to $k/4\pi$ on infinitely many disjoint intervals in (10.2), and therefore diverges. It follows that $\omega(E) = 0$, and since E coincides with $\partial\Omega_\infty$, we have $\omega(\partial\Omega_\infty) = 0$. □

As a consequence of the preceding example, the implication here is that Levinson's theorem alone is not sufficient to establish weighted completeness for the most general domains, including those studied by Beurling in [5], and that Vol'berg's extension [22] of Levinson's theorem must necessarily play an essential role as it does in [8].

References

[1] Ахиезер Н. И., *О взвешенном приближении непрерывных функций многочленами на всей числовой оси*, Успехи мат. наук **11** (1956), №4, 3–43.

[2] Bernstein S. N., *Le problème de l'approximation des fonctions continues sur l'axe réel et l'une des ses applications*, Bull. Soc. Math. France **52** (1924), 399–410.

[3] Beurling A., *On quasianalyticity and general distributions*, Multilithed Lecture Notes, Summer School, Stanford Univ., Stanford, CA, 1961.

[4] Beurling A., *Analytic continuation across a linear boundary*, Acta Math. **128** (1972), no. 3–4, 153–182.

[5] Beurling A., *The collected works of Arne Beurling.* Vol. I. *Complex analysis*, Birkhäuser, Boston, MA, 1989.

[6] Brennan J. E., *Point evaluations, invariant subspaces, and approximation in the mean by polynomials*, J. Funct. Anal. **34** (1979), no. 3, 407–420.

[7] Brennan J. E., *Functions with rapidly decreasing negative Fourier coefficients*, Lecture Notes in Math., vol. 1275, Springer, Berlin, 1987, pp. 31–43.

[8] Brennan J. E., *Weighted polynomial approximation and quasianalyticity for general sets*, Алгебра и анализ **6** (1994), no. 4, 69–89.

[9] Brennan J. E., *Three problems in function theory*, Oper. Theory. Adv. Appl., vol. 261, Birkhäuser, Basel, 2017, pp. 191–227.

[10] Дынькин Е. М., *Функции с данной оценкой $\partial f/\partial\bar z$ и теорема Н. Левинсона*, Мат. сб. **89** (1972), №2, 182–190.

[11] Garnett J. B., Marshall D. E., *Harmonic measure*, New Math. Monogr., vol. 2, Cambridge Univ. Press, Cambridge, 2005.

[12] Хавин В. П., *Аппроксимация аналитическими функциями в среднем*, Докл. АН СССР **178** (1968), №5, 1025–1028.

[13] Keldysh M. V., *О замкнутости ортогональных с весом систем полиномов*, Докл. АН СССР **30** (1941), №9, 771–773.

[14] Keldysh M. V., *Sur l'approximation en moyenne par polynomes des fonctions d'une variable complexe*, Мат. сб. **16** (1945), №1, 1–20.

[15] Koosis P., *The logarithmic integral.* I, II, Cambridge stud. Adv. Math., vol. 12, 21, Cambridge Univ. Press, Cambridge, (1988), (1992).

[16] Levinson N., *On a class of non-vanishing functions*, Proc. London Math. Soc.(2) **41** (1936), no. 5, 393–407.

[17] Мергелян С. Н., *Равномерные приближения функций комплексного переменного*, Успехи мат. наук **7** (1952), 31–122.

[18] Мергелян С. Н, *О полноте систем аналитических функций*, Успехи мат. наук **8** (1953), №4, 3–63.

[19] Мергелян С. Н., *Весовые приближения многочленами*, Успехи мат. наук **11** (1956), №5, 107–152.

[20] Pommerenke Ch., *Boundary behavior of conformal maps*, Grundlehren Math. Wiss., Bd. 299, Springer-Verlag, Berlin, 1992.

[21] Синанян С. О., *Аппроксимация аналитическими функциями и полиномами в среднем по площади*, Мат. сб. **69** (1966), №4, 546–578.

[22] Вольберг А. Л., *Логарифм почти аналитической функции суммируем*, Докл. АН СССР **265** (1982), №6, 1297–1302.

[23] Вольберг А. Л., Ёрикке Б., *Суммируемость логарифма почти аналитической функции и обобщение теоремы Левинсона–Картрайт*, Мат. сб. **130** (1986), №3, 335–348.

[24] Wermer J., *Seminar über Funktionen-Algebren*, Lecture Notes in Math., vol. 1, Springer-Verlag, Berlin, 1964.

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