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ON THE INVARIANCE OF WELSCHINGER INVARIANTS

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Some observations about original Welschinger invariants defined in the paper *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. **162** (2005), no. 1, 195–234, are collected. None of their proofs is difficult, nevertheless these remarks do not seem to have been made before. The main result is that when $X_{\mathbb{R}}$ is a real rational algebraic surface, Welschinger invariants only depend on the number of real interpolated points, and some homological data associated with $X_{\mathbb{R}}$. This strengthened the invariance statement initially proved by Welschinger.

This main result follows easily from a formula relating Welschinger invariants of two real symplectic manifolds that differ by a surgery along a real Lagrangian sphere. In its turn, once one believes that such a formula may hold, its proof is a mild adaptation of the proof of analogous formulas previously obtained by the author on the one hand, and by Itenberg, Kharlamov, and Shustin on the other hand.

The two aforementioned results are applied to complete the computation of Welschinger invariants of real rational algebraic surfaces, and to obtain vanishing, sign, and sharpness results for these invariants that generalize previously known statements. Some hypothetical relationship of the present work with tropical refined invariants defined in the papers *Refined curve counting with tropical geometry*, Compos. Math. **152** (2016), no. 1, 115–151, and *Refined broccoli invariants*, J. Algebraic Geom. **28** (2019), no. 1, 1–41, is also discussed.

§1. Main results

A *real symplectic manifold*

$$X_{\mathbb{R}} = (X, \omega_X, \tau_X)$$

is a symplectic manifold (X, ω_X) equipped with an antisymplectic involution τ_X. The *real part* of (X, ω_X, τ_X), denoted by $\mathbb{R}X$, is by definition the

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fixed point set of τ_X. An almost complex structure J on X is said to be τ_X-*compatible* if it is tamed by ω, and if τ_X is J-antiholomorphic. In this note, the manifold $X_{\mathbb{R}}$ will always be compact of dimension 4 with a nonempty real part, and we denote by $H_2^{-\tau_X}(X;\mathbb{Z})$ the space of τ_X-anti-invariant classes. A nonsingular projective real algebraic variety is always implicitly assumed to be equipped with some Kähler form, which turns it into a real symplectic manifold. All algebraic surfaces considered here are assumed to be projective and nonsingular.

Let $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ be a real compact symplectic manifold of dimension 4, and denote by $L_1, \ldots, L_k$ the connected components of $\mathbb{R}X$. Choose a class $d \in H_2(X;\mathbb{Z})$, and a vector $\rho = (r_1, \ldots, r_k) \in \mathbb{Z}^k_{\geqslant 0}$ such that

$$c_1(X) \cdot d - 1 - \sum_{i=1}^{k} r_i = 2s \in 2\mathbb{Z}_{\geqslant 0}.$$

Choose a configuration $\underline{x}$ made of r_i points in L_i for $i = 1, \cdots k$, and s pairs of τ_X-conjugated points in $X \setminus \mathbb{R}X$. Given a τ_X-compatible almost complex structure J, we denote by $\mathcal{C}_{X_{\mathbb{R}}}(d, \underline{x}, J)$ the set of real rational J-holomorphic curves in X realizing the class d, and passing through $\underline{x}$. Then we define the integer

$$W_{X_{\mathbb{R}},\rho}(d;s) = \sum_{C \in \mathcal{C}_{X_{\mathbb{R}}}(d,\underline{x},J)} (-1)^{m(C)},$$

where $m(C)$ is the number of nodes of C in $\mathbb{R}X$ with two τ_X-conjugated branches. For a generic choice of J, the set $\mathcal{C}_{X_{\mathbb{R}}}(d, \underline{x}, J)$ is finite, and $W_{X_{\mathbb{R}},\rho}(d;s)$ depends neither on $\underline{x}$, J, nor on the deformation class of $X_{\mathbb{R}}$ (see [Wel05, Wel15]). We call these numbers the *Welschinger invariants of* $X_{\mathbb{R}}$. The main result of this paper, Theorem 1.3 below, is that when $X_{\mathbb{R}}$ is a real rational algebraic surface, Welschinger invariants eventually only depend on s and some homological data of $X_{\mathbb{R}}$.

Remark 1.1. The set $\mathcal{C}_{X_{\mathbb{R}}}(d, \underline{x}, J)$ is clearly empty when either $c_1(X) \cdot d \leqslant 0$, or $d \notin H_2^{-\tau_X}(X;\mathbb{Z})$, or the partition ρ contains two positive elements. This implies that $W_{X_{\mathbb{R}},\rho}(d;s) = 0$ in these cases.

Remark 1.2. We restrict ourselves to the original Welschinger invariants as defined in [Wel05], and do not consider the more general modified Welschinger invariants defined in [IKS13, IKS15, IKS17], and also considered in [BP15, Bru18]. Further, because of Remark 1.1, the invariants $W_{X_{\mathbb{R}},\rho}(d;s)$ are usually considered with a partition ρ containing a single positive entry, which corresponds to a connected component L of $\mathbb{R}X$. In this case, the invariant $W_{X_{\mathbb{R}},\rho}(d;s)$ in this text corresponds to the invariant $W_{X_{\mathbb{R}},L,[\mathbb{R}X\setminus L]}(d;s)$

in [BP15, Bru18], to the invariant $W_s(X_{\mathbb{R}}, d, L, 0)$ in [IKS17], and to the invariant $W(X_{\mathbb{R}}, d, L, 0)$ in [IKS13, IKS15] if $s = 0$. Our motivation to consider arbitrary partitions ρ comes from Theorem 1.3.

Two real rational algebraic surfaces $X_{1,\mathbb{R}}$ and $X_{2,\mathbb{R}}$ are said to be *homologically equivalent* if both are obtained, up to deformation, as a real blow-up $\pi_i : X_{i,\mathbb{R}} \to X_{0,\mathbb{R}}$ of a real minimal algebraic surface $X_{0,\mathbb{R}}$ at p distinct real points and q distinct pairs of τ_{X_0}-conjugated points. We emphasize that the distributions of the p real points among connected components of $\mathbb{R}X_0$ may not coincide for π_1 and π_2. Note nevertheless that

$$\chi(\mathbb{R}X_1) = \chi(\mathbb{R}X_2) = \chi(\mathbb{R}X_0) - p.$$

Denoting by $E_1, \dots E_p$ (respectively, $F_1, \overline{F}_1, \dots, F_q, \overline{F}_q$) the exceptional divisors coming from the blow-ups at real points (respectively, at pairs of τ_{X_0}-conjugated points), we see that the map π_i induces the following decomposition

$$H_2(X_i;\mathbb{Z}) = H_2(X_0;\mathbb{Z}) \oplus \bigoplus_{j=1}^{p} \mathbb{Z}[E_j] \oplus \bigoplus_{j=1}^{q} \Big(\mathbb{Z}[F_j] \oplus \mathbb{Z}[\overline{F}_j]\Big)$$

which is orthogonal with respect to the intersection form. Furthermore, the action of τ_{X_i} is given by

$$\begin{aligned} \tau_{X_i,*}|_{H_2(X_0;\mathbb{Z})} &= \tau_{X_0,*}, \\ \tau_{X_i,*}([E_j]) &= -[E_j] \text{ for } j = 1, \dots, p, \\ \tau_{X_i,*}([F_j]) &= -[\overline{F}_j] \text{ for } j = 1, \dots, q. \end{aligned}$$

In particular, the two maps π_1 and π_2 provide an identification of the groups $H_2(X_1;\mathbb{Z})$ and $H_2(X_2;\mathbb{Z})$ commuting with both intersection forms and the action of the antisymplectic involutions. We denote by $[X_{\mathbb{R}}]$ the homological equivalence class of a real rational algebraic surface $X_{\mathbb{R}}$.

Theorem 1.3. *If $X_{\mathbb{R}}$ is a real rational algebraic surface, then $W_{X_{\mathbb{R}},\rho}(d;s)$ does not depend on ρ, nor on a particular representative of $[X_{\mathbb{R}}]$.*

As a consequence of Theorem 1.3, we simply denote by $W_{[X_{\mathbb{R}}]}(d;s)$ the invariant $W_{X_{\mathbb{R}},\rho}(d;s)$. Note that some particular instances of Theorem 1.3 have already been noticed in [Bru15, Corollary 6.11 and Theorem 7.5] and [Bru18, Corollary 4.5]. Further, from the classification of real rational algebraic surfaces up to deformation established in [DK02, Main Theorem and Theorem 2.4.1], it follows that all Welschinger invariants of projective real rational algebraic surfaces are determined by the following ones[1]:

[1]Recall that all real algebraic surfaces considered here are assumed to have a nonempty real part.

- $W_{[\mathbb{C}P^2_{p,q}]}(d;s)$, where $\mathbb{C}P^2_{p,q}$ is a real blow-up of $\mathbb{C}P^2$ at p real points and q pairs of complex conjugated points;
- $W_{[QH_{0,q}]}(d;s)$, where $QH_{0,q}$ is the real blow-up of the quadric hyperboloid in $\mathbb{C}P^3$ at q pairs of complex conjugated points;
- $W_{[QE_{0,q}]}(d;s)$, where $QE_{0,q}$ is a real blow-up of the quadric ellipsoid in $\mathbb{C}P^3$ at q pairs of complex conjugated points;
- $W_{[CBN_{p,q}]}(d;s)$, where $CBN_{p,q}$ is a real blow-up at p real points and q pairs of complex conjugated points of a real minimal conic bundle whose real part consists of the disjoint union of $N \geqslant 2$ spheres S^2;
- $W_{[DP2_{p,q}]}(d;s)$, where $DP2_{p,q}$ is a real blow-up at p real points and q pairs of complex conjugated points of a real minimal del Pezzo surface of degree 2 whose real part consists of the disjoint union of 4 spheres S^2;
- $W_{[DP1_{p,q}]}(d;s)$, where $DP1_{p,q}$ is a real blow-up at p real points and q pairs of complex conjugated points of a real minimal del Pezzo surface of degree 1 whose real part consists of the disjoint union of $\mathbb{R}P^2$ and 4 spheres S^2.

Remark 1.4. Loosely speaking, Theorem 1.3 states that $W_{X_\mathbb{R},\rho}(d;s)$ only depends on s and the lattice $H_2(X;\mathbb{Z})$ equipped with the intersection form and the action of $\tau_{X,*}$. It may be interesting to work this out more rigorously. It may also be interesting to study generalizations of Theorem 1.3 to modified Welschinger invariants introduced in [IKS13], as well as to to higher genus Welschinger invariants introduced in [Shu14], or to the higher dimensional invariants recently defined in [Geo16, GZ18].

Theorem 1.3 easily implies the next corollary, which generalizes [BP15, Theorem 1.1(1)] in the case where $F = [\mathbb{R}X_\mathbb{R} \setminus L]$.

Corollary 1.5. *Let $X_\mathbb{R}$ be a compact real rational algebraic surface with a disconnected real part. Suppose that $X_\mathbb{R}$ is a real blow-up of another real rational algebraic surface at at least two real points, and denote by E_1 and E_2 the corresponding exceptional divisors. Then for any $d \in H_2(X;\mathbb{Z})$ such that both $d \cdot [E_1]$ and $d \cdot [E_2]$ are odd, one has $W_{[X_\mathbb{R}]}(d;s) = 0$.*

Combining Theorem 1.3 with [Wel07, Theorem 1.1] and Corollary 1.5, we obtain the following.

Theorem 1.6. *Let $X_\mathbb{R}$ be a compact real rational algebraic surface with a disconnected real part, and assume that $c_1(X) \cdot d - 1 - 2s > 0$. Then one has*

$$(-1)^{\frac{d^2 - c_1(X)\cdot d + 2}{2}} \cdot W_{[X_\mathbb{R}]}(d;s) \geqslant 0.$$

Furthermore, the invariant $W_{[X_\mathbb{R}]}(d;s)$ is sharp in the following sense: there exists a compact real rational algebraic surface $Y_\mathbb{R}$ in $[X_\mathbb{R}]$, a real configuration $\underline{x}$

of $c_1(X)\cdot d-1$ *points in* Y *with* $|\underline{x}\cap\mathbb{R}Y| = c_1(Y)\cdot d-2s$, *and a generic* τ_Y*-compatible almost complex structure* J *on* Y *such that*

$$\mathrm{Card}(\mathcal{C}_{Y_\mathbb{R}}(d,\underline{x},J)) = |W_{[X_\mathbb{R}]}(d;s)|.$$

Remark 1.7. A configuration $\underline{x}$ and a τ_Y-compatible almost complex structure J as in Theorem 1.6 may not exist for any representative $Y_\mathbb{R}$ of $[X_\mathbb{R}]$, even up to deformation, see [Bru15, Remark 6.13].

One of the main ingredients in our proof of Theorem 1.3 is Theorem 2.1 that relates Welschinger invariants of two real symplectic 4-manifolds differing by a so-called *surgery along a real Lagrangian sphere.* We refer to §2 for more details about this operation, and for the statement of Theorem 2.1 together with its proof. The last-mentioned theorem partially generalizes both [IKS15, Corollary 4.2] and [Bru18, Theorem 1.1, Remark 1.3]. We point out that its proof is an easy adaptation of the proof of [IKS15, Corollary 4.2], by using [BP15, Theorem 2.5(1)]. It merely required to believe in the correctness of the statement to prove it.

Combining [DK02, Main Theorem, and Theorem 2.4.1] together with the classical rigid isotopy classifications of plane real quartics, of real cubic sections of the quadratic cone in $\mathbb{C}P^3$, and of real quadrics in $\mathbb{C}P^3$ (see for example [DK00]), one easily classifies real rational algebraic surfaces up to deformation, real blow-up, and surgery along a real Lagrangian sphere: any real rational surface is obtained by a finite sequence of these three operations starting from either $\mathbb{C}P^2$ or the quadric hyperboloid QH. In particular we have the following result.

Theorem 1.8. *Let* $X_\mathbb{R}$ *be a real algebraic rational surface. Then by finitely many successive applications of Theorem* 2.1, *all Welschinger invariants of* $X_\mathbb{R}$ *can be computed out of Welschinger invariants of either* $\mathbb{C}P^2_{p,0}$ *or* QH.

Since all Welschinger invariants of QH and $CP^2_{p,0}$ have been computed, see for example [Mik05, BM08, HS12, Che18], Theorem 1.8 completes the computation of Welschinger invariants $W_{[X_\mathbb{R}]}(d;s)$ of real rational algebraic surfaces.

The next statement can be viewed as a property for $W_{[X_\mathbb{R}]}$ to increase with respect to $\chi(\mathbb{R}X_\mathbb{R})$, and goes in a somewhat different direction than [BP13, Theorem 3.4], [BP15, Proposition 2.8], and [Bru15, Corollaries 4.4 and 6.10].

Theorem 1.9. *Let* $X_\mathbb{R}$ *and* $Y_\mathbb{R}$ *be two compact real rational algebraic surfaces with a disconnected real part, differing by a surgery along a real Lagrangian sphere, and such that* $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$. *Then for any* $d \in H_2^{-\tau_Y}(Y;\mathbb{Z})$ *and* $s \in \mathbb{Z}_{\geqslant 0}$ *such that* $c_1(X)\cdot d - 1 - 2s > 0$, *one has*

$$|W_{[Y_\mathbb{R}]}(d;s)| \geqslant |W_{[X_\mathbb{R}]}(d;s)|.$$

Theorem 2.1 is obtained thanks to a real version of a (very simple instance) of the symplectic sum formula from [IP04, LR01, TZ14], see also [Li02, Li04] for an analogous formula in the complex algebraic category. It turns out that the same strategy provides a formula similar to Theorem 2.1 for relative Gromov–Witten invariants of symplectic 4-manifolds. This observation suggests a possible relationship of our work with tropical refined invariants defined in [BG16, GS19]. We discuss this aspect in §4. In particular, we provide there an alternative explanation for the specializations in $q = \pm 1$ of the tropical refined descendant invariants from [GS19]. We also show that a refined version of a conjecture by Itenberg, Kharlamov and Shustin [IKS04, Conjecture 6] holds true, although it was known to be wrong in the nonrefined case.

The remaining part of the paper is organised as follows. We introduce surgeries along real Lagrangian spheres in §2, then state and prove Theorem 2.1. All statements given in the present section can easily be derived from Theorem 2.1 and previously known results. Proofs are given in §3. Finally, in §4 we discuss relationship of our work with tropical refined invariants of algebraic surfaces and refined Severi degrees.

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§2. Surgery along a real Lagrangian sphere

Let $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ be a real compact symplectic manifold of dimension 4, and let $S \subset X$ be a Lagrangian sphere globally invariant under τ_X. Locally, a neighborhood V of S is given by a neighborhood in the real affine quadric (Q, ω_Q, τ) in $\mathbb{C}^3$ defined by the equation

$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = 1 \qquad \text{with } \varepsilon_i \in \{0,1\},$$

of the sphere S_Q in $i^{\varepsilon_1}\mathbb{R} \times i^{\varepsilon_2}\mathbb{R} \times i^{\varepsilon_3}\mathbb{R}$ with the equation

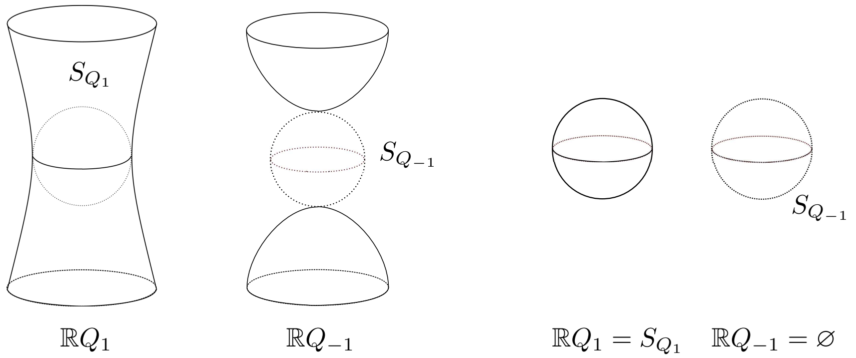
$$x^2 + y^2 + z^2 = 1.$$

As explained in [Bru18, Subsection 2.2], one can modify the symplectic and real structure of $X_{\mathbb{R}}$ in V so that V is now given by a neighborhood of S_Q in the real affine quadric in $\mathbb{C}^3$ with equation

$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = -1.$$

The resulting real symplectic manifold $Y_{\mathbb{R}}$ is called a *surgery of $X_{\mathbb{R}}$ along S* (see Figure 1 for a local picture). Note that, with the convention that $\chi(\varnothing) = 0$, we have

$$\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) \pm 2.$$



a) Q_t with equation $x^2+y^2-z^2=t$ b) Q_t with equation $x^2+y^2+z^2=t$

Figure 1. Surgery of a real symplectic 4-manifold along a real Lagrangian sphere.

Furthermore the class $[S]$ in $H_2(X;\mathbb{Z})$ is τ_X-anti-invariant if and only if it is τ_Y-invariant (in which case we have $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$).

Suppose that $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$, and let ρ be a vector whose entries are indexed by connected components of $\mathbb{R}Y$. If $S \subset \mathbb{R}Y$, we further assume that the entry of ρ corresponding to S vanishes. We denote by S^{τ_X} (respectively, S^{τ_Y}) the fixed point set of the involution $\tau_X|_S$ (respectively, $\tau_Y|_S$). In particular, we have either $S^{\tau_X} = S^1$ and S^{τ_Y} consists in 2 points, or $S^{\tau_X} = \varnothing$ and $S^{\tau_Y} = S$. The connected components of $\mathbb{R}Y \setminus S^{\tau_Y}$ are canonically in one-to-one correspondence with the connected components of $\mathbb{R}X \setminus S^{\tau_X}$. Hence one can associate to ρ a vector $\widetilde{\rho}$ whose entries are indexed by the connected components of $\mathbb{R}X$: the entry corresponding to the connected component L of $\mathbb{R}X$ is the sum of the entries corresponding to the connected component of $\mathbb{R}Y \setminus S^{\tau_Y}$ corresponding to $L \setminus S^{\tau_X}$.

The next theorem partially generalizes both [IKS15, Corollary 4.2] and [Bru18, Theorem 1.1, Remark 1.3].

Theorem 2.1. *Let $X_\mathbb{R}$ be a compact real symplectic manifold of dimension 4. Let S be a real Lagrangian sphere in $X_\mathbb{R}$, and $Y_\mathbb{R}$ a surgery of $X_\mathbb{R}$ along S. Suppose that $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$. Then for any class $d \in H_2^{-\tau_Y}(X;\mathbb{Z})$, the following identity holds:*

$$W_{Y_\mathbb{R},\rho}(d;s) = \sum_{k\in\mathbb{Z}} (-1)^k \; W_{X_\mathbb{R},\widetilde{\rho}}(d-k[S];s),$$

whenever the entry of ρ corresponding to S vanishes if $S \subset \mathbb{R}Y$.

Proof. Note first that the identity we want to prove is independent on the orientation chosen on S to define an element $[S] \in H_2(X;\mathbb{Z})$. By [Bru18, Remark 1.3], this identity can be rewritten in the following form once such an orientation is chosen:

$$W_{Y_\mathbb{R},\rho}(d;s) = W_{X_\mathbb{R},\widetilde{\rho}}(d;s) + 2\sum_{k\geqslant 1} (-1)^k \; W_{X_\mathbb{R},\widetilde{\rho}}(d-k[S];s). \tag{1}$$

Recall (see [Bru18, Section 2.2]) that $X_\mathbb{R}$ can be deformed to a real symplectic 4-manifold (Z,ω_Z,τ_Z) for which S, equipped with the chosen orientation, becomes symplectic. Choose a real configuration of points $\underline{x}$ in Z with s pairs of τ_Z-conjugated points, and ρ_L points in L for each connected component L of $\mathbb{R}Z \setminus \mathbb{R}S$. Choose also a τ_Z-compatible almost complex structure J on Z for which S is J-holomorphic. Given an integer $k \geqslant 0$, we denote by $\mathcal{C}^\beta(d-k[S],\underline{x},J)$ the set of all irreducible rational real J-holomorphic curves in (Z,ω_Z,τ_Z) passing through all points in $\underline{x}$, realizing the class $d-k[S]$, and intersecting $S\setminus\mathbb{R}S$ in exactly β pairs of τ_Z-conjugated points. For a generic choice of J satisfying the above conditions, the set $\mathcal{C}^\beta(d-k[S],\underline{x},J)$ is finite and only contains nodal curves by [BP15, Lemma 3.1 and Proposition 3.3]. We define

$$W^\beta_{Z_\mathbb{R},\widetilde{\rho}}(d-k[S];s) = \sum_{C\in\mathcal{C}^\beta(d-k[S],\underline{x},J)} (-1)^{m(C)}.$$

Note that $W^\beta_{Z_\mathbb{R},\widetilde{\rho}}(d-k[S];s)$ may depend on the choices of $\underline{x}$ and J, nevertheless we will not record this dependence in our notation in order to lighten the exposition. By [BP15, Theorem 2.5(1)], we have

$$W_{Y_\mathbb{R},\rho}(d;s) = \sum_{l\geqslant 0}(-2)^l \; W^l_{Z_\mathbb{R},\widetilde{\rho}}(d-l[S];s),$$

and

$$W_{X_\mathbb{R},\widetilde{\rho}}(d;s) = \sum_{j,b,\beta\geqslant 0} \begin{pmatrix} d\cdot[S]+2j-2b \\ j-2\beta \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix} W^b_{Z_\mathbb{R},\widetilde{\rho}}(d-j[S];s).$$

Denoting by A the right-hand side of (1), and using the last identity, we obtain

$$\begin{aligned} A = &\sum_{j,b,\beta\geqslant 0} \binom{2j-2b}{j-2\beta}\binom{b}{\beta} W^b_{Z_\mathbb{R},\widetilde{\rho}}(d-j[S];s) \\ &+ 2\sum_{k\geqslant 1}(-1)^k \sum_{j,b,\beta\geqslant 0}\binom{2k+2j-2b}{j-2\beta}\binom{b}{\beta} W^b_{Z_\mathbb{R},\widetilde{\rho}}(d-(j+k)[S];s). \end{aligned}$$

By the change of variables $j = l - k$, we obtain

$$A = \sum_{l,b,\beta\geqslant 0} \binom{2l-2b}{l-2\beta}\binom{b}{\beta} W^b_{Z_{\mathbb{R}},\widetilde{\rho}}(d-l[S];s)$$
$$+ 2\sum_{k,l\geqslant 1}\sum_{b,\beta\geqslant 0}(-1)^k \binom{2l-2b}{l-k-2\beta}\binom{b}{\beta} W^b_{Z_{\mathbb{R}},\widetilde{\rho}}(d-l[S];s).$$

Hence the coefficient of $W^b_{Z_{\mathbb{R}},\widetilde{\rho}}(d-l[S];s)$ in A is

$$\sum_{\beta\geqslant 0}\binom{2l-2b}{l-2\beta}\binom{b}{\beta} + 2\sum_{k\geqslant 1}\sum_{\beta\geqslant 0}(-1)^k\binom{2l-2b}{l-k-2\beta}\binom{b}{\beta},$$

that is to say

$$\sum_{\beta\geqslant 0}\left(\binom{2l-2b}{l-2\beta} + 2\sum_{k\geqslant 1}(-1)^k\binom{2l-2b}{l-k-2\beta}\right)\binom{b}{\beta}.$$

We denote by $u_{l,b,\beta}$ the coefficient of $\binom{b}{\beta}$ in the last sum. We have

$$u_{l,b,b-\beta} = \binom{2l-2b}{l-2\beta} + 2\sum_{k\geqslant 1}(-1)^k\binom{2l-2b}{l+k-2\beta}.$$

Hence we get

$$u_{l,b,\beta} + u_{l,b,b-\beta} = 2\times(-1)^l \sum_{p\geqslant 0}^{2l-2b}(-1)^p\binom{2l-2b}{p},$$

and so

$$u_{l,b,\beta} + u_{l,b,b-\beta} = 0$$

if $l > b$, and

$$u_{l,l,\beta} + u_{l,l,l-\beta} = 2\times(-1)^l.$$

This implies that

$$\sum_{\beta\geqslant 0} u_{l,b,\beta}\binom{b}{\beta} = 0$$

if $l > b$, and

$$\sum_{\beta\geqslant 0} u_{l,l,\beta}\binom{l}{\beta} = (-2)^l,$$

which is precisely what we have to prove. □

Remark 2.2. Theorem 2.1 can clearly be generalised to modified Welschinger invariants introduced in [IKS13], at the cost of much heavier notation. It may be nevertheless interesting to work out such a generalisation.

As an example of application of Theorem 2.1, we have the following.

Proposition 2.3. *Let $X_\mathbb{R}$ be a compact real symplectic manifold of dimension 4. Let $\widetilde{X}_\mathbb{R}$ be the blow up of $X_\mathbb{R}$ at two disjoint real balls, and $\widetilde{Y}_\mathbb{R}$ the blow up of $X_\mathbb{R}$ at two τ_X-conjugated disjoint balls. In both cases we denote by E_1 and E_2 the two exceptional divisors. Then for any class $d \in H^{-\tau_X}(X;\mathbb{Z})$ and any $l \in \mathbb{Z}_{\geqslant 0}$, one has*

$$W_{\widetilde{Y}_\mathbb{R},\rho}(d-l[E_1]-l[E_2];s) = W_{\widetilde{X}_\mathbb{R},\rho}(d-l[E_1]-l[E_2];s) + 2\sum_{\lambda=1}^{l}(-1)^\lambda\, W_{\widetilde{X}_\mathbb{R},\rho}(d-(l-\lambda)[E_1]-(l+\lambda)[E_2];s).$$

The notation ρ both for $\widetilde{X}_\mathbb{R}$ and $\widetilde{Y}_\mathbb{R}$ in the theorem makes sense because the connected components of $\mathbb{R}\widetilde{X}$ and $\mathbb{R}\widetilde{Y}$ are in a canonical one-to-one correspondence.

Proof. This is Theorem 2.1 applied to the class $[E_1]-[E_2]$. □

Proposition 2.3 applied with $l = 1$, combined with [DH18, Theorems 1.1 and 1.2], specializes to the Welschinger formula [Wel05, Theorem 0.4].

Corollary 2.4. *Let $X_\mathbb{R}$ be a compact real symplectic manifold of dimension 4, and let $\widetilde{X}_\mathbb{R}$ be the blow up of $X_\mathbb{R}$ at one real ball. We denote by E the exceptional divisor. Then for any $d \in H^{-\tau_X}(X;\mathbb{Z})$ and $s \in \mathbb{Z}_{\geqslant 0}$ such that $c_1(X)\cdot d - 2s \geqslant 3$, one has*

$$W_{X_\mathbb{R},\rho}(d;s+1) = W_{X_\mathbb{R},\rho}(d;s) - 2W_{\widetilde{X}_\mathbb{R},\rho}(d-2E;s).$$

§3. Proofs of the main results

As mentioned in §1, a classification of real rational algebraic surfaces up to deformation, real blow-up, and surgery along a real Lagrangian sphere is easily obtained by combination of [DK02, Main Theorem and Theorem 2.4.1] together with the classical rigid isotopy classifications of plane real quartics, of real cubic sections of the quadratic cone in $\mathbb{C}P^3$, and of real quadrics in $\mathbb{C}P^3$ (see for example [DK00]). We implicitly use this classification in the following five proofs.

Proof of Theorem 1.3. The surface $X_\mathbb{R}$ can be degenerated to a nodal real algebraic rational surface $\overline{X}_\mathbb{R}$ with a connected real part and having only real nodes. Blowing up these nodes, we obtain a nonsingular real algebraic rational surface $Z_\mathbb{R}$ with a connected real part. By construction $X_\mathbb{R}$ is obtained, up to deformation, by surgeries along the disjoint union of the exceptional divisors of the desingularization $Z_\mathbb{R} \to \overline{X}_\mathbb{R}$. Since $\mathbb{R}Z$ is connected, the Welschinger invariants of $Z_{\mathbb{R}}$ depend neither on the choice of $\underline{x}$ nor on the position of the real blown-up points on a minimal model of Z. Now the proposition is an immediate consequence of Theorem 2.1. $\square$

Proof of Corollary 1.5. Up to choosing another representative of $[X_{\mathbb{R}}]$, we may assume that $\mathbb{R}E_1$ and $\mathbb{R}E_2$ do not lie on the same connected component of $\mathbb{R}X$. In particular, by the connectedness of $\mathbb{R}P^1$, the set $\mathcal{C}_{X_{\mathbb{R}}}(d, \underline{x}, J)$ is clearly empty for any configuration $\underline{x}$. $\square$

Proof of Theorem 1.6. Up to choosing another representative of $[X_{\mathbb{R}}]$, we may assume that a connected component L of $\mathbb{R}X$ is homeomorphic to the sphere S^2. If $c_1(X) \cdot d - 1 - 2s \geqslant 2$, the result follows from Remark 1.1. If $c_1(X) \cdot d - 1 - 2s = 1$, then we choose the configuration $\underline{x}$ such that $\underline{x} \cap \mathbb{R}X \subset L$. Now the result follows from [Wel07, Theorem 1.1]. $\square$

Remark 3.1. If $c_1(X) \cdot d - 1 - 2s = 0$, then [Wel07, Theorem 1.1] states that one can find $\underline{x}$ and J so that there are no curve C in $\mathcal{C}_{X_{\mathbb{R}}}(d, \underline{x}, J)$ with $\mathbb{R}C \subset L$. Nevertheless, there may exist curves C in $\mathcal{C}_{X_{\mathbb{R}}}(d, \underline{x}, J)$ with $\mathbb{R}C \subset \mathbb{R}X \setminus L$.

Proof of Theorem 1.8. Any real rational algebraic surface can be obtained by a finite sequence of deformations, real blow-ups, and surgery along real Lagrangian spheres, starting with either $\mathbb{C}P^2$ or the quadric hyperboloid QH in $\mathbb{C}P^3$.

Let $Y_{1,\mathbb{R}}$ and $Y_{2,\mathbb{R}}$ be two real algebraic surfaces obtained by blowing up a real rational surface $Y_{\mathbb{R}}$ at respectively two real and τ_Y-conjugated points. When we denote by E_1 and E_2 the two exceptional divisors, the real surface $Y_{2,\mathbb{R}}$ is obtained by a surgery along a real Lagrangian sphere realizing the class $[E_1] - [E_2]$. Since QH blown-up at $l \geqslant 1$ real points is also $\mathbb{C}P^2$ blown-up at $l + 1$ real points, the result follows. $\square$

Proof of Theorem 1.9. Let S be a real Lagrangian sphere allowing to pass from $X_{\mathbb{R}}$ to $Y_{\mathbb{R}}$ by surgery. Since one has $[S]^2 = -2$ and $c_1(X) \cdot [S] = 0$, from Theorem 1.6 it follows that $W_{[X_{\mathbb{R}}]}(d; s)$ and $(-1)^k\ W_{[X_{\mathbb{R}}]}(d - k[S]; s)$ have the same sign. Now the result follows from Theorem 2.1. $\square$

§4. Hypothetical relationship with tropical refined invariants

We end this note by pointing out a possible relationship of Theorem 2.1 with tropical refined invariants of algebraic surfaces, and refined Severi degrees.

4.1. Tropical refined invariants and surgeries along real Lagrangian spheres. A nondegenerate convex polygon $\Delta \subset \mathbb{R}^2$ with vertices in $\mathbb{Z}^2$ defines a complex toric surface X_Δ together with a complete linear system d. Block and Göttsche proposed in [BG16] to enumerate irreducible tropical curves with Newton polygon Δ and genus g as proposed in [Mik05], but replacing Mikhalkin's complex multiplicity with its quantum analog. In this way, one obtains a Laurent polynomial in the variable q, called the *tropical refined invariant* and denoted by $G_{X_\Delta}(d,g)$, which does not depend on the configuration of points chosen to define it [IM13]. By [Mik05], the value $G_{X_\Delta}(d,g)(1)$ recovers the number of complex irreducible algebraic curves of genus g in X_Δ realizing the class d, and passing through a generic configuration of $c_1(X_\Delta)\cdot d-1+g$ points. Any complex toric surface has a standard real structure induced by the standard real structure on $(\mathbb{C}^*)^2$, and we denote by $X_{\Delta,\mathbb{R}}$ the corresponding real toric surface. It also follows from [Mik05] that when $X_{\Delta,\mathbb{R}}$ is a real unnodal toric del Pezzo surface (i.e., $\mathbb{C}P^1\times\mathbb{C}P^1$ or $\mathbb{C}P^2$ blown up at at most 3 real points not contained in a line), one has

$$G_\Delta(0)(-1)=W_{[X_{\Delta,\mathbb{R}}]}(d;0).$$

In other words, tropical refined invariants interpolate between Gromov–Witten invariants (for $q=1$) and Welschinger invariants with $s=0$ (for $q=-1$) of $X_{\Delta,\mathbb{R}}$ when both are defined. Tropical refined invariants are conjectured to agree with the χ_y-refinement of Severi degrees introduced in [GS14].

In the case of a real unnodal toric del Pezzo surface $X_{\Delta,\mathbb{R}}$, Göttsche and Schroeter defined in [GS19] some tropical refined descendant invariants, denoted by $G_{X_\Delta}(d,0;s)$, interpolating between some genus 0 tropical descendent invariants and Welschinger invariants $W_{[X_{\Delta,\mathbb{R}}]}(d;s)$, and this for arbitrary values of s. This work was then generalized to all genus 0 tropical descendant invariants by Blechman and Shustin in [BS19], leaving open the general interpretation of the value at $q=-1$ of these new tropical refined descendant invariants.

Despite recent progress, see [Mik17, NPS18, Bou17], a general enumerative interpretation of tropical refined invariants unifying their complex and real aspects remains unknown.

Nevertheless, tropical refined invariants seem to behave nicely under degenerations of the ambient algebraic surface (or symplectic 4-manifold). In particular, the relationship of our results with tropical refined invariants may be suggested by the following observation: relative Gromov–Witten invariants of symplectic 4-manifolds satisfy a formula similar to that from Theorem 2.1. To make this precise, we need first to introduce some additional notation.

Let (X,ω_X) be a compact symplectic manifold of dimension 4, containing a finite union $U=E_1\cup\ldots\cup E_\kappa$ of pairwise disjoint embedded symplectic spheres with $[E_i]^2=-2$. Let also J be an almost complex structure on X tamed by ω_X for which all curves $E_1,\ldots,E_\kappa$ are J-holomorphic. Given $d\in H_2(X;\mathbb{Z})$, let us choose a configuration $\underline{x}$ of $c_1(X)\cdot d-1$ distinct points in $X\setminus\bigcup_{i=1}^{\kappa}E_i$. We define $GW^U_{(X,\omega)}(d)$ as the number of irreducible J-holomorphic rational curves $f : C \to X$ with $f_*[C] = d$ passing through all points in $\underline{x}$ and whose image is not contained in $\bigcup_{i=1}^{\kappa} E_i$. For a generic choice of J, this number is finite and does not depend on $\underline{x}$, it is called the *Gromov–Witten invariant of* (X, ω) *relative to* U. Given an element E of U, we define $\widehat{U} = U \setminus \{E\}$. Given two integers m and k, we define

$$u_{m,k} = (-1)^k \left(\binom{m+k}{m} + \binom{m+k-1}{m} \right).$$

Proposition 4.1. *One has*

$$GW^U_{(X,\omega)}(d) = \sum_{k \geqslant 0} u_{d\cdot[E],k}\; GW^{\widehat{U}}_{(X,\omega)}(d - k[E]).$$

In particular, if $d \cdot [E] = 0$, *then one has*

$$GW^U_{(X,\omega)}(d) = \sum_{k \in \mathbb{Z}} (-1)^k GW^{\widehat{U}}_{(X,\omega)}(d - k[E]).$$

Proof. From [BP15, Corollary 3.8], it follows immediately that

$$GW^{\widehat{U}}_{(X,\omega)}(d) = \sum_{k \geqslant 0} \binom{d\cdot[E] + 2k}{k} GW^U_{(X,\omega)}(d - k[E]).$$

So the first formula follows from [Bru18, Proposition 3.12]. If $d \cdot [E] = 0$, this formula becomes

$$GW^U_{(X,\omega)}(d) = GW^{\widehat{U}}_{(X,\omega)}(d) + 2\sum_{k \geqslant 0} (-1)^k GW^{\widehat{U}}_{(X,\omega)}(d - k[E]).$$

Thanks to the identity $GW^U_{(X,\omega)}(d') = GW^U_{(X,\omega)}(d' + (d' \cdot [E])[E])$ for any $d' \in H_2(X;\mathbb{Z})$, we obtain the desired result. □

Combination of Theorem 2.1 with Proposition 4.1 implies the following "refined flavored" corollary.

Corollary 4.2. *Let* $X_{\mathbb{R}}$ *be either the quadric hyperboloid* QH *in* $\mathbb{C}P^3$, *or* $\mathbb{C}P^2$ *blown-up at finitely many real points. Let also* $U = E_1 \cup \ldots \cup E_\kappa$ *be a finite collection of pairwise disjoint real embedded symplectic spheres with* $[E_i]^2 = -2$. *Suppose that we are given a function* $\Gamma_{X_{\mathbb{R}}} : H_2(X;\mathbb{Z}) \to \mathbb{Z}[q, q^{-1}]$ *such that for any* $d \in H_2(X;\mathbb{Z})$, *one has*

$$\Gamma_{X_{\mathbb{R}}}(d)(1) = GW_{(X,\omega)}(d) \qquad \text{and} \qquad \Gamma_{X_{\mathbb{R}}}(d)(-1) = W_{[X_{\mathbb{R}}]}(d;0).$$

Let $Y_{\mathbb{R}}$ be the surgery of $X_{\mathbb{R}}$ along the real Lagrangian spheres $E_1, \ldots, E_\kappa$, and define the following function

$$\begin{array}{cccc} \Gamma_{Y_{\mathbb{R}}}\colon H_2(X;\mathbb{Z}) & \longrightarrow & \mathbb{Z}[q,q^{-1}] \\ d & \longmapsto & \displaystyle\sum_{k_1,\cdots,k_\kappa\geqslant 0}\left(\prod_{i=1}^{\kappa} u_{d\cdot[E_i],k_i}\right)\Gamma_{X_{\mathbb{R}}}(d-k_1[E_1]-\ldots k_\kappa[E_\kappa]). \end{array}$$

Then $\Gamma_{Y_{\mathbb{R}}}$ satisfies the two following properties:

$$\forall d\in H_2(X;\mathbb{Z}),\ \Gamma_{Y_{\mathbb{R}}}(d)(1) = GW^U_{(X,\omega)}(d);$$
$$\forall d\in H_2^{-\tau_Y}(X;\mathbb{Z}),\ \Gamma_{Y_{\mathbb{R}}}(d)(-1) = W_{Y_{\mathbb{R}}}(d;0).$$

Following Corollary 4.2, in the next two subsections we discuss a refined version of Corollary 2.4, and of [BP15, Proposition 2.7].

4.2. A refined Corollary 2.4. Refined tropical descendant invariants defined in [GS19] can be computed by using floor diagrams from [BGM12], equipped with refined weights as indicated in [BS19]. The next proposition, whose proof we omit, is an easy consequence of this floor diagrammatic computation. Recall that QH is the real quadric hyperboloid in $\mathbb{C}P^3$.

Proposition 4.3. *Let $X_{\mathbb{R}}$ be either QH or $\mathbb{C}P^2$ blown-up at at most two distinct real points, and let $\widetilde{X}_{\mathbb{R}}$ be the blow-up of $X_{\mathbb{R}}$ at one real point. We denote by E the exceptional divisor. Then for any $d\in H_2(X;\mathbb{Z})$ and $s\in\mathbb{Z}_{\geqslant 0}$ such that $c_1(X)\cdot d-2s\geqslant 3$, one has*

$$G_{X_{\mathbb{R}}}(d,0;s+1) = G_{X_{\mathbb{R}}}(d,0;s) - 2G_{\widetilde{X}_{\mathbb{R}}}(d-2E,0;s).$$

By Corollary 4.2, Proposition 4.3 is clearly a refined version of Corollary 2.4. In particular, induction on s provides an alternative proof of [GS19, Corollaries 3.14 and 3.29] (however it does not explain the tropical invariance of the numbers $G_{X_{\mathbb{R}}}(d,0;s)$).

Example 4.4. One computes easily, for example using floor diagrams, that the coefficient of degree $\frac{(d-1)(d-2)}{2}-1$ of $G_{\mathbb{C}P^2}(d,0;0)$ is $3d+1$. Since this coefficient is 1 for $G_{\widetilde{\mathbb{C}P^2}}(d-2E,0;s)$, we deduce from Proposition 4.3 that the coefficient of degree $\frac{(d-1)(d-2)}{2}-1$ of $G_{\mathbb{C}P^2}(d,0;s)$ is $3d+1-2s$. In particular, when s is maximal, this coefficient is equal to 2 or 3 depending on the parity of d.

Since all coefficients of $G_{X_{\mathbb{R}}}(d,0;s)$ are positive for an unnodal real del Pezzo toric surface $X_{\mathbb{R}}$, Proposition 4.3 immediately implies the following.

Corollary 4.5. *Let $X_{\mathbb{R}}$ be either QH or $\mathbb{C}P^2$ blown-up at at most two distinct real points. For $d \in H_2(X;\mathbb{Z})$, we denote by $a_{r,s}$ the coefficient of degree r of $G_{X_{\mathbb{R}}}(d,0;s)$. Then one has*

$$\forall r \in \left\{1, \cdots, \frac{d^2 - c_1(X)\cdot d + 2}{2}\right\}, \quad a_{r,0} \geqslant a_{r,1} \geqslant \cdots \geqslant a_{r,\left[\frac{c_1(X)\cdot d - 1}{2}\right]} > 0.$$

Corollary 4.5 may be regarded as a refined version of [IKS04, Conjecture 6]. It is amusing that although this conjecture was proved to be wrong in [Wel07] and [ABLdM11], its refined version is true eventually.

4.3. Refined invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and Σ_2. Recall that the quadric hyperboloid QH can be deformed to the second Hirzebruch surface Σ_2 equipped with its toric real structure. This can be done algebraically by degenerating the quadric hyperboloid QH to a nodal quadric, and by blowing up the node. The surgery of $X_{\mathbb{R}}$ along the expcetional curve in Σ_2 is, up to deformation, the quadric ellipsoid QE in $\mathbb{C}P^3$. Since both $\mathbb{C}P^1 \times \mathbb{C}P^1$ and Σ_2 are toric, Corollary 4.2 suggests the following conjecture.[2] [3]

Conjecture 4.6. *We denote by $\square_{a,b}$ (respectively, $\triangle_{a,b}$) the class in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$ (respectively, Σ_2) defined by the convex polygon in $\mathbb{R}^2$ with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) (respectively, $(0,0)$, $(2a+b,0)$, $(0,a)$, and (b,a)), see Figure 2. Then for any integers $a,b,g \geqslant 0$, the tropical refined invariants*

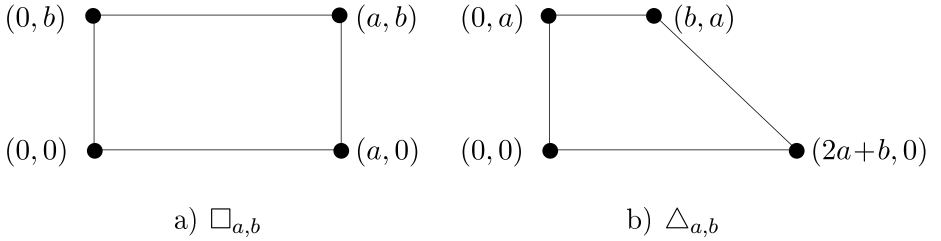


Figure 2

$G_{QH}(\square, g)$ and $G_{\Sigma_2}(\triangle, g)$ satisfy the following relations:

$$G_{\Sigma_2}(\triangle_{a,b}, g) = \sum_{k\geqslant 0} u_{b,k} G_{QH}(\square_{a+b+k,a-k}, g).$$

[2]As stated, Corollary 4.2 only suggests the conjecture for $g = 0$. Nevertheless, [Bru18, Corollary 3.17] suggests its extension to any genus.

[3]This conjecture has been proved by Bousseau by now. See [Bou19].

Note that this can be regarded as a refined version of [BP15, Proposition 2.7]: denoting by h the hyperplane section class of the quadric ellipsoid QE, one has

$$\forall a \geqslant 0,\ G_{\Sigma_2}(\triangle_{a,0}, 0)(-1) = W_{QE}(ah; 0).$$

In the next examples, we check that Conjecture 4.6 holds true in a few cases.

Example 4.7. The cases of $a = 1$ or $g = (a-1)(a-1+b)$ hold trivially.

The value of all refined invariants needed in the next examples are provided in Appendix A.

Example 4.8. In the case $(a, b) = (2, 0)$ and $g = 0$, one has

$$G_{\Sigma_2}(\triangle_{2,0}, 0; s) = G_{QH}(\square_{2,2}, 0; s) - 2G_{QH}(\square_{3,1}, 0; s).$$

Example 4.9. The case of $(a, b) = (2, 2)$ gives

$$\begin{aligned} g = 2: &\quad G_{\Sigma_2}(\triangle_{2,2}, 2) = G_{QH}(\square_{4,2}, 2);\\ g = 1: &\quad G_{\Sigma_2}(\triangle_{2,2}, 1) = G_{QH}(\square_{4,2}, 1);\\ g = 0: &\quad G_{\Sigma_2}(\triangle_{2,2}, 0; s) = G_{QH}(\square_{4,2}, 0; s) - 4G_{QH}(\square_{5,1}, 0; s). \end{aligned}$$

Example 4.10. A more interesting case is given by $(a, b) = (3, 0)$ and $g \in \{0, 1, 2, 3\}$. We obtain the following:

$$\begin{aligned} g=3: &\quad G_{\Sigma_2}(\triangle_{3,0}, 3) = G_{QH}(\square_{3,3}, 3) - 2G_{QH}(\square_{4,2}, 3);\\ g=2: &\quad G_{\Sigma_2}(\triangle_{3,0}, 2) = G_{QH}(\square_{3,3}, 2) - 2G_{QH}(\square_{4,2}, 2);\\ g=1: &\quad G_{\Sigma_2}(\triangle_{3,0}, 1) = G_{QH}(\square_{3,3}, 1) - 2G_{QH}(\square_{4,2}, 1);\\ g=0: &\quad G_{\Sigma_2}(\triangle_{3,0}, 0; s) = G_{QH}(\square_{3,3}, 0; s) - 2G_{QH}(\square_{4,2}, 0; s) + 2G_{QH}(\square_{5,1}, 0; s). \end{aligned}$$

A refined version of the strategy used in [BM16] may lead to a combinatorial proof of Conjecture 4.6, however it is not clear that such a combinatorial proof will be geometrically meaningful.

Appendix §A. Some computations of tropical refined invariants

In this appendix we provide a few values of the tropical refined invariants of tropical projective plane and Hirzebruch surfaces of small degree. All nontrivial computations of absolute refined invariants have been done by using floor diagrams, see [BM08, BG16, BIMS15]. The computations of tropical refined descendant invariants have been done by using floor diagrams from [BGM12] equipped with refined weights as indicated in [BS19].

A.1. $\mathbb{T}P^1 \times \mathbb{T}P^1$.

Here we give the values of $G_{QH}(\square_{a,b}, g)$ for a few a and b. Note that

$$G_{QH}(\square_{a,b}) = G_{QH}(\square_{b,a})$$

- $a = 1$:

$$G_{QH}(\square_{1,b}, 0) = 1$$

- $(a, b) = (2, 0)$:

$$\begin{aligned} g = 1: &\quad G_{QH}(\square_{2,2}, 1) = 1 \\ g = 0: &\quad G_{QH}(\square_{2,2}, 0; s) = q^{-1} + (10 - 2s) + q \end{aligned}$$

- $(a, b) = (2, 4)$:

$$\begin{aligned} g = 3: &\quad G_{QH}(\square_{2,4}, 3) = 1 \\ g = 2: &\quad G_{QH}(\square_{2,4}, 2) = 3q^{-1} + 22 + 3q \\ g = 1: &\quad G_{QH}(\square_{2,4}, 1) = 3q^{-2} + 36q^{-1} + 162 + 36q + 3q^2 \\ g = 0: &\quad G_{QH}(\square_{2,4}, 0; 0) = q^{-3} + 14q^{-2} + 95q^{-1} + 420 + 95q + 14q^2 + q^3 \\ &\quad G_{QH}(\square_{2,4}, 0; 1) = q^{-3} + 12q^{-2} + 71q^{-1} + 280 + 71q + 12q^2 + q^3 \\ &\quad G_{QH}(\square_{2,4}, 0; 2) = q^{-3} + 10q^{-2} + 51q^{-1} + 180 + 51q + 10q^2 + q^3 \\ &\quad G_{QH}(\square_{2,4}, 0; 3) = q^{-3} + 8q^{-2} + 35q^{-1} + 112 + 35q + 8q^2 + q^3 \\ &\quad G_{QH}(\square_{2,4}, 0; 4) = q^{-3} + 6q^{-2} + 23q^{-1} + 68 + 23q + 6q^2 + q^3 \\ &\quad G_{QH}(\square_{2,4}, 0; 5) = q^{-3} + 4q^{-2} + 15q^{-1} + 40 + 15q + 4q^2 + q^3 \end{aligned}$$

- $(a, b) = (3, 3)$:

$$\begin{aligned} g=4: &\quad G_{QH}(\square_{3,3}, 4)=1 \\ g=3: &\quad G_{QH}(\square_{3,3}, 3)=4q^{-1}+26+4q \\ g=2: &\quad G_{QH}(\square_{3,3}, 2)=6q^{-2}+64q^{-1}+256+64q+6q^2 \\ g=1: &\quad G_{QH}(\square_{3,3}, 1)=4q^{-3}+52q^{-2}+332q^{-1}+1144+332q+52q^2+4q^3 \\ g=0: &\quad G_{QH}(\square_{3,3}, 0; 0)=q^{-4}+14q^{-3}+109q^{-2}+592q^{-1}+2078+592q+109q^2+14q^3+q^4 \\ &\quad G_{QH}(\square_{3,3}, 0; 1)=q^{-4}+12q^{-3}+83q^{-2}+404q^{-1}+1270+404q+83q^2+12q^3+q^4 \\ &\quad G_{QH}(\square_{3,3}, 0; 2)=q^{-4}+10^{-3}+61q^{-2}+264^{-1}+742+264q+61q^2+10q^3+q^4 \\ &\quad G_{QH}(\square_{3,3}, 0; 3)=q^{-4}+8q^{-3}+43q^{-2}+164q^{-1}+414+164q+43q^2+8q^3+q^4 \\ &\quad G_{QH}(\square_{3,3}, 0; 4)=q^{-4}+6q^{-3}+29q^{-2}+96q^{-1}+222+96q+29q^2+6q^3+q^4 \\ &\quad G_{QH}(\square_{3,3}, 0; 5)=q^{-4}+4q^{-3}+19q^{-2}+52q^{-1}+118+52q+19q^2+4q^3+q^4 \end{aligned}$$

A.2. $\mathbb{T}\Sigma_2$**.**
Here we give the values of $G_{\Sigma_2}(\triangle_{a,b}, g)$ for a few a and b.

- $a = 1$**:**
$$G_{\Sigma_2}(\triangle_{1,b}, 0) = 1$$
- $(a, b) = (2, 0)$**:**
$$g = 1: \quad G_{\Sigma_2}(\triangle_{2,0}, 1) = 1$$
$$g = 0: \quad G_{\Sigma_2}(\triangle_{2,0}, 0; s) = q^{-1} + (8 - 2s) + q$$
- $(a, b) = (2, 2)$**:**

$$g = 3: \quad G_{\Sigma_2}(\triangle_{2,2}, 3) = 1$$
$$g = 2: \quad G_{\Sigma_2}(\triangle_{2,2}, 2) = 3q^{-1} + 22 + 3q$$
$$g = 1: \quad G_{\Sigma_2}(\triangle_{2,2}, 1) = 3q^{-2} + 36q^{-1} + 162 + 36q + 3q^2$$
$$g = 0: \quad G_{\Sigma_2}(\triangle_{2,2}, 0; 0) = q^{-3} + 14q^{-2} + 95q^{-1} + 416 + 95q + 14q^2 + q^3$$
$$G_{\Sigma_2}(\triangle_{2,2}, 0; 1) = q^{-3} + 12q^{-2} + 71q^{-1} + 276 + 71q + 12q^2 + q^3$$
$$G_{\Sigma_2}(\triangle_{2,2}, 0; 2) = q^{-3} + 10q^{-2} + 51q^{-1} + 176 + 51q + 10q^2 + q^3$$
$$G_{\Sigma_2}(\triangle_{2,2}, 0; 3) = q^{-3} + 8q^{-2} + 35q^{-1} + 108 + 35q + 8q^2 + q^3$$
$$G_{\Sigma_2}(\triangle_{2,2}, 0; 4) = q^{-3} + 6q^{-2} + 23q^{-1} + 64 + 23q + 6q^2 + q^3$$
$$G_{\Sigma_2}(\triangle_{2,2}, 0; 5) = q^{-3} + 4q^{-2} + 15q^{-1} + 36 + 15q + 4q^2 + q^3$$

- $(a, b) = (3, 0)$**:**

$$g=4: \quad G_{\Sigma_2}(\triangle_{3,0}, 4)=1$$
$$g=3: \quad G_{\Sigma_2}(\triangle_{3,0}, 3)=4q^{-1}+24+4q$$
$$g=2: \quad G_{\Sigma_2}(\triangle_{3,0}, 2)=6q^{-2}+58q^{-1}+212+58q+6q^2$$
$$g=1: \quad G_{\Sigma_2}(\triangle_{3,0}, 1)=4q^{-3}+46q^{-2}+260q^{-1}+820+260q+46q^2+4q^3$$
$$g=0: \quad G_{\Sigma_2}(\triangle_{3,0}, 0; 0)=q^{-4}+12q^{-3}+81q^{-2}+402q^{-1}+1240+402q+81q^2+12q^3+q^4$$
$$G_{\Sigma_2}(\triangle_{3,0}, 0; 1)=q^{-4}+10q^{-3}+59q^{-2}+262q^{-1}+712+262q+59q^2+10q^3+q^4$$
$$G_{\Sigma_2}(\triangle_{3,0}, 0; 2)=q^{-4}+8q^{-3}+41q^{-2}+162q^{-1}+384+162q+41q^2+8q^3+q^4$$
$$G_{\Sigma_2}(\triangle_{3,0}, 0; 3) = q^{-4} + 6q^{-3} + 27q^{-2} + 94q^{-1} + 192 + 94q + 27q^2 + 6q^3 + q^4$$
$$G_{\Sigma_2}(\triangle_{3,0}, 0; 4) = q^{-4} + 4q^{-3} + 17q^{-2} + 50q^{-1} + 88 + 50q + 17q^2 + 4q^3 + q^4$$
$$G_{\Sigma_2}(\triangle_{3,0}, 0; 5) = q^{-4} + 2q^{-3} + 11q^{-2} + 22q^{-1} + 40 + 22q + 11q^2 + 2q^3 + q^4$$

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Université de Nantes,
Laboratoire de Mathématiques Jean Leray,
2 rue de la Houssinière,
F-44322 Nantes Cedex 3, France
E-mail: `erwan.brugalle@math.cnrs.fr`

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