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**DISCRETE UNIVERSALITY OF THE RIEMANN
ZETA-FUNCTION AND UNIFORM DISTRIBUTION
MODULO 1**

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It is proved that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\varphi(k))$, $k \geq k_0$, $k \in \mathbb{N}$, of the Riemann zeta-function. Here the function $\varphi(t)$ has a continuous nonvanishing derivative on $[k_0, \infty)$ satisfying the estimate $\varphi(2t) \max_{t \leq u \leq 2t} (\varphi'(u))^{-1} \ll t$, and the sequence $\{a\varphi(k) : k \geq k_0\}$ with every real $a \neq 0$ is uniformly distributed modulo 1. Examples of $\varphi(t)$ are given.

§1. Introduction

As usual, let $\zeta(s)$, $s = \sigma + it$, be the Riemann zeta-function. In 1975, S. M. Voronin discovered [11] the universality property of the function $\zeta(s)$ in approximation of a wide class of analytic functions by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. For the precise statement of the Voronin theorem, it is convenient to use the following notation. Let

$$D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\},$$

let \mathcal{K} be the class of compact subsets of the strip D with connected complements, and let $H_0(K)$ with $K \in \mathcal{K}$ denote the class of continuous nonvanishing functions on K that are analytic in the interior of K . Then the last version of the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Here $\text{meas} A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The proof of the above statement can be found, for example, in [1, 4, 6, 10].

The above version of the Voronin theorem is of continuous type: τ in $\zeta(s + i\tau)$ can take an arbitrary real value. Also, there exists a discrete version of the Voronin universality theorem when τ takes values in a certain discrete set.

Key words: Riemann zeta-function, uniform distribution modulo 1, universality, weak convergence.

Denote by $\#A$ the cardinality of the set A . Then, for the same K and $f(s)$ as above, for every $\varepsilon > 0$ and $h > 0$ we have

$$\liminf_{N \rightarrow \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\} > 0.$$

The discrete universality was proposed by A. Reich in [9]. He proved that the Dedekind zeta-functions of number fields have a discrete universality property. In the case of the field of rational numbers \mathbb{Q} , the Reich theorem gives the above statement for the Riemann zeta-function.

The discrete set $\{kh : h = 0, 1, 2, \dots, h > 0\}$ is very simple. In [3], this set was replaced by $\{k^\alpha h : k = 0, 1, 2, \dots, h > 0\}$ with a fixed $0 < \alpha < 1$. There exists a problem to describe the sets for which a discrete universality theorem is valid. Our aim in this paper is to generalize the result of [3]. We denote by $U_1(k_0)$, $k_0 \in \mathbb{N}$, the class of real monotone increasing functions $\varphi(t)$ having a continuous nonvanishing derivative on $[k_0, \infty)$ and satisfying the estimate

$$\varphi(2t) \max_{t \leq u \leq 2t} (\varphi'(u))^{-1} \ll t$$

for $t \geq k_0$. Moreover, let $U_2(k_0)$ be the class of real functions $\varphi(t)$ on $[k_0, \infty)$ such that the sequence

$$\{a\varphi(k) : k \geq k_0\}$$

with every real $a \neq 0$ is uniformly distributed modulo 1. We remind the reader that the sequence

$$\{a_k : k \in \mathbb{N}\} \subset \mathbb{R}$$

is said to be uniformly distributed modulo 1 if for every real numbers a, b with $0 \leq a < b \leq 1$ we have

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq k \leq N : \{a_k\} \in [a, b]\}}{N} = b - a,$$

where $\{u\}$ is the fractional part of $u \in \mathbb{R}$.

Theorem 1. *Suppose that*

$$\varphi(t) \in U_1(k_0) \cap U_2(k_0).$$

Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then for every $\varepsilon > 0$ and $h > 0$ we have

$$\liminf_{N \rightarrow \infty} \frac{1}{N - k_0 + 1} \# \left\{ k_0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ih\varphi(k)) - f(s)| < \varepsilon \right\} > 0.$$

It is known (see [5, Excercise 3.9]) that the sequence $\{ak^\alpha : k \in \mathbb{N}\}$, where $\alpha > 0$ is not an integer and $a \neq 0$ is real, is uniformly distributed modulo 1. Therefore,

$$t^\alpha \in U_1(1) \cap U_2(1).$$

Let a and α be the same as above, and let $\beta \in \mathbb{R}^+$ be arbitrary. Then the sequence

$$\{ak^\alpha \log^\beta k : k \geq 2\}$$

is uniformly distributed modulo 1, see [5, Exercise 3.10]. It is easy to check that $t^\alpha \log^\beta t \in U_1(2)$. Therefore, Theorem 1 is valid with

$$\varphi(k) = k^\alpha \log^\beta k.$$

The functions

$$\varphi(t) = t^\alpha \log^\beta t,$$

with $\alpha \in \mathbb{N}$ and $\beta > 1$,

$$\varphi(t) = t \log^\beta t \text{ and } \varphi(t) = t^2 \log^\beta t,$$

with $0 < \beta \leq 1$, $t \geq 2$, see [5, Exercises 3.11 and 3.14], also satisfy the hypothesis of Theorem 1.

Theorem 1 admits the following modification.

Theorem 2. *Suppose that*

$$\varphi(t) \in U_1(k_0) \cap U_2(k_0).$$

Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$. Then the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N - k_0 + 1} \# \left\{ k_0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ih\varphi(k)) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

§2. Lemmas

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p . Denote by $\mathcal{B}(X)$ the Borel σ -field of the space X . Since the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group, the space $(\Omega, \mathcal{B}(\Omega))$ admits the probability Haar measure m_H , and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p , $p \in \mathbb{P}$, where \mathbb{P} is the set of all prime numbers.

We start with a limit theorem on the torus Ω . For $A \in \mathcal{B}(\Omega)$, let

$$Q_N(A) = \frac{1}{N - k_0 + 1} \# \left\{ k_0 \leq k \leq N : \left(p^{-ih\varphi(k)} : p \in \mathbb{P} \right) \in A \right\}.$$

Lemma 1. *Suppose that $\varphi(t) \in U_2(k_0)$. Then Q_N converges weakly to the Haar measure m_H as $N \rightarrow \infty$.*

Proof. We consider the Fourier transform $g_N(\underline{k})$, $\underline{k} = (k_p \in \mathbb{Z} : p \in \mathbb{P})$, of the measure Q_N . It is well known that

$$\begin{aligned} g_N(\underline{k}) &= \int_{\Omega} \prod_p \omega^{k_p}(p) \, dQ_N = \frac{1}{N - k_0 + 1} \sum_{k=k_0}^N \prod_p p^{-ik_p h \varphi(k)} \\ &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^N \exp \left\{ -ih\varphi(k) \sum_p k_p \log p \right\}, \end{aligned} \quad (1)$$

where only a finite number of integers k_p are distinct from zero. Obviously,

$$g_N(\underline{0}) = 1. \quad (2)$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over \mathbb{Q} , we have

$$\sum_p k_p \log p \neq 0$$

for $\underline{k} \neq \underline{0}$. Therefore, by the hypothesis of the lemma, the sequence

$$\left\{ -\frac{h}{2\pi} \varphi(k) \sum_p k_p \log p \right\}$$

with $\underline{k} \neq \underline{0}$ is uniformly distributed modulo 1. Hence, by the Weyl criterion (see, e.g., [5, Theorem 2.1]), from (1) and (2) we see that

$$\lim_{N \rightarrow \infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of the latter identity is the Fourier transform of the Haar measure m_H , we obtain the assertion of the lemma by using a continuity theorem for probability measures on compact groups. \square

Lemma 2. *Suppose that $\varphi(t) \in U_1(k_0)$, $\frac{1}{2} < \sigma < 1$, $\tau \in \mathbb{R}$ and $h > 0$. Then*

$$\int_{k_0}^T |\zeta(\sigma + i\tau + ih\varphi(t))|^2 \, dt \ll_{h,\sigma} T(1 + |\tau|).$$

Proof. The definition of the class $U_1(k_0)$ shows that, for $V \geq k_0$,

$$\begin{aligned} \int_V^{2V} |\zeta(\sigma + i\tau + ih\varphi(t))|^2 dt &= \int_V^{2V} \frac{1}{\varphi'(t)} |\zeta(\sigma + i\tau + ih\varphi(t))|^2 d\varphi(t) \\ &\ll_{h,\sigma} \max_{V \leq t \leq 2V} \frac{1}{\varphi'(t)} \int_V^{2V} d \left(\int_1^{\tau+h\varphi(t)} |\zeta(\sigma + iu)|^2 du \right) \\ &\ll_{h,\sigma} \max_{V \leq t \leq 2V} \frac{1}{\varphi'(t)} \int_1^{\tau+h\varphi(t)} |\zeta(\sigma + iu)|^2 du \Big|_V^{2V}. \end{aligned}$$

Hence, using the well-known estimate

$$\int_1^T |\zeta(\sigma + it)|^2 dt \ll_\sigma T,$$

which is valid for $\frac{1}{2} < \sigma < 1$, we obtain

$$\begin{aligned} \int_V^{2V} |\zeta(\sigma + i\tau + ih\varphi(t))|^2 dt &\ll_{h,\sigma} V + |\tau| \max_{V \leq t \leq 2V} (\varphi'(t))^{-1} \\ &\ll_{h,\sigma} V + \frac{|\tau|}{\varphi(2V)} \varphi(2V) \max_{V \leq t \leq 2V} (\varphi'(t))^{-1} \\ &\ll_{h,\sigma} V + V \frac{|\tau|}{\varphi(2V)} \ll_{h,\sigma} V + V|\tau|. \end{aligned}$$

Now, taking $V = 2^{-k-1}T$ and summing over $k = 0, 1, \dots$, we arrive at the estimate of the lemma. \square

To estimate the discrete mean square

$$J(N, t) \stackrel{\text{def}}{=} \sum_{k=k_0}^N |\zeta(\sigma + it + ih\varphi(k))|^2, \quad t \in \mathbb{R},$$

we need the following Gallagher lemma, see [8, Lemma 1.4].

Lemma 3. *Let T_0 and $T \geq \delta > 0$ be real numbers, and let \mathcal{T} be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < \delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T_0 + T]$ having a continuous derivative on $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 dx + \left(\int_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \right)^{\frac{1}{2}}.$$

Lemma 4. Suppose that $\varphi(t) \in U_1(k_0)$. Then for $\frac{1}{2} < \sigma < 1$ and $h > 0$ we have

$$J(N, t) \ll_{\sigma} N(1 + |t|).$$

Proof. We take $\mathcal{T} = \{k : k_0 \leq k \leq N\}$ in Lemma 3. Then, after a suitable choice of δ , T_0 and T , we get

$$\begin{aligned} J(N, t) &\ll \int_{k_0 + \frac{1}{2}}^{N + \frac{1}{2}} |\zeta(\sigma + it + ih\varphi(\tau))|^2 d\tau \\ &+ \left(\int_{k_0 + \frac{1}{2}}^{N + \frac{1}{2}} |\zeta(\sigma + it + ih\varphi(\tau))|^2 d\tau \int_{k_0 + \frac{1}{2}}^{N + \frac{1}{2}} |\zeta'(\sigma + it + ih\varphi(\tau))|^2 d\tau \right)^{\frac{1}{2}}. \end{aligned} \quad (3)$$

Lemma 2 implies the estimate

$$\int_{k_0 + \frac{1}{2}}^{N + \frac{1}{2}} |\zeta(\sigma + it + ih\varphi(\tau))|^2 d\tau \ll_{h, \sigma} N(1 + |t|).$$

Moreover, by the Cauchy integral formula, for $\frac{1}{2} < \sigma < 1$,

$$\zeta'(\sigma + it + ih\varphi(\tau)) = \frac{1}{2\pi i} \int_{|z - \sigma| = \delta} \frac{\zeta(z + it + ih\varphi(\tau))}{(z - \sigma)^2} dz \ll |\zeta(\sigma_1 + it + ih\varphi(\tau))|$$

with a suitable $\delta > 0$ and σ_1 satisfying $\frac{1}{2} < \sigma_1 < 1$. Hence, in view of Lemma 2,

$$\int_{k_0 + \frac{1}{2}}^{N + \frac{1}{2}} |\zeta'(\sigma + it + ih\varphi(\tau))|^2 d\tau \ll_{h, \sigma} N(1 + |t|).$$

We note that the last estimate also follows from the well-known bound

$$\int_0^T |\zeta'(\sigma + it)|^2 dt \ll_{\sigma} T, \quad \frac{1}{2} < \sigma < 1,$$

in the same way as that used in the proof of Lemma 2. Combined with (3), this proves the lemma. \square

For a fixed $\theta > \frac{1}{2}$ and $m, n \in \mathbb{N}$, we put

$$v_n(m) = \exp \left\{ - \left(\frac{m}{n} \right)^{\theta} \right\},$$

and define the functions

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}$$

and

$$\zeta_n(s, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m)v_n(m)}{m^s},$$

where, for $\omega \in \Omega$,

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p).$$

We note that the series for $\zeta_n(s)$ and $\zeta_n(s, \omega)$ are absolutely convergent for $\sigma > \frac{1}{2}$ [6].

Let $H(D)$ denote the space of analytic functions on D equipped with the topology of uniform convergence on compact sets.

Let $\{K_l : l \in \mathbb{N}\}$ be a sequence of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric in the space $H(D)$ inducing its topology of uniform convergence on compact sets.

Lemma 5. *Suppose that $\varphi(t) \in U_1(k_0)$ and $h > 0$. Then*

$$\liminf_{n \rightarrow \infty} \frac{1}{N - k_0 + 1} \sum_{k=k_0}^N \rho(\zeta(s + ih\varphi(k)), \zeta_n(s + ih\varphi(k))) = 0.$$

Proof. Let K be a compact subset of the strip D . For $\sigma > \frac{1}{2}$, the function $\zeta_n(s)$ has the representation

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z) l_n(z) \frac{dz}{z},$$

where

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s, \quad n \in \mathbb{N},$$

and $\Gamma(s)$ is the Euler gamma-function, see the proof of Theorem 5.4.2 in [6]. This and the residue theorem imply the estimate

$$\begin{aligned} & \frac{1}{N - k_0 + 1} \sum_{k=k_0}^N \sup_{s \in K} |\zeta(s + ih\varphi(k)) - \zeta_n(s + ih\varphi(k))| \\ & \ll \int_{-\infty}^{\infty} |l_n(\sigma_1 + i\tau)| \left(\frac{1}{N - k_0 + 1} \sum_{k=k_0}^N |\zeta(\sigma + it + ih\varphi(k) + i\tau)| \right) d\tau + o(1) \end{aligned}$$

as $N \rightarrow \infty$, where $\sigma_1 < 0$, $\frac{1}{2} < \sigma < 1$, and t is bounded by a constant depending on K . Therefore, by Lemma 4, the left-hand side of the above relation is estimated as

$$\ll_K \int_{-\infty}^{\infty} l_n(\sigma_1 + i\tau)(1 + |\tau|) d\tau + o(1)$$

as $N \rightarrow \infty$. Now, this and the properties of $l_n(s)$ give the identity

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N - k_0 + 1} \sum_{k=k_0}^N \sup_{s \in K} |\zeta(s + ih\varphi(k)) - \zeta_n(s + ih\varphi(k))| = 0.$$

Recalling the definition of the metric ρ , we get the assertion of the lemma. \square

§3. A limit theorem

For $\omega \in \Omega$, define

$$\zeta(s, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m)}{m^s}.$$

Then $\zeta(s, \omega)$ is an $H(D)$ -valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, because for almost all $\omega \in \Omega$ the above series converges uniformly on compact subsets of the strip D , see [6]. Denote by P_ζ the distribution of $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Theorem 3. *Suppose that $\varphi(t) \in U_1(k_0) \cap U_2(k_0)$ and $h > 0$. Then*

$$P_N(A) \stackrel{\text{def}}{=} \frac{1}{N - k_0 + 1} \# \{k_0 \leq k \leq N : \zeta(s + ih\varphi(k)) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the measure P_ζ as $N \rightarrow \infty$. Moreover, the support of P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Proof. We start with a limit theorem for

$$P_{N,n}(A) \stackrel{\text{def}}{=} \frac{1}{N - k_0 + 1} \# \{k_0 \leq k \leq N : \zeta_n(s + ih\varphi(k)) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Let $u_n : \Omega \rightarrow H(D)$ be defined by the formula

$$u_n(\omega) = \zeta_n(s, \omega), \quad \omega \in \Omega.$$

The absolute convergence of the series for $\zeta_n(s, \omega)$ ensures that the function u_n is continuous. Moreover, $P_{N,n} = Q_N u_n^{-1}$, where Q_N is from Lemma 1. Therefore, denoting $\widehat{P}_n = m_H u_n^{-1}$ and using Lemma 1 and the continuity of u_n , we see that $P_{N,n}$ converges weakly to \widehat{P}_n as $N \rightarrow \infty$.

Let θ_N be a random variable defined on a certain probability space with measure μ and having the distribution

$$\mu \{\theta_N = h\varphi(k)\} = \frac{1}{N - k_0 + 1}, \quad k = k_0, k_0 + 1, \dots, N.$$

We introduce the $H(D)$ -valued random elements

$$X_{N,n}(s) = \zeta_n(s + i\theta_N)$$

and

$$X_N(s) = \zeta(s + i\theta_N).$$

Using these random elements, the weak convergence for $P_{N,n}$, Lemma 5, and a standard method based on the Prokhorov theorem (see [2, Theorem 6.1] and Theorem 4.2 of [2]), we conclude that, on $(H(D), \mathcal{B}(H(D)))$, there exists a

probability measure P such that P_N converges weakly to P as $N \rightarrow \infty$, and that \widehat{P}_n also converges weakly to P as $n \rightarrow \infty$. On the other hand,

$$\frac{1}{T} \text{meas} \{ \tau \in [0, T] : \zeta(s + i\tau) \in A \}, \quad A \in \mathcal{B}(H(D)),$$

also converges weakly to the limit measure P of \widehat{P}_n as $n \rightarrow \infty$, and P coincides with P_ζ , see [3]. Therefore, P_N also converges weakly to P_ζ as $N \rightarrow \infty$. Moreover, it is known that the support of P_ζ is the set S , see [6]. The theorem is proved. \square

§4. Proof of universality theorems

Proof of Theorem 1. By the Mergelyan theorem on the approximation of analytic functions by polynomials, see [7] and [12], there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2}. \quad (4)$$

Define the set

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\}.$$

Then G is an open set in $H(D)$. Moreover, since $e^{p(s)} \in S$, Theorem 3 implies that the set G is an open neighborhood of the support of the measure P_ζ . Thus, $P_\zeta(G) > 0$. Therefore, Theorem 3 together with the equivalent of the weak convergence of probability measure in terms of open sets, see [2, Theorem 2.1], yields the inequality

$$\liminf_{N \rightarrow \infty} P_N(G) \geq P_\zeta(G) > 0.$$

Hence, by the definitions of the set G and P_N , we have

$$\liminf_{N \rightarrow \infty} \frac{1}{N - k_0 + 1} \# \left\{ k_0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ih\varphi(k)) - e^{p(s)}| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with (4) gives the assertion of Theorem 1. \square

Proof of Theorem 2. For a given $\varepsilon > 0$, let

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then we see that the boundary ∂G_ε is the set

$$\left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ whenever the positive numbers ε_1 and ε_2 are different. It follows that $P_\zeta(\partial G_\varepsilon) > 0$ for an at most countable set of values of

$\varepsilon > 0$, i.e., the set G_ε is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Therefore, using the equivalent of the weak convergence of probability measures in terms of continuity sets, see [2, Theorem 2.1], and recalling Theorem 3, we have

$$\lim_{N \rightarrow \infty} P_N(G_\varepsilon) = P_\zeta(G_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of P_N and G_ε give the relation

$$\lim_{N \rightarrow \infty} \frac{1}{N - k_0 + 1} \# \left\{ k_0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ih\varphi(k)) - f(s)| < \varepsilon \right\} = P_\zeta(G_\varepsilon) \quad (5)$$

for all but at most countably many $\varepsilon > 0$. By Theorem 3, the function $e^{p(s)}$, where $p(s)$ is a polynomial, is an element of the support of the measure P_ζ . By the Mergelyan theorem, the polynomial $p(s)$ can be chosen to satisfy inequality (4). Define yet another set

$$\widehat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\}.$$

Then $P_\zeta(\widehat{G}_\varepsilon) > 0$. Moreover, inequality (4) shows that $\widehat{G}_\varepsilon \subset G_\varepsilon$. Thus,

$$P_\zeta(G_\varepsilon) \geq P_\zeta(\widehat{G}_\varepsilon) > 0,$$

and the assertion of Theorem 2 follows from (5). \square

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