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On Analytic and Subharmonic Functions in Unit Disc Growing Near a Part of the Boundary

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In the paper there was found an analog of the Blaschke condition for analytic and subharmonic functions in the unit disc, which grow at most as a given function φ near some subset of the boundary.

Key words: analytic function, subharmonic function, Riesz measure, Blaschke condition.

Mathematics Subject Classification 2010: 30D50, 31A05.

1. Introduction

It is well known that the zeroes $\{z_n\}$ of any bounded analytic function in the unit disk \mathbb{D} satisfy the Blaschke condition

$$\sum (1 - |z_n|) < \infty. \quad (1)$$

There have been a number of papers published where there can be found the analogous conditions for various classes of unbounded analytic and subharmonic functions (see, for example, [6–9]). In [5], there was studied the case of analytic functions in \mathbb{D} of an exponential growth near a finite subset $E \subset \partial\mathbb{D}$. In [1], the case of an arbitrary compact subset $E \subset \partial\mathbb{D}$ was considered. Namely, there were considered subharmonic functions v in \mathbb{D} such that

$$v(z) \leq \text{dist}(z, E)^{-q}, \quad z \in \mathbb{D}, \quad (2)$$

for some $0 < q < \infty$. The Riesz measures (generalized Laplacians) $\mu = \frac{1}{2\pi} \Delta v$ are proven to satisfy the condition

$$\int_{\mathbb{D}} (1 - |\lambda|) (\text{dist}(\lambda, E))^{(q-\alpha)_+} d\mu(\lambda) < \infty \quad (3)$$

for α such that

$$\int_0^2 \frac{m\{s \in \partial\mathbb{D} : \text{dist}(s, E) < t\} dt}{t^{\alpha+1}} < \infty. \quad (4)$$

Here $x_+ = \max\{0, x\}$, and m is the normalized Lebesgue measure on $\partial\mathbb{D}$, that is, $m(\partial\mathbb{D}) = 1$. If $m(E) > 0$, then (3) and (4) are valid for any $\alpha < 0$. If (4) is valid with some $\alpha > 0$ and $q \leq \alpha$, then μ satisfies the Blaschke condition for bounded from above in \mathbb{D} subharmonic functions

$$\int (1 - |\lambda|) d\mu(\lambda) < \infty. \quad (5)$$

It was also proved in [1] that (3) is not valid for the subharmonic function $v_0(z) = \text{dist}(z, E)^{-q}$ in the case of divergent integral in (4).

If $f(z)$ is an analytic function, then $\log |f(z)|$ is a subharmonic function with the Riesz measure $\sum_n k_n \delta_{z_n}$, where δ_{z_n} are the unit masses in the zeros $\{z_n\}$ of $f(z)$, and k_n are the multiplicities of the zeros. Hence, if this is the case, (2) has the form

$$|f(z)| \leq \exp \text{dist}(z, E)^{-q}, \quad (6)$$

while (3) has the form

$$\sum_{z_n} (1 - |z_n|) \text{dist}(z_n, E)^{(q-\alpha)_+} < \infty. \quad (7)$$

Note that condition (6) seems to be too restrictive to be applied to the operator theory [1]. It is natural to consider subharmonic (and analytic) functions such that

$$v(z) \leq \varphi(\text{dist}(z, E)), \quad (8)$$

where $\varphi(t)$ is the monotonically decreasing on \mathbb{R}^+ function, $\varphi(t) \rightarrow +\infty$ as $t \rightarrow +0$. It is clear that for any subharmonic function v in \mathbb{D} , which grows as $\text{dist}(z, E) \rightarrow 0$, condition (8) is valid for $\varphi(t) = \sup\{v(z) : \text{dist}(z, E) \geq t\}$.

The Main Results

To formulate our results we need some notations. Let

$$\rho(z) = \text{dist}(z, E), \quad F(t) = F_E(t) = m\{\zeta \in \partial\mathbb{D} : \rho(\zeta) < t\}.$$

Consider

$$I(\varphi, E) := \int_0^2 \varphi(s) dF(s).$$

Note that $F(t)$ is continuous on $[0, 2]$ and has a discontinuity at $t = 0$ if and only if $m(E) > 0$.

Theorem 1. *Let E be a closed subset of $\partial\mathbb{D}$, v be a subharmonic function in \mathbb{D} , $v \not\equiv -\infty$, and $v(z)$ satisfy (8). If $I(\varphi, E) < \infty$, then the Riesz measure μ of the function $v(z)$ satisfies (5).*

E x a m p l e 1. Let $E = \{\zeta_1, \dots, \zeta_k\}$, $\varphi(t) = t^{-1} \log^{-\alpha}(1/t)$, $\alpha > 1$. It is readily seen that $F(t) = 2kt/(2\pi) + o(1)$ as $t \rightarrow 1$. The assumption of Theorem 1 is fulfilled and the Riesz measure of any subharmonic function $v(z) \leq \rho^{-1}(z) \log^{-\alpha}(1/\rho(z))$ satisfies the Blaschke condition (5).

Theorem 2. *Let E , v and μ satisfy the assumption of Theorem 1, $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an absolutely continuous nonnegative monotonically decreasing function, $\varphi(t) \rightarrow +\infty$ as $t \rightarrow +0$, and $\psi(t)$ be an absolutely continuous monotonically increasing function on $[0, 2]$ such that $\psi(0) = 0$. If*

$$\int_0^{1/25} \psi(50y)\varphi'(y)F(y)dy > -\infty, \tag{9}$$

then

$$\int_{\mathbb{D}} \psi(\rho(\lambda))(1 - |\lambda|)d\mu(\lambda) < \infty. \tag{10}$$

R e m a r k 1. In the case $m(E) > 0$, we always get $I(\varphi, E) = \infty$ and thus the assumption of Theorem 1 is not satisfied. Further, (9) takes the form

$$\int_0^{1/25} \psi(50y)\varphi'(y)dy > -\infty.$$

In the case $E = \partial\mathbb{D}$, the result coincides with that obtained in [8].

E x a m p l e 2. Let $E = \{\zeta_1, \dots, \zeta_k\}$, $\varphi(t) = e^{1/t}$. The assumption of Theorem 2 is satisfied with $\psi(t) = e^{-c/t}$ for $c > 50$. Hence, if v is a subharmonic function $v(z)$ such that $v(z) \leq \exp(1/\rho(z))$, then for $c > 50$ the integral

$$\int \exp(-c/\rho(z))(1 - |\lambda|)d\mu(\lambda)$$

converges.

The following theorem shows the accuracy of Theorem 2.

Theorem 3. Let E , φ , ψ be the same as above and, in addition, $\varphi(1/t)$ be convex with respect to $\log t$. If

$$\int_0^2 \psi(y)\varphi'(y)F(y)dy = -\infty, \tag{11}$$

then for the Riesz measure μ_0 of the function $v_0(z) = \varphi(\rho(z))$ we get

$$\int \psi(\rho(\lambda))(1 - |\lambda|)d\mu_0(\lambda) = +\infty.$$

Now consider the case $v(z) = \log |f(z)|$ with the analytic function $f(z)$.

Theorem 1'. Let E be a closed subset of $\partial\mathbb{D}$, $\varphi(t)$ be an absolutely continuous nonnegative monotonically decreasing function on \mathbb{R}^+ , $\varphi(t) \rightarrow +\infty$ as $t \rightarrow +0$, $I(\varphi, E) < \infty$, f be an analytic function in \mathbb{D} with zeros z_n . If

$$|f(z)| \leq \exp(\varphi(\rho(z))),$$

then its zeros satisfy the Blaschke condition (1).

Theorem 2'. Let E , f , φ be the same as in Theorem 1', but the condition $I(\varphi, E) < \infty$ for some absolutely continuous monotonically increasing function ψ on $[0, 2]$ such that $\psi(0) = 0$ be replaced by (9). Then

$$\sum_{z_n} \psi(\rho(z_n, E))(1 - |z_n|) < \infty.$$

R e m a r k 2. For $\varphi(t) = t^{-a}$, $\psi(t) = t^s$, Theorems 1–3, 1', 2' were proved earlier in [1].

2. Demonstrations

We begin with the lemmas.

Lemma 1. Let ν be a nonnegative finite measure on X , $g(x)$ be a Borel function on X , $\varphi(t)$ be a Borel function on \mathbb{R} . Then

$$\int_X (\varphi \circ g)(x)d\nu(x) = \int_{-\infty}^{\infty} \varphi(y)dH(y),$$

where $H(y) = \nu\{x : g(x) < y\}$.

The proof of this lemma can be easily reduced to the case $\nu(X) = 1$ which is well known (see, for example, [2, formula (15.3.1)]).

Lemma 2. For each $z \in \mathbb{D}$ and $\tau \in [0, 1]$ we have $\rho(z) \leq 2\rho(\tau z)$.

P r o o f. Consider $\zeta = z/|z|$ and $z' \in [0, \zeta]$ such that $\rho(z') = \text{dist}(E, [0, \zeta])$. The cases $z' = 0$ and $z \in [0, z']$ are trivial. Suppose $z \in (z', \zeta)$. We have

$$\rho(z) \leq \rho(z') + |z' - z| \leq \rho(z') + |z' - \zeta|.$$

Consider $\zeta' \in E$ such that $\rho(z') = |z' - \zeta'|$ and the triangle with vertexes on z', ζ, ζ' . The angle in z' is right, and the angle in ζ is greater than $\pi/4$. Thus, $|z' - \zeta| \leq |z' - \zeta'|$ and therefore $\rho(z) \leq 2\rho(z') \leq 2\rho(\tau z)$.

Note that the result of the lemma was used in [1, p.41] without proof.

P r o o f of Theorem 1. First, suppose $v(0) = 0$. Let $I(\varphi, E) < \infty$. Using Lemma 1 with the measure $m(\zeta)$ on $\partial\mathbb{D}$ and taking into account the equalities $F(y) \equiv 0$ for $y < 0$ and $F(y) \equiv 1$ for $y > 2$, we get

$$\int_{\partial\mathbb{D}} (\varphi \circ \rho)(\zeta) dm(\zeta) = \int_0^\infty \varphi(y) dF(y) = \int_0^2 \varphi(y) dF(y) = I(\varphi, E) < \infty. \quad (12)$$

First prove that there exists a harmonic majorant for v in \mathbb{D} . Using the properties of the Poisson integral

$$U(z) = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\rho(\zeta)) dm(\zeta),$$

we obtain

$$\lim_{z \rightarrow \zeta} U(z) = \varphi(\rho(\zeta)), \quad \zeta \in \partial\mathbb{D} \setminus E.$$

Hence $\overline{\lim}_{z \rightarrow \zeta} (v(z) - U(z)) \leq 0$ for $\zeta \in \partial\mathbb{D} \setminus E$ and $\lim_{z \rightarrow \zeta \in E} U(z) = +\infty$.

Let $\Omega_t, t \in (0, 1)$, be the connected component of the set $\{z \in \mathbb{D} : \rho(z) > t\}$ such that $0 \in \Omega_t$. Put

$$\Gamma_t = \{z \in \mathbb{D} : \rho(z) = t\}, \quad E_t = \{\zeta \in \partial\mathbb{D} : \rho(z) < t\}, \quad E_t^c = \partial\mathbb{D} \setminus E_t.$$

Since E_t is a finite union of disjoint open sets, it is seen that E_t^c is a finite union of disjoint closed sets. Moreover, $\partial\Omega_t \subset E_t^c \cup \Gamma_t$. Note that for the connected $\{z \in \mathbb{D} : \rho(z) > t\}$ we have $\partial\Omega_t = E_t^c \cup \Gamma_t$.

For $z \in \Gamma_t$, take $\zeta' \in E$ such that $|z - \zeta'| = t$. Put $\gamma_z = \{\zeta \in \partial\mathbb{D} : |\zeta - \zeta'| \leq t\}$. Clearly, $\gamma_z \subset \overline{E_t}$. Since the function $\varphi(t)$ is positive and $\varphi(\rho(\zeta)) \geq \varphi(t)$ on E_t , we get

$$U(z) \geq \int_{\gamma_z} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\rho(\zeta)) dm(\zeta) \geq \varphi(t) \omega(z, \gamma_z, \mathbb{D}),$$

where

$$\omega(\lambda, \gamma_z, \mathbb{D}) = \int_{\gamma_z} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} dm(\zeta)$$

is the harmonic measure at the point $\lambda = z$ with respect to the arc γ_z (see, for example, [3, §4.3]). On the other hand, the harmonic measure at any point $\lambda \in \mathbb{D}$ with respect to the arc γ_z equals $\beta/(2\pi)$, where $\beta = \beta(\lambda)$ is the length of the arc of the unit circle formed by the strait lines passing through λ and the ends of γ_z (see [10, chapter I, §5]). By direct calculations, $\beta(z) = \pi - 2 \arcsin(t/2)$. Since $0 < t \leq 1$, we obtain $\omega(z, \gamma_z, \mathbb{D}) \geq 1/3$ and

$$U(z) \geq \varphi(t) \omega(z, \gamma_z, \mathbb{D}) \geq \frac{1}{3} \varphi(t) \geq \frac{v(z)}{3}, \quad z \in \Gamma_t.$$

Therefore,

$$\overline{\lim}_{z \rightarrow \zeta} [v(z) - 3U(z)] \leq 0, \quad \zeta \in \Gamma_t, \quad \overline{\lim}_{z \rightarrow \zeta} [v(z) - U(z)] \leq 0, \quad \zeta \in E_t^c.$$

Using the maximum principle, we get $v(z) \leq 3U(z)$ for all $z \in \Omega_t$.

Furthermore, by the Green formula [3, Theorem 4.5.4], we have

$$v(z) = u_t(z) - \int_{\Omega_t} G_{\Omega_t}(z, \lambda) d\mu(\lambda), \tag{13}$$

where u_t is the least harmonic majorant of v in Ω_t . We have $u_t(z) \leq 3U(z)$ for $z \in \Omega_t$. The Green function $G_{\Omega_t}(z, \lambda)$ in Ω_t is equal to

$$G_{\Omega_t}(z, \lambda) = \log 1/(|z - \lambda|) - h_t(z, \lambda),$$

where $h_t(z, \lambda)$ is the harmonic function in Ω_t with the boundary values $\log 1/|z - \lambda|$. By [1, proof of Theorem 1], $G_{\Omega_t}(0, \lambda) \geq (1 - |\lambda|)/6$ for $\lambda \in \Omega_{kt}$ with $k = 25 > 6\pi + 3$. Consequently,

$$\begin{aligned} \int_{\Omega_{kt}} (1 - |\lambda|) d\mu(\lambda) &\leq 6 \int_{\Omega_t} G_{\Omega_t}(0, \lambda) d\mu(\lambda) = 6u_t(0) \\ &\leq 18U(0) = 18 \int_{\partial\mathbb{D}} \varphi(\rho(\zeta)) dm(\zeta) = 18I(\varphi, E). \end{aligned}$$

The later inequality is valid for all $t > 0$, hence we obtain the statement of the theorem for the case $v(0) = 0$.

If $v(0) > 0$, consider the function $v(z) - v(0)$ instead of $v(z)$.

If $-\infty < v(0) < 0$, consider the function $v_1(z) = \varphi(2) \frac{v(z) - v(0)}{\varphi(2) - v(0)}$. We have

$$v_1(z) \leq \varphi(\rho(z)) \frac{1 - v(0)/\varphi(\rho(z))}{1 - v(0)/\varphi(2)} \leq \varphi(\rho(z)).$$

Notice that the Riesz measure of the function $v_1(z)$ coincides with the Riesz measure of the function $v(z)$ up to a constant depending only on $v(0)$. Hence the Blaschke condition (5) for v_1 implies the same condition for v .

If $v(0) = -\infty$, consider the harmonic function $h(z)$ in the disk $\{|z| < 1/2\}$ such that $h(z) = v(z)$ for $|z| = 1/2$ and put

$$v_1(z) = \begin{cases} \max(v(z), h(z)), & |z| < \frac{1}{2} \\ v(z), & |z| \geq \frac{1}{2}. \end{cases}$$

Clearly,

$$v_1(z) \leq \max_{|z|=1/2} v(z) \leq \varphi(1/2) \quad \text{for } |z| \leq 1/2 \quad \text{and} \quad v_1(0) \neq -\infty.$$

In addition, $v_1(z)$ is subharmonic in \mathbb{D} (see [3, Theorem 2.4.5]) and the restriction of its Riesz measure μ_1 to the set $\{z \in \mathbb{D} : |z| > \frac{1}{2}\}$ is equal to μ . Applying the proved statement to the function $\varphi_1(z) = \max\{\varphi(z), \varphi(1/2)\}$, we obtain

$$\int_{\mathbb{D}} (1 - |\lambda|) d\mu_1(\lambda) < \infty.$$

Therefore the integral in (5) is also finite. Theorem 1 is proved.

P r o o f of Theorem 2. Arguing as above, we can consider only the case $v(0) = 0$.

Let $\Omega_t, \Gamma_t, E_t, E_t^c, \gamma_z, \omega(\lambda, \gamma_z, \mathbb{D})$ be the same as in the proof of Theorem 1. For $z \in \mathbb{D}$, put

$$V_t(z) = \int_{E_t^c} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\rho(\zeta)) dm(\zeta) + \varphi(t) \int_{E_t} \frac{1 - |z|^2}{|\zeta - z|^2} dm(\zeta). \quad (14)$$

Since $\gamma_z \subset \overline{E_t}$, we get

$$V_t(z) \geq \varphi(t) \omega(z, \gamma_z, \mathbb{D}) \geq \frac{1}{3} \varphi(t).$$

Therefore,

$$\limsup_{z \rightarrow \zeta} v(z) \leq \lim_{z \rightarrow \zeta} 3V_t(z) = 3V_t(\zeta), \quad \zeta \in \partial\Omega_t.$$

Using the maximum principle, we get $v(z) \leq 3V_t(z)$ for all $z \in \Omega_t$, in particular, $v(0) \leq 3V_t(0)$. Clearly, we have

$$\begin{aligned} V_t(0) &= \int_{\partial\mathbb{D}} V_t(\zeta) dm(\zeta) = \int_{E_t} \varphi(t) dm(\zeta) + \int_{E_t^c} \varphi(\rho(\zeta)) dm(\zeta) \\ &= \varphi(t)F(t) + \int_{E_t^c} \varphi(\rho(\zeta)) dm(\zeta). \end{aligned}$$

Using Lemma 1 with $g(\zeta) = \rho(\zeta)$ and taking into account the equality

$$H(y) = m\{\zeta : \rho(\zeta) < y\} - m\{\zeta : \rho(\zeta) \leq t\} = F(y) - F(t),$$

we get

$$\varphi(t)F(t) + \int_{E_t^c} \varphi(\rho(\zeta))dm(\zeta) = \varphi(t)F(t) + \int_t^2 \varphi(y)dF(y).$$

Integrating by parts, we get

$$\begin{aligned} V_t(0) &= \varphi(t)F(t) + \varphi(2)F(2) - \varphi(t)F(t) - \int_t^2 \varphi'(y)F(y)dy \\ &= \varphi(2) - \int_t^2 \varphi'(y)F(y)dy. \end{aligned} \tag{15}$$

Therefore, using the Green formula (13) and the estimate $u_t(z) \leq 3V_t(z)$ of the least harmonic majorant u_t in Ω_t , we get

$$\int_{\Omega_t} G_{\Omega_t}(0, \lambda)d\mu(\lambda) = u_t(0) \leq 3V_t(0) = 3 \left(\varphi(2) - \int_t^2 \varphi'(y)F(y)dy \right). \tag{16}$$

By [1, proof of Theorem 1], $G_{\Omega_t}(0, \lambda) \geq (1 - |\lambda|)/6$ for $\lambda \in \Omega_{kt}$ with $k = 25 > 6\pi + 3$. Therefore,

$$\int_{\Omega_{kt}} (1 - |\lambda|)d\mu(\lambda) \leq 18 \left(\varphi(2) - \int_t^2 \varphi'(y)F(y)dy \right) \tag{17}$$

if $kt \in (0, 1)$. In particular, the measure $(1 - |\lambda|)d\mu(\lambda)$ of the set $\{\lambda \in \mathbb{D} : \rho(\lambda) \geq \varepsilon\}$ is finite for each $\varepsilon > 0$. Applying Lemma 1 with the function $g = \rho$ and taking into account that $\rho(\lambda) \leq 2$, we get

$$\int_{\{\lambda \in \mathbb{D} : \rho(\lambda) \geq \varepsilon\}} \psi(\rho(\lambda))(1 - |\lambda|)d\mu(\lambda) = \int_\varepsilon^2 \psi(t)dG(t)$$

with $G(t) = \int_{\{\lambda \in \mathbb{D} : \varepsilon \leq \rho(\lambda) < t\}} (1 - |\lambda|) d\mu(\lambda)$. We have

$$\begin{aligned} \int_{\varepsilon}^2 \psi(t) dG(t) &= - \int_{\varepsilon}^2 \psi(t) d \left(\int_{\{\lambda : \rho(\lambda) > t\}} (1 - |\lambda|) d\mu(\lambda) \right) \\ &= \psi(\varepsilon) \int_{\{\lambda : \rho(\lambda) \geq \varepsilon\}} (1 - |\lambda|) d\mu(\lambda) + \int_{\varepsilon}^2 \psi'(t) \left(\int_{\{\lambda : \rho(\lambda) > t\}} (1 - |\lambda|) d\mu(\lambda) \right) dt. \end{aligned} \quad (18)$$

We claim that under the condition

$$\int_0^2 \psi'(t) \left(\int_{\{\lambda : \rho(\lambda) > t\}} (1 - |\lambda|) d\mu(\lambda) \right) dt < \infty$$

we have

$$\psi(\varepsilon) \int_{\{\lambda : \rho(\lambda) \geq \varepsilon\}} (1 - |\lambda|) d\mu(\lambda) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (19)$$

Indeed, for any $\eta > 0$ and sufficiently small positive $\delta < \varepsilon < 2$,

$$\int_{\delta}^{\varepsilon} \psi'(t) \left(\int_{\{\lambda : \rho(\lambda) > t\}} (1 - |\lambda|) d\mu(\lambda) \right) dt \leq \eta.$$

Therefore,

$$\begin{aligned} (\psi(\varepsilon) - \psi(\delta)) \int_{\{\lambda : \rho(\lambda) \geq \varepsilon\}} (1 - |\lambda|) d\mu(\lambda) &= \int_{\delta}^{\varepsilon} \psi'(t) dt \int_{\{\lambda : \rho(\lambda) \geq \varepsilon\}} (1 - |\lambda|) d\mu(\lambda) \\ &\leq \int_{\delta}^{\varepsilon} \psi'(t) \left(\int_{\{\lambda : \rho(\lambda) > t\}} (1 - |\lambda|) d\mu(\lambda) \right) dt \leq \eta. \end{aligned}$$

Passing to the limit $\delta \rightarrow 0$, we obtain

$$\psi(\varepsilon) \int_{\{\lambda : \rho(\lambda) \geq \varepsilon\}} (1 - |\lambda|) d\mu(\lambda) \leq \eta,$$

which proves (19).

By Lemma 2, if $\rho(z) > t$, then $\rho(\tau z) > t/2$ for all $z \in \mathbb{D}$ and $0 < \tau < 1$. Hence the interval $[0, z]$ belongs to the set $\{\zeta : \rho(\zeta) > t/2\}$, and $\{z : \rho(z) > t\} \subset \Omega_{t/2}$. Therefore,

$$\int_{\varepsilon}^2 \psi'(t) \left(\int_{\{\rho(\lambda) > t\}} (1 - |\lambda|) d\mu(\lambda) \right) dt \leq \int_{\varepsilon}^2 \psi'(t) \left(\int_{\Omega_{t/2}} (1 - |\lambda|) d\mu(\lambda) \right) dt.$$

By (18), to prove the convergence of the integral

$$\int_{\mathbb{D}} \psi(\rho(\lambda))(1 - |\lambda|) d\mu(\lambda),$$

it is sufficient to show the convergence of the integral

$$\int_0^2 \psi'(t) \left(\int_{\Omega_{t/2}} (1 - |\lambda|) d\mu(\lambda) \right) dt = 2k \int_0^{1/k} \psi'(2kt) \left(\int_{\Omega_{kt}} (1 - |\lambda|) d\mu(\lambda) \right) dt.$$

We have

$$\begin{aligned} \int_0^{1/k} \psi'(2kt) \left(\int_t^2 \varphi'(y) F(y) dy \right) dt &= \frac{1}{2k} \psi(2) \int_{1/k}^2 \varphi'(y) F(y) dy \\ &- \lim_{t \rightarrow 0} \frac{\psi(2kt)}{2k} \int_t^2 \varphi'(y) F(y) dy + \frac{1}{2k} \int_0^{1/k} \psi(2ky) \varphi'(y) F(y) dy \\ &\geq \text{const} + \frac{1}{2k} \int_0^{1/k} \psi(2ky) \varphi'(y) F(y) dy. \end{aligned}$$

By the condition of the theorem, the last integral is finite. The proof is complete.

P r o o f of Theorem 3. Note that the function $-\log \rho(z)$ is subharmonic, hence the function $\varphi(\rho(z)) = \varphi\left(\frac{1}{e^{-\log \rho(z)}}\right)$ is subharmonic as well.

Using the Green formula (13) for the function $v_0(z)$ in the domain Ω_t , we get

$$\varphi(1) = v_0(0) = u_t^0(0) - \int_{\Omega_t} (\log 1/|\lambda| - h_t(0, \lambda)) d\mu_0(\lambda), \quad (20)$$

where $u_t^0(z)$ is the least harmonic majorant for $v_0(z)$ in Ω_t , and $h_t(0, \lambda)$ is the solution of Dirichlet problem in Ω_t with the boundary values $\log 1/|\lambda|$. Clearly,

$h_t(0, \lambda) \geq 0$. On the other hand, if $V_t(z)$ is the defined in (14) harmonic function in \mathbb{D} , then $\lim_{z \rightarrow \zeta} V_t(z) = v_0(\zeta)$ for $\zeta \in E_t^c$ and $V_t(z) \leq \varphi(t)$ in \mathbb{D} . Hence, $V_t(z) \leq v_0(z)$ on Γ_t and $V_t(z) \leq u_t^0(z)$ in Ω_t . Combining equality (15) for $V_t(0)$ with (20), we get

$$\varphi(2) - \int_t^2 \varphi'(y)F(y)dy \leq \varphi(1) + \int_{\Omega_t} \log \frac{1}{|\lambda|} d\mu_0(\lambda).$$

Note that $\Omega_t \subset \{\lambda : \rho(\lambda) > t\}$ and $\log \frac{1}{|\lambda|} \leq 2(1 - |\lambda|)$ for $|\lambda| \geq 1/2$. Hence,

$$- \int_t^2 \varphi'(y)F(y)dy \leq 2 \int_{\{\lambda: \rho(\lambda) > t\}} (1 - |\lambda|)d\mu_0(\lambda) + \text{const.} \tag{21}$$

On the other hand, arguing as in the proof of Theorem 2, we get

$$\begin{aligned} & \int_{\{\lambda: \rho(\lambda) \geq \varepsilon\}} \psi(\rho(\lambda))(1 - |\lambda|)d\mu_0(\lambda) \\ &= \psi(\varepsilon) \int_{\{\lambda: \rho(\lambda) \geq \varepsilon\}} (1 - |\lambda|)d\mu_0(\lambda) + \int_{\varepsilon}^2 \psi'(t) \left(\int_{\{\lambda: \rho(\lambda) > t\}} (1 - |\lambda|)d\mu_0(\lambda) \right) dt. \end{aligned}$$

Using (21), we get

$$\begin{aligned} & -\psi(\varepsilon) \int_{\varepsilon}^2 \varphi'(y)F(y)dy - \int_{\varepsilon}^2 \psi'(t) \left(\int_t^2 \varphi'(y)F(y)dy \right) dt \\ & \leq 2 \int_{\{\lambda: \rho(\lambda) \geq \varepsilon\}} \psi(\rho(\lambda))(1 - |\lambda|)d\mu_0(\lambda) + \text{const.} \end{aligned}$$

Integrating by parts in t , we obtain

$$- \int_{\varepsilon}^2 \psi(t)\varphi'(t)F(t)dt \leq 2 \int_{\{\lambda: \rho(\lambda) \geq \varepsilon\}} \psi(\rho(\lambda))(1 - |\lambda|)d\mu_0(\lambda) + \text{const.}$$

Proceeding here to the limit as $\varepsilon \rightarrow 0$ and using the assumption of the theorem, we complete the proof.

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