



Общероссийский математический портал

Alexey A. Kytmanov, Integral Representations and Volume Forms on Hirzebruch Surfaces, *Журн. СФУ. Сер. Матем. и физ.*, 2008, том 1, выпуск 2, 125–132

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением  
<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 18.97.14.91

17 марта 2025 г., 17:32:12



УДК 517.55

## Integral Representations and Volume Forms on Hirzebruch Surfaces

Alexey A.Kytmanov\*

Institute of Space and Information Technologies,  
Siberian Federal University,  
Kirenskogo 26, Krasnoyarsk, 660074,  
Russia

Received 10.01.2008, received in revised form 20.02.2008, accepted 05.03.2008

We construct a class of integral representations for holomorphic functions in a polyhedron in  $\mathbb{C}^4$ , associated with Hirzebruch surfaces. The kernels of the integral representations are closed differential forms in  $\mathbb{C}^4$  associated with volume forms on Hirzebruch surfaces.

Keywords: integral representation, Hirzebruch surface, toric variety.

### Introduction

The kernel of the Bochner–Martinelli integral representation in  $\mathbb{C}^{n+1}$  is well known to be closely connected with the Fubini–Studi form for the projective space  $\mathbb{P}^n = \mathbb{C}\mathbb{P}^n$  as follows:

$$\omega(z) = \frac{1}{2\pi i} \frac{d\lambda}{\lambda} \wedge \omega_0([\xi]) \quad (1)$$

(see, for instance, [1, Ch. 3]; [2, Ch. 4]). Here  $\omega$  is the Bochner–Martinelli form,

$$\omega(z) = \frac{n!}{(2\pi i)^{n+1}} \sum_{k=1}^{n+1} (-1)^{k-1} \frac{\bar{z}_k}{|z|^{2n+2}} d\bar{z}[k] \wedge dz,$$

$dz = dz_1 \wedge \dots \wedge dz_{n+1}$ , and  $d\bar{z}[k]$  results from deleting the differential  $d\bar{z}$  in  $d\bar{z}_k$ . The form  $\omega_0([\xi])$  is the volume form for the Fubini–Studi metric in  $\mathbb{P}^n$  (see [3, p. 21])

$$\omega_0([\xi]) = \frac{n!}{(2\pi i)^n} \frac{E(\xi) \wedge \overline{E(\xi)}}{|\xi|^{2(n+1)}}, \quad (2)$$

where

$$E(\xi) = \sum_{k=1}^{n+1} (-1)^{k-1} \xi_k d\xi[k]$$

is the Euler form and  $\xi = (\xi_1, \dots, \xi_{n+1})$  are the homogeneous coordinates of a point  $[\xi] \in \mathbb{P}^n$ . Moreover,  $\xi, z \in \mathbb{C}^{n+1}$  and  $\lambda \in \mathbb{C}$  are connected by the relation  $z = \lambda\xi$ .

The Bochner–Martinelli form is a “canonical” form of degree  $2n + 1$  in  $\mathbb{C}^{n+1} \setminus \{0\}$ . The latter set is a bundle over  $\mathbb{P}^n$  whose fiber is the one-dimensional torus  $\mathbb{C}_*$ . In other words,

\*e-mail: aakytm@akadem.ru

$\mathbb{P}^n = [\mathbb{C}^{n+1} \setminus \{0\}]/G$ , where  $G = \{(\lambda, \dots, \lambda) \in \mathbb{C}^{n+1} : \lambda \in \mathbb{C}_*\}$  is the transformation group of diagonal matrices. The projective space is a particular instance of a toric variety. In the general case, each  $n$ -dimensional toric variety is some quotient space (see [4, 5, 6])

$$\mathbb{X} = [\mathbb{C}^d \setminus Z(\Sigma)] / G.$$

Here  $Z(\Sigma)$  is the union of some coordinate subspaces in  $\mathbb{C}^d$  constructed from a fan  $\Sigma \subset \mathbb{R}^n$  with  $d$  generators and  $G$  is a group isomorphic to the torus  $(\mathbb{C}_*)^r$ ,  $r = d - n$ , which is also constructed from  $\Sigma$ .

In his report at the “Nordan” conference on complex analysis (Stockholm, April 1999) A. K. Tsikh posed the problem of calculating the volume forms  $\omega_0([\xi])$  on toric varieties  $\mathbb{X}_k$  (the Fubini–Studi forms) and the canonical forms  $\omega(z)$  on  $\mathbb{C}^d \setminus Z(\Sigma)$  with the property

$$\omega(z) \sim \frac{1}{(2\pi i)^r} \frac{d\lambda_1}{\lambda_1} \wedge \dots \wedge \frac{d\lambda_r}{\lambda_r} \wedge \omega_0([\xi]),$$

generalizing (1), where the sign  $\sim$  means that the forms have the same residues with respect to  $\lambda_1 = \dots = \lambda_r = 0$ . Moreover, he noted that the forms  $\omega$  may serve as kernels of integral representations in  $\mathbb{C}^d$ .

In the present work we consider a class of toric varieties of complex dimension 2 called Hirzebruch surfaces. We construct volume forms for this class and canonical forms in  $\mathbb{C}^4 \setminus Z'$  where the set  $Z'$  is, in general, not the same as the singular set  $Z(\Sigma)$ . It is shown that the constructed canonical forms define an integral representation in 4-circular polyhedra  $G \subset \mathbb{C}^4$ . In [7] author considered toric varieties, defined by convex fans. Convexity of a fan provides that the singular set of a canonical form  $\omega$  coincides with  $Z(\Sigma)$ . As we will see below in the case of Hirzebruch surfaces fan fails to be convex if  $k > 2$ .

## 1. Hirzebruch Surfaces, Moment Maps and Integration Cycles

Hirzebruch surface  $\mathbb{X}_k$  is the toric variety defined by the 2-dimensional fan, spanned by the vectors  $v_1=(1, 0)$ ,  $v_2=(0, 1)$ ,  $v_3=(-1, 0)$ ,  $v_4=(-k, -1)$ , where  $k \in \mathbb{Z}_+$ .

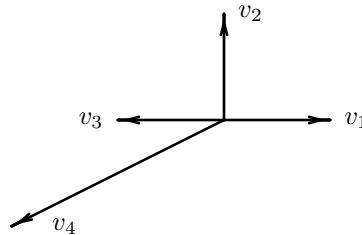


Fig. 1. The fan of  $\mathbb{X}_2$ .

To each vector  $v_j$  we assign a complex variable  $\zeta_j$  so that  $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$  plays role of homogeneous coordinates of Hirzebruch surfaces  $\mathbb{X}_k$ . Each pair of nonneighboring vectors

$v_i, v_j$  (i.e., those not defining a two-dimensional cone) defines a coordinate plane in  $Z(\Sigma)$  (see [7]) so that

$$Z(\Sigma) = \{\zeta_1 = \zeta_3 = 0\} \cup \{\zeta_2 = \zeta_4 = 0\}.$$

The group  $G$  is determined by the relations  $\sum_j \mu_j v_j = 0$  on the vectors  $v_j$ . The following equations

$$\begin{cases} v_1 + v_3 = 0, \\ kv_1 + v_2 + v_4 = 0, \end{cases}$$

are all linearly independent relations between the vectors  $v_k$ . Consequently, the vectors  $\mu_1 = (1, 0, 1, 0)$ ,  $\mu_2 = (k, 1, 0, 1)$  constitute a basis for the lattice of relations. The group  $G$  is the 2-parameter surface  $\{(\lambda_1 \lambda_2^k, \lambda_2, \lambda_1, \lambda_2) : \lambda_j \in \mathbb{C}_*\} \subset (\mathbb{C}_*)^4$ , so that

$$\zeta \sim \eta \Leftrightarrow \exists \lambda_1, \lambda_2 : \zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (\lambda_1 \lambda_2^k \eta_1, \lambda_2 \eta_2, \lambda_1 \eta_3, \lambda_2 \eta_4).$$

The moment map (see, for instance, [5, 8])  $\mu : \mathbb{C}^4 \rightarrow \mathbb{R}^4 / \mathbb{R}^2 \simeq \mathbb{R}^2$  looks like

$$\mu(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (\rho_1, \rho_2),$$

where

$$\begin{cases} \rho_1 = |\zeta_1|^2 + |\zeta_3|^2, \\ \rho_2 = k|\zeta_1|^2 + |\zeta_2|^2 + |\zeta_4|^2. \end{cases} \quad (3)$$

For a fixed  $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ , the relations (3) define the set  $\Gamma_0^k(\rho) = \mu^{-1}(\rho)$ .

The Kähler cone (see, for instance, [5]) for  $\mathbb{X}_k$  is defined by the following inequalities:

$$\begin{cases} \rho_1 > 0, \\ \rho_2 > k\rho_1. \end{cases} \quad (4)$$

The fact that the inequalities (4) hold provides that the integration cycle  $\Gamma_0^k$  does not intersect the singular set  $Z(\Sigma)$ .

## 2. A Canonical Form and a Volume Form

We write down a form  $\omega$  in  $\mathbb{C}^d \setminus Z(\Sigma)$  that is an analog of the Bochner–Martinelli form and establish its basic properties.

The sought form has bidegree (4, 2) and looks like

$$\omega(\zeta) = \frac{h(\bar{\zeta}) \wedge d\zeta}{g(\zeta, \bar{\zeta})}. \quad (5)$$

The numerator is a form of type (4, 2), where  $d\zeta = d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4$ , and

$$h(\zeta) = \zeta_3 \zeta_4 d\zeta_1 \wedge d\zeta_2 - \zeta_2 \zeta_3 d\zeta_1 \wedge d\zeta_4 + \zeta_1 \zeta_4 d\zeta_2 \wedge d\zeta_3 + k\zeta_1 \zeta_3 d\zeta_2 \wedge d\zeta_4 + \zeta_1 \zeta_2 d\zeta_3 \wedge d\zeta_4 \quad (6)$$

is an analog of the Euler form. The denominator  $g$  is the function

$$g(\zeta, \bar{\zeta}) = |\zeta_1|^4 |\zeta_2|^{4-2k} + |\zeta_1|^4 |\zeta_4|^{4-2k} + |\zeta_2|^{2k+4} |\zeta_3|^4 + |\zeta_3|^4 |\zeta_4|^{2k+4}.$$

Here we have to make one important remark.

Note that  $g$  may contain negative powers of  $\zeta$ . In this case we define the form  $\omega$  as in (5), whose numerator and denominator are multiplied by the least power of  $\zeta$  such that the denominator of the resulting form contains no negative powers of  $\zeta$ . This procedure does not affect the transformation laws of the form  $\omega$  that we will derive below.

However, the singular set  $Z_\omega$  of the form  $\omega$  depends on  $k$ . More precisely, we have the following three cases:

1. If  $k = 0$  or  $k = 1$  then  $Z_\omega$  coincides with  $Z(\Sigma) = \{\zeta_1 = \zeta_3 = 0\} \cup \{\zeta_2 = \zeta_4 = 0\}$ ;
2. If  $k = 2$  then  $Z_\omega = Z' := \{\zeta_1 = \zeta_3 = 0\} \cup \{\zeta_1 = \zeta_2 = \zeta_4 = 0\}$ ;
3. If  $k > 2$  then  $Z_\omega = Z'' := \{\zeta_1 = \zeta_3 = 0\} \cup \{\zeta_2 = \zeta_4 = 0\} \cup \{\zeta_1 = \zeta_2 = 0\} \cup \{\zeta_1 = \zeta_4 = 0\}$ .

Each fixed element  $\delta = (\lambda_1 \lambda_2^k, \lambda_2, \lambda_1, \lambda_2) \in G$  defines the mapping  $\delta : \mathbb{C}^4 \setminus Z(\Sigma) \rightarrow \mathbb{C}^4 \setminus Z(\Sigma)$  by the formula  $\zeta \rightarrow \delta \cdot \zeta$ , i.e.,

$$\begin{cases} \zeta_1 \rightarrow \lambda_1 \lambda_2^k \zeta_1, \\ \zeta_2 \rightarrow \lambda_2 \zeta_2, \\ \zeta_3 \rightarrow \lambda_1 \zeta_3, \\ \zeta_4 \rightarrow \lambda_2 \zeta_4. \end{cases} \quad (7)$$

**Proposition 1.** *The differential form  $\omega$  is invariant under the action of  $\delta$ .*

PROOF. By direct substitution, we obtain the following transformation laws for  $h(\bar{\zeta})$ ,  $d\zeta$ , and  $g(\zeta, \bar{\zeta})$ :

$$h(\bar{\zeta}) \rightarrow \bar{\lambda}_1^2 \bar{\lambda}_2^{k+2} h(\bar{\zeta}), \quad d\zeta \rightarrow \lambda_1^2 \lambda_2^{k+2} d\zeta, \quad g(\zeta, \bar{\zeta}) \rightarrow (\lambda_1 \bar{\lambda}_1)^2 (\lambda_2 \bar{\lambda}_2)^{k+2} g(\zeta, \bar{\zeta}).$$

Inserting them in  $\omega$ , we arrive at the assertion of the proposition.  $\square$

We now describe the behavior of  $\omega$  under the action of the group  $G : (\mathbb{C}^4 \setminus Z(\Sigma)) \times \mathbb{C}_*^2 \rightarrow \mathbb{C}^4 \setminus Z(\Sigma)$ , defined by (7).

**Lemma 1.** *The form  $d\zeta$  transforms as follows under the action of (7):*

$$d\zeta \rightarrow \lambda_1 \lambda_2^{k+1} d\lambda_1 \wedge d\lambda_2 \wedge h(\zeta) + \psi(\lambda, \zeta),$$

where  $h$  is determined by (6), and the form  $\psi$  has higher degree in  $\zeta$  than  $h(\zeta)$ .

**Lemma 2.** *The form  $h(\bar{\zeta})$  transforms by the following rule under the action of (7):*

$$h(\bar{\zeta}) \rightarrow \bar{\lambda}_1^2 \bar{\lambda}_2^{k+2} h(\bar{\zeta}).$$

It is not hard to prove lemmas 1 and 2 by direct substitution of the action of  $G$  into the forms  $d\zeta$  and  $h(\bar{\zeta})$ .

Let us note that since the denominator  $g$  is a function (not differential form), it transforms by the same rule as in Proposition 1 under the action of (7).

We thus come to the following

**Theorem 1.** *Under the action of (7) the form  $\omega$  transforms as follows:*

$$\omega \rightarrow \frac{d\lambda_1}{\lambda_1} \wedge \frac{d\lambda_2}{\lambda_2} \wedge \omega_0 + \omega_1 \quad (8)$$

with the positive form

$$\omega_0 = \frac{h(\bar{\zeta}) \wedge h(\zeta)}{g(\zeta, \bar{\zeta})}$$

of homogeneity degree zero under the action of the group  $G$  and with some form  $\omega_1$ , involving no conjugate differentials  $d\bar{\lambda}_i$  and having at most one differential  $d\lambda_j$  in each summand.

The form  $\omega_0$  is an analog of the Fubini–Studi form (2) for the projective space.

Recall that  $\Gamma_0^k = \Gamma_0^k(\rho)$  is the set (3). We now treat it as an integration cycle. The cycle  $\Gamma_0^k$  foliates over  $\mathbb{X}_k$  with fibers isomorphic to the real tori  $\mathbb{T}^2$  ( $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ ), i.e.,

$$\Gamma_0^k(\rho)/G_{\mathbb{R}} = \mathbb{X}_k, \quad (9)$$

where  $G_{\mathbb{R}} := \{(\lambda_1 \lambda_2^k, \lambda_2, \lambda_1, \lambda_2) : |\lambda_j| = 1, j = 1, 2\}$  (see [5, Theorem 4.1]). From this and Theorem 1 we see that the form  $\omega_0$  depends only on the orbits of the group  $G$  and consequently is well defined on  $\mathbb{X}_k$ . Moreover,  $\Gamma_0^k$  is not homologous to zero in  $\mathbb{C}^4 \setminus Z(\Sigma)$ .

At this point let us note that if  $k \geq 2$  then the singular set  $Z_\omega$  does not coincide with  $Z(\Sigma)$ . (This happens because the fan  $\Sigma$  is not strictly convex.) If  $k = 2$  then the singular set  $Z_\omega$  is a subset of  $Z(\Sigma)$ , and therefore the cycle  $\Gamma_0^k$  does not intersect  $Z_\omega$ . If  $k > 2$  then the cycle  $\Gamma_0^k$  can intersect the planes  $\{\zeta_1 = \zeta_2 = 0\}$  и  $\{\zeta_1 = \zeta_4 = 0\}$ . In this case we need to prove the following

**Proposition 2.** *The form  $\omega$  is bounded in the neighborhood of the planes  $\{\zeta_1 = \zeta_2 = 0\}$  and  $\{\zeta_1 = \zeta_4 = 0\}$ .*

PROOF. Let us show that the form  $\omega$  is bounded in the neighborhood of the plane  $\{\zeta_1 = \zeta_2 = 0\}$ . Let  $|\zeta_1| = \varepsilon_1$ , and  $|\zeta_2| = \varepsilon_2$ . Equalities (3) imply  $|\zeta_3|^2 = \rho_1 - \varepsilon_1^2 > \frac{\rho_1}{2}$  and  $|\zeta_4|^2 = \rho_2 - k\varepsilon_1^2 - \varepsilon_2^2 > \frac{\rho_2}{2}$  when  $\varepsilon_1$  and  $\varepsilon_2$  are sufficiently small. Note that for such  $|\zeta_k|$  we have that  $g \geq \frac{\rho_1^2 \rho_2^{k+2}}{2^{k+4}}$ , and the numerator  $h(\bar{\zeta}) \wedge d\zeta$  is bounded. Therefore, the form  $\omega$  is bounded in the neighborhood of  $\{\zeta_1 = \zeta_2 = 0\}$ . Similarly one can show that  $\omega$  is bounded in the neighborhood of  $\{\zeta_1 = \zeta_4 = 0\}$ .  $\square$

Proposition 2 implies that the form  $\omega$  is integrable over the cycle  $\Gamma_0^k$ .

**Corollary 1.** *The equality  $\int_{\Gamma_0^k} \omega = C$  holds, where  $C$  is some nonzero constant.*

PROOF. (8) and (9) imply

$$\int_{\Gamma_0^k} \omega = \int_{|\lambda_1|=1} \frac{d\lambda_1}{\lambda_1} \int_{|\lambda_2|=1} \frac{d\lambda_2}{\lambda_2} \int_{\mathbb{X}_k} \omega_0 = (2\pi i)^2 \int_{\mathbb{X}_k} \omega_0.$$

The last integral is a positive number by positivity of the form  $\omega_0$ , as required.  $\square$

Now, we prove the following

**Proposition 3.** *The form  $\omega$  is closed.*

PROOF. In fact we have to demonstrate that  $(g/\tilde{g})\bar{\partial}h - \bar{\partial}(g/\tilde{g}) \wedge h = 0$ . This would imply that

$$(g/\tilde{g})d(h \wedge d\zeta) - d(g/\tilde{g}) \wedge (h \wedge d\zeta) = (g/\tilde{g})dh \wedge d\zeta - d(g/\tilde{g}) \wedge h \wedge d\zeta = ((g/\tilde{g})\bar{\partial}h - \bar{\partial}(g/\tilde{g}) \wedge h) \wedge d\zeta = 0,$$

i.e., the form  $\omega$  is closed. By direct calculation of  $\bar{\partial}h$  and  $\bar{\partial}(g/\tilde{g})$  we get the statement of the proposition.  $\square$

**Proposition 4.** *Let  $f(\zeta)$  be a holomorphic function in a neighborhood  $U$  about the origin and let  $\rho_1, \rho_2$  be small enough to guarantee  $\Gamma_0^k \subset U$ . Then the following integral representation is valid:*

$$f(0) = \frac{1}{C} \int_{\Gamma_0^k} f(\zeta) \omega(\zeta), \quad (10)$$

where  $C$  is the normalization constant:  $\int_{\Gamma_0^k} \omega = C \neq 0$ .

PROOF. Since the form  $f\omega$  is  $\bar{\partial}$ -closed, the integral in (10) is independent of  $\rho_1, \dots, \rho_r$ . We rewrite it as

$$\begin{aligned} \int_{\Gamma_0^k} f(\zeta) \omega(\zeta) &= \int_{\Gamma_0^k} f(0) \omega(\zeta) + \int_{\Gamma_0^k} (f(\zeta) - f(0)) \omega(\zeta) = \\ &= C f(0) + \int_{\Gamma_0^k} (f(\zeta) - f(0)) \omega(\zeta). \end{aligned}$$

Let us show that the last integral vanishes. By substituting  $\zeta \rightarrow \tau\zeta$ , we obtain:

$$\begin{cases} \zeta_1 \rightarrow \tau^{k+1} \zeta_1, \\ \zeta_2 \rightarrow \tau \zeta_2, \\ \zeta_3 \rightarrow \tau \zeta_3, \\ \zeta_4 \rightarrow \tau \zeta_4. \end{cases}$$

Then the cycle  $\Gamma_0^k$  goes into the cycle  $\Gamma_\tau^k$ :

$$\begin{cases} |\tau^{k+1} \zeta_1|^2 + |\tau \zeta_3|^2 = \rho_1, \\ k |\tau^{k+1} \zeta_1|^2 + |\tau \zeta_2|^2 + |\tau \zeta_4|^2 = \rho_2. \end{cases}$$

The integral goes to

$$\int_{\Gamma_0^k} (f(\zeta) - f(0)) \omega(\zeta) = \lim_{\tau \rightarrow 0} \int_{\Gamma_\tau^k} (f(\zeta) - f(0)) \omega(\zeta) = \lim_{\tau \rightarrow 0} \int_{\Gamma_0^k} (f(\zeta\tau) - f(0)) \omega(\zeta\tau).$$

By Proposition 1 the form  $\omega$  is invariant under the substitution  $\omega(\zeta\tau) = \omega(\zeta)$ . Since all  $s_k$  are positive, we have  $\lim_{\tau \rightarrow 0} f(\zeta\tau) = f(0)$ . Thus

$$\lim_{\tau \rightarrow 0} \int_{\Gamma_0^k} (f(\zeta\tau) - f(0)) \omega(\zeta\tau) = \lim_{\tau \rightarrow 0} \int_{\Gamma_0^k} (f(\zeta\tau) - f(0)) \omega(\zeta) = 0.$$

The proof of the proposition is now completed.  $\square$

### 3. Integral Representation

We now consider the question of finding a domain  $D$ , such that the following integral representation is valid for every point  $z \in D$

$$f(z) = \frac{1}{C} \int_{\mu^{-1}(\rho)} f(\zeta) \omega(\zeta - z). \quad (11)$$

Consider the domain  $D = D_\rho$ :

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ |\zeta_2|^2 + |\zeta_4|^2 < \rho_2 - k\rho_1. \end{cases} \quad (12)$$

We will show that it is the required domain. Note that  $D$  is nonempty if the Kähler conditions (4) are satisfied.

Denote by  $Z_z(\Sigma)$  the translate  $z + Z(\Sigma)$ :

$$Z_z(\Sigma) = \{\zeta_1 - z_1 = \zeta_3 - z_3 = 0\} \cup \{\zeta_2 - z_2 = \zeta_4 - z_4 = 0\},$$

and let  $\Gamma_z^k$  be the translate  $z + \Gamma_0^k$ :

$$\Gamma_z^k : \begin{cases} |\zeta_1 - z_1|^2 + |\zeta_3 - z_3|^2 = \rho_1, \\ k|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2 + |\zeta_4 - z_4|^2 = \rho_2. \end{cases}$$

Denote by  $W = W_\rho$  2-circular polyhedron defined by the system

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ k|\zeta_1|^2 + |\zeta_2|^2 + |\zeta_4|^2 < \rho_2. \end{cases} \quad (13)$$

By  $W_{2\rho}$  we denote the domain like (13), where the right-hand sides of the inequalities are  $2\rho_1, 2\rho_2$ .

**Lemma 3.** *For each  $z \in D$  the cycle  $\Gamma_z^k$  lies in  $W_{2\rho}$ . Moreover, if the Kähler conditions (4) are satisfied then the homology  $\Gamma_z \sim \Gamma_0^k$  holds in the domain  $W_{2\rho} \setminus Z_z(\Sigma)$ .*

PROOF. Consider the following homotopy of the cycles  $\Gamma_0^k$  and  $\Gamma_z^k$ :

$$\begin{cases} |\zeta_1 - tz_1|^2 + |\zeta_3 - tz_3|^2 = \rho_1, \\ k|\zeta_1 - tz_1|^2 + |\zeta_2 - tz_2|^2 + |\zeta_4 - tz_4|^2 = \rho_2, \end{cases} \quad (14)$$

where  $0 \leq t \leq 1$ . We will prove that the cycle (14) is disjoint from  $Z_z(\Sigma)$  for any  $t$  in the interval  $[0, 1]$ .

Let us show that the cycle (14) is disjoint from the plane  $\{\zeta_1 - z_1 = \zeta_3 - z_3 = 0\}$  in  $Z_z(\Sigma)$ . Substituting it to (14) we get  $(1-t)^2(|\zeta_1|^2 + |\zeta_3|^2) = \rho_1$ . The last equality is false since  $(1-t)^2 \leq 1$  and  $|\zeta_1|^2 + |\zeta_3|^2 < \rho_1$ .

Similarly we show that the cycle (14) is disjoint from the plane  $\{\zeta_2 - z_2 = \zeta_4 - z_4 = 0\}$  in  $Z_z(\Sigma)$ . Substituting it to (14) we get  $k|\zeta_3 - tz_3|^2 = -(\rho_2 - k\rho_1) + (1-t)^2(|\zeta_2|^2 + |\zeta_4|^2)$  that never holds since  $(1-t)^2 \leq 1$  and  $|\zeta_2|^2 + |\zeta_4|^2 < \rho_2 - k\rho_1$ . This completes the proof of the lemma.  $\square$



We have thus proven the integral representation (11) for functions holomorphic in  $W_{2\rho}$ . Note that it suffices to take the holomorphy domain of the function  $f(z)$  in (11) to be  $W = W_\rho$ , since the latter is a convex domain whose boundary contains the cycle  $\Gamma_0^k$ . It follows from convexity of  $W$  that a function holomorphic in  $W$  and continuous in the closure of  $W$  can be approximated by polynomials in the closure of  $W$  for which the integral representation (11) is proven. Thus, we arrive at the following

**Theorem 2.** *Suppose that  $f(\zeta)$  is a holomorphic function in the domain  $W$  defined by (13) and  $f$  is continuous in the closure of  $W$ . Then the integral representation (11), with the cycle  $\Gamma_0^k$  defined by (3) is valid in the domain  $D$  defined by (12).*

*The author was supported by Grants MK-914.2007.1 and NSh-2427.2008.1 from the President of Russian Federation.*

## References

- [1] P.Griffiths, J.Harris, Principles of algebraic geometry, John Wiley & Sons, New-York et al.,1978.
- [2] A.M.Kytmanov, The Bochner-Martinelli Integral and its Applications, Birkhäuser Verlag, Basel et al., 1995.
- [3] B.V.Shabat, Value distribution of holomorphic mappings, Nauka, Moscow, 1982.
- [4] M.Audin, The Topology of Torus Actions on Symplectic Manifolds, *Progress in Math.*, **93**(1991), Birkhäuser Verlag, Boston et al.
- [5] D.A.Cox, Recent Developments in Toric geometry, *Proc. Symp. Pure Math.*, Providence, RI, AMS, **62**(1997), №2, 389-436.
- [6] D.A.Cox, Toric Residues, *Ark. Mat.*, **34**(1996), №1, 73-96.
- [7] A.A.Kytmanov, An analog of the Fubini-Studi form for two-dimensional toric varieties, *Sib. Mat. Zh.* **44**(2003), №2, 358-371; transl. in *Sib. Math. J.*
- [8] V.Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian  $T^n$ -spaces, *Progress in Math.*, **122**(1994), BirkhäuserVerlag, Boston et al.