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PRODUCTS OF CONJUGACY CLASSES IN PERFECT LINEAR GROUPS. EXTENDED COVERING NUMBER

Abstract. In this paper we obtain estimates of extended covering numbers for some classes of perfect linear groups.

1. Introduction

This paper is a continuation of the investigation of products of conjugacy classes in finite and in algebraic groups [10, 9, 11, 12]. The products of conjugacy classes of such groups were considered in a number of articles, in particular in [1, 4, 5, 9, 7, 10, 13, 14, 15, 16, 19], and others.

In most of these papers the authors deal with simple and quasisimple groups (i.e. central extensions of simple groups). However, the questions which were investigated for simple groups are of interest in a more general situation, namely, for perfect groups (by a perfect group we mean here a group G such that $G = [G, G]$). In the present paper we concentrate on the question of the extended covering number $\mathrm{ecn}(G)$ of a perfect group G. Recall (see [11]) that a *generating conjugacy class* C of a perfect group G is a conjugacy class such that $G = \langle C\rangle$ and the *extended covering number* of G is

$$\mathrm{ecn}(G) \overset{def}{=} \min\{n \mid C_1C_2\cdots C_n = G\}$$

for any n generating conjugacy classes $C_1, C_2, \ldots, C_n \subset G$. In [11] it has been shown that there exists a general constant d such that $\mathrm{ecn}(G) \leq d \cdot \mathrm{rank}\, G$ for every Chevalley group G (proper or twisted in the sense of [18]). In [11] it has been conjectured that there exists a constant c such that $\mathrm{ecn}(G) \leq c \dim V$ for every perfect linear group $G \leq \mathrm{GL}(V)$ such that the extended covering number $\mathrm{ecn}(G)$ exists. However, N. Nikolov found a counterexample [17]. Also, the question of estimates of $\mathrm{ecn}(G)$ as well as other questions that are connected with

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products of conjugacy classes for perfect groups seem to be much more difficult than the ones for quasisimple groups. This paper is an attempt to find approaches to study such products that presumably will allow us to gain a deeper understanding of the nature of perfect groups.

The first step for the extension of our theory from quasisimple groups to perfect groups is to find a relation between $\operatorname{ecn}(\Gamma)$ and $\operatorname{ecn}(G)$, where Γ is a perfect group and $G = \Gamma/N$ for some $N \lhd \Gamma$. We emphasize two cases (Section 3). First we consider a nilpotent normal subgroup N such that $V = N/[N, N]$ is an *augmentative* $\mathbb{Z}[G]$-module (i.e. $I_G V = V$ where I_G is the augmentation ideal of $\mathbb{Z}[G]$). In particular, if Γ is a finite group, we show that

$$\operatorname{ecn}(\Gamma) \leq (\operatorname{gen}(\Gamma) + 1)(\operatorname{gen}(N) + 1) \operatorname{gen}(G) \tag{1.1}$$

(here $\operatorname{gen}(\Gamma)$ and $\operatorname{gen}(G)$ are the minimal numbers of generators of Γ and G, respectively). Actually, we prove more general estimates (see Theorem 1, Corollaries 1, 2). In the second case $N = \prod_{j=1}^{j=m} Q_j$, where the Q_j are quasisimple finite groups and the natural action of G on $\{Q_j\}_{j=1}^{j=m}$ is transitive. In this case we show (Theorem 2) the existence of a general constant c such that

$$\operatorname{ecn}(\Gamma) \leq (1 + 2 \min\{c, \operatorname{ecn}(Q_1)\}) \operatorname{ecn}(G). \tag{1.2}$$

Using (1.1) and (1.2) we give in Section 4 estimates of $\operatorname{ecn}(\Gamma)$ for some classes of finite perfect groups.

In Section 5 we consider connected perfect algebraic groups. For a simple algebraic group G let $l(G) = \operatorname{rank} G + 1$ if all roots of the corresponding root system have the same length, or $l(G) = 2 \operatorname{rank} G$ in the other case. For a connected perfect algebraic group Γ let $l(\Gamma)$ be the maximum of the numbers $l(G_i)$, where G_i runs through all simple components of the group $\Gamma/R_u(\Gamma)$ (where $R_u(\Gamma)$ is the unipotent radical of Γ). For a connected perfect firm (see Definition 2.1) algebraic group Γ we show that

$$\operatorname{ecn}(\Gamma) \leq 2l(\Gamma) + 1$$

(Corollary 3).

2. Notation, terminology, conventions

2.1. Let G be a group. Here:

$[a, b] = aba^{-1}b^{-1}$ is a commutator;

if $x_1, \dots, x_n, y_1, \dots, y_n \in G$, then the following identity holds

$$(x_1 y_1 x_1^{-1})(x_2 y_2 x_2^{-1}) \cdots (x_n y_n x_n^{-1}) y_n^{-1} y_{n-1}^{-1} \cdots y_1^{-1} =$$

$$= [x_1, y_1](y_1 [x_2, y_2] y_1^{-1}) \cdots y_1 y_2 \cdots y_{n-1} [x_n, y_n] y_{n-1}^{-1} \cdots y_1^{-1} \cdots y_2^{-1} y_1^{-1}; \tag{*}$$

$Z(G)$ is the center of the group G;

$\mathrm{gen}(G)$ is the minimal number of generators of the group G;

if G_1, G_2 are groups, then $G_1 \circ G_2$ is the central product of G_1, G_2 (i.e. $G_1 \circ G_2 = (G_1 \times G_2)/H$ for some $H \leq Z(G_1 \times G_2)$); the symbol $\circ \prod G_i$ denotes the central product of several groups; a group G is called *quasisimple* if it is perfect and $G/Z(G)$ is simple; a *generating conjugacy class* C of the group Γ is a class such that $\langle C \rangle = \Gamma$.

2.2. Let A be a commutative ring and let V be an $A[G]$-module.

Let $I_{A[G]} = \{a_1(g_1 - 1) + a_2(g_2 - 1) + \dots + a_n(g_n - 1) \mid a_i \in A,\, g_i \in G\}$ be the *augmentation ideal* of $A[G]$. If $A = \mathbb{Z}$, we shall write I_G instead of $I_{\mathbb{Z}[G]}$. An $A[G]$-module V is called *augmentative* if $I_{A[G]} V = V$. Obviously, $I_{A[G]} V = I_G V$ and therefore an $A[G]$-module V is augmentative if and only if it is augmentative as a $\mathbb{Z}[G]$-module.

Definition 2.1. *A group Γ is called firm if there exists a normal series $1 = \Gamma_s \lhd \Gamma_{s-1} \lhd \dots \lhd \Gamma_1 \lhd \Gamma_0 = \Gamma$ such that each factor Γ_i/Γ_{i+1} is either a central product of quasisimple groups or it is abelian and an augmentative I_{Γ/Γ_i}-module.*

2.3. Here all algebraic groups are linear algebraic groups defined over a field K ([2]). We omit the field of definition K if it is assumed to be algebraically closed.

If G is an algebraic group and $X \subset G$ then $\overline{X}$ is the Zariski closure of X. Here $\overline{\mathrm{ecn}}(G)$ is a topological extended covering number of G (see [11]):

$$\overline{\mathrm{ecn}}(G) \stackrel{\mathrm{def}}{=} \min\{n | \overline{C_1 C_1 \dots C_n} = G\}$$

for any generating conjugacy classes $C_1, \dots, C_n$ of G.

3. Theorems of reduction

3.1. Nilpotent A-subgroups. Let N be a nilpotent group. Further, let $1 = N_{k+1} \lhd N_k \lhd \dots \lhd N_1 = N$ be a central series of N (here $N_i = [N_{i-1}, N]$). Put $V_i = N_i/N_{i+1}$.

Definition 3.1. *The group N is called an A-group if for every i the abelian group V_i is an A-module over a commutative ring A and for all $n \in N_k, m \in N_l$ and for every $a \in A$ the following congruence holds*

$$[an, m] \equiv a[n, m] \equiv [n, am] (\operatorname{mod} N_{l+k+1}).$$

If, in addition, $V = V_1$ is a finitely generated A-module, then we say that N is a finitely generated A-group.

Remark 3.1. The condition of the definition always holds for $A = \mathbb{Z}$. Also, if N is a connected unipotent algebraic group that is defined and split over a field K, then the condition holds for $A = K$.

Definition 3.2. *The set $\{n_1, \dots, n_c\} \subset N_1$ is a commutator generating set if*

$$N_2 = [N, N] = [n_1, N][n_2, N] \cdots [n_c, N].$$

The minimal length c of a commutator generating set will be denoted by $c_g(N)$ and will be called the commutator generating length of N.

For a finitely generated nilpotent A-group we can use its graded Lie algebra (over A). Indeed, let N be a finitely generated nilpotent A-group and let

$$L = L(N) = \sum_{1 \le i \le k} V_i$$

be a graded Lie algebra of N over A, where the Lie brackets are induced by commutators (we denote this bracket also by $[x, y]$).

Further, let

$$\rho : V_1 \longrightarrow \operatorname{End}_A(L)$$

be the map that is given by the formula

$$\rho(v)(l) = [v, l] \text{ for } v \in V_1, l \in L.$$

Under these assumptions the following proposition holds.

Proposition 1. *Let $m_1, \dots, m_r \in N$ be elements such that*

$$\rho(m_1 (\operatorname{mod} N_2))(L) + \cdots + \rho(m_r (\operatorname{mod} N_2))(L) = [L, L] = \sum_{2 \le i \le k} V_i.$$

Then $\{m_1, \dots, m_r\}$ *is a commutator generating set.*

Proof. Take $v_k = m_k (\operatorname{mod} N_2) \in L$ for every $k = 1, \dots, r$. Let $n \in N_i \setminus N_{i+1},\ i \geq 2$, and let $v \equiv n (\operatorname{mod} N_{i+1}) \in V_i$. There exists a sequence $l_1, \dots, l_r \in V_{i-1}$ such that

$$\rho(v_1)(l_1) + \cdots + \rho(v_r)(l_r) = v. \tag{3.1}$$

Let $q_1, \dots, q_r \in N_{i-1}$ be elements such that $l_k \equiv q_k (\operatorname{mod} N_{i-1})$ for every $k \leq r$. Then from (3.1) we get

$$n = [m_1, q_1][m_2, q_2] \cdots [m_r, q_r] n',\ \ n' \in N_{i+1}. \tag{3.2}$$

Now suppose that in (3.2) we have $n' \in N_j \setminus N_{j+1},\ j \geq i+1$. In the same way as we obtained (3.2) we get

$$n' = [m_1, q_1'][m_2, q_2'] \cdots [m_r, q_r'] n'',\ \ n'' \in N_{j+1} \tag{3.3}$$

for some $q_1', \dots, q_r' \in N_{j-1}$. Since $[[m_t, q_t], [m_f, q_f']] \in N_{i+j}$ and $[q_t, [m_f, q_f']] \in N_{i-1+j}$ for every $t, f \leq r$ and since $i-1+j \geq j+1, i+j > j+1$, we have from (3.2) and (3.3) that

$$n = [m_1, q_1 q_1'][m_2, q_2 q_2'] \cdots [m_r, q_r q_r'] n''',\ \ n''' \in N_{j+1}. \tag{3.4}$$

Comparing (3.2) and (3.4) shows that we can get our statement by induction. □

Remark 3.2. It is a well-known and simple fact that if $\{n_1, \dots, n_k\}$ is a set of generators of a nilpotent group N, then $[N, N] = [n_1, N]$ $[n_2, N] \cdots [n_k, N]$. It is easy to see that instead of generators we can take here A-generators if N is an A-group. Thus, $c_g(N) \leq \operatorname{gen}_A(N)$. Moreover, it is easy to see that if the elements $n_1, \dots, n_k \in N$ such that $\rho(n_1 (\operatorname{mod} N_2)), \dots, \rho(n_k (\operatorname{mod} N_2))$ generate the A-module $\rho(V_1)$, then

$$\rho(n_1 (\operatorname{mod} N_2))(L) + \cdots + \rho(n_k (\operatorname{mod} N_2))(L) = [L, L]$$

and according to Proposition 1 the set $\{n_1, \dots n_k\}$ is a commutator generating set and therefore $c_g(N) \leq k$.

Example 3.1. The difference between $c_g(N)$ and the numbers $\operatorname{gen}_A(\rho(V_1)) \leq \operatorname{gen}_A(V)$ can be large. Let $N = U$ be the maximal unipotent group of a simple algebraic group G defined over an algebraically closed field K. Then $\operatorname{gen}_K(\rho(V_1)) = \operatorname{gen}_K(N) = \dim_K U/[U, U] = \operatorname{rank} G$, but $c_g(N) = 1$ because $[U, U] = [u, U]$ for a regular unipotent element $u \in U$.

3.2. Covering of normal nilpotent A-groups.

Theorem 1. *Let Γ be a perfect group and let $N \lhd \Gamma$ be a normal nilpotent finitely generated A-subgroup. Further, let k be the minimal number such that $V = V_1 = N_1/N_2$ is an augmentative $A[\langle\gamma_1, \dots, \gamma_k\rangle]$ -module for some $\gamma_1, \dots, \gamma_k \in \Gamma$. Then*

$$\operatorname{ecn}(\Gamma) \le (k+1)(c_g(N)+1)\operatorname{ecn}(\Gamma/N).$$

Proof. In the following lemma we use the notation of **3.1**.

Lemma 1. *Let $m_1, \dots, m_r \in N$ be a commutator generating set and let $O_1, \dots, O_r$ be the conjugacy classes of $m_1, \dots, m_r$ in N. Further, let $m = m_1 m_2 \cdots m_r$. Then*

$$O_1 O_2 \cdots O_r = N_2 m.$$

Proof. From (*) we have

$$(x_1 m_1 x_1^{-1})(x_2 m_2 x_2^{-1}) \cdots (x_r m_r x_r^{-1}) m_r^{-1} m_{r-1}^{-1} \cdots m_1^{-1} =$$
$$= [x_1, m_1](m_1[x_2, m_2]m_1^{-1}) \cdots m_1 m_2 \cdots m_{r-1}[x_r, m_r]m_{r-1}^{-1} \cdots m_1^{-1} \cdots m_2^{-1} m_1^{-1}$$

for $x_i \in N$. By Proposition 1, the set

$$\{m_1, m_1 m_2 m_1^{-1}, \dots, (m_1 m_2 \cdots m_{r-1}) m_r (m_{r-1}^{-1} \cdots m_1^{-1})\}$$

is also a commutator generating set. Thus

$$O_1 O_2 \cdots O_r m^{-1} = N_2.$$

□

Now we prove Theorem 1.

Put $e = \operatorname{ecn}(\Gamma/N), c = c_g(N)$. Let $X, Y_1, \dots, Y_k, Z_1, \dots, Z_c$ be products of e generating conjugacy classes of Γ and let $\tilde{Z}_1, \dots, \tilde{Z}_c$ be products of ke generating conjugacy classes. We can fix $\gamma_1 \in Y_1, \dots, \gamma_k \in Y_k$ such that V is an augmentative $\mathbb{Z}\langle\gamma_1, \dots, \gamma_k\rangle$-module. From (*) (see [11], Proposition 1) it follows that

$$V \subset \gamma Y_1 Y_2 \cdots Y_k (\operatorname{mod} N_2), \tag{3.5}$$

where $\gamma = \gamma_k^{-1}\gamma_{k-1}^{-1}\cdots\gamma_1^{-1}$. For the same reason as for (3.5) there are element $\delta_1 \in Z_1, \dots, \delta_c \in Z_c$ such that

$$V \subset \delta_i \tilde{Z}_i (\operatorname{mod} N_2) \quad \text{for every} \quad i \leq c. \tag{3.6}$$

Now (3.6) implies that we can fix a minimal commutator generating set

$$\{m_1 \in \delta_1 \tilde{Z} \cap N, \dots, m_c \in \delta_c \tilde{Z}_c \cap N\}. \tag{3.7}$$

Put $m = m_1 m_2 \cdots m_c$. By Lemma 1,

$$N_2 \subset m^{-1} Z_1 \tilde{Z}_1 Z_2 \tilde{Z}_2 \cdots Z_c \tilde{Z}_c. \tag{3.8}$$

Since $m^{-1} \in N$, we see that (3.5) and (3.8) imply

$$N \subset \gamma(Y_1 Y_2 \cdots Y_k)(Z_1 \tilde{Z}_1 Z_2 \tilde{Z}_2 \cdots Z_c \tilde{Z}_c). \tag{3.9}$$

Now let $\sigma \in \Gamma$. We can find elements $\sigma_1, \tau \in \Gamma$ such that

$$\sigma_1 \tau \in X, \;\; \sigma = \sigma_1 n_1, \gamma = n_2 \tau, \;\; n_1, n_2 \in N. \tag{3.10}$$

From (3.9), (3.10)

$$\sigma_1 \tau = (\sigma_1 n_1)(n_1^{-1} n_2^{-1})(n_2 \tau) = \sigma n \gamma \in X, \;\; n \in N. \tag{3.11}$$

From (3.9) , (3.11)

$$\sigma \in X(Y_1 Y_2 \cdots Y_k)(Z_1 \tilde{Z}_1 Z_2 \tilde{Z}_2 \cdots Z_c \tilde{Z}_c).$$

The number of generating conjugacy classes in the product above is equal to $e + ke + c(ke + e) = (k+1)(c+1)e$. □

In Theorem 1 we do not suppose that V is an augmentative $\mathbb{Z}[G]$-module (thus $k = \infty$ is possible). In the following corollaries we keep the assumptions of Theorem 1. Moreover, we assume that V is an augmentative $\mathbb{Z}[G]$-module.

Corollary 1.

$$\operatorname{ecn}(\Gamma) \leq (\operatorname{gen}(\Gamma) + 1)(c_g(N) + 1)\operatorname{ecn}(\Gamma/N).$$

Proof. Obviously, $k \leq \operatorname{gen}(\Gamma)$. □

Corollary 2. *Suppose A is either a field or a local ring , or $A = \mathbb{Z}$ and $V = N_1/N_2$ is a finite abelian group. Then*

$$\begin{aligned}\operatorname{ecn}(\Gamma) &\le (k+1)(\operatorname{gen}_A(N)+1)\operatorname{ecn}(\Gamma/N)\\ &\le (\operatorname{gen}(\Gamma)+1)(\operatorname{gen}_A(N)+1)\operatorname{ecn}(\Gamma/N).\end{aligned}$$

Proof. This follows from Remark 3.2 and Corollary 1. □

3.3. Normal semisimple subgroups.

Let Γ be a perfect group and let $Q \lhd \Gamma$ be a finite normal subgroup of the form

$$Q = \prod_{j=1}^{j=m} Q_i,$$

where Q_i is a finite quasisimple group. Further, let

$$\pi : \Gamma \longrightarrow S_m$$

be the homomorphism that is induced by the action of Γ on the set $\{Q_j\}$.

Suppose $m = 1$. Then $\operatorname{Aut}(Q)$ is a soluble group (this follows from the Classification). Since Γ is a perfect group, the natural homomorphism $\Gamma \longrightarrow \operatorname{Aut}(Q)$ is trivial. Hence $\Gamma = Q \circ \Delta$, where $\Delta/Z = \Gamma/Q$ for some $Z \le Z(Q)$. In this case, $e(\Gamma) = \min\{\operatorname{ecn}(\Delta), \operatorname{ecn}(Q)\}$. Moreover, one can easily check that $e(\Gamma) \le e_M(1 + e_M \operatorname{ecn}(Q))$, where $e_M = \max\{\operatorname{ecn}(Q), \operatorname{ecn}(\Gamma/Q)\}$. In Theorem 2 below we exclude the case $m = 1$.

Theorem 2. *There exists a general constant c such that for every perfect group Γ and for every semisimple normal subgroup $Q \lhd \Gamma$ with a nontrivial ($m > 1$) transitive permutation representation π the following inequality holds*

$$\operatorname{ecn}(\Gamma) \le (1 + 2\min\{c, \operatorname{ecn}(Q)\})\operatorname{ecn}(\Gamma/Q).$$

Proof. Below we use the notion of a *big element.*

The elements of a finite quasisimple group G are called big if a product of any c conjugacy classes of them coincides with G, where c is a general constant that does not depend on G.

It is possible to define big elements since the existence of such c has been shown in [11]. Indeed, let us fix a constant c such that $c > \operatorname{ecn}(G)$ for every quasisimple sporadic group G and for every group of Lie type of Lie rank ≤ 8. If G is one of those groups, then every non-central element is big. If G is a central extension of an alternating group, then the preimages of cycles of maximal length are big [7]. Now let G be a group of Lie type corresponding to the simple root system $\{\alpha_1, \ldots, \alpha_r\}$, $r > 8$ (the numeration of roots corresponds here to [3]; for twisted groups of type ${}^2A_{2r}, {}^2A_{2r+1}$ we assign root systems B_r, C_r, respectively, as in [6]), let $G = BNB$ be the Bruhat decomposition, and let $B = HU$, $W = N/H$. Further, denote by

$$\dot{w}_a = \prod_{i \leq a} \dot{w}_{\alpha_i}$$

the product of the first a elements (in any order) of the form $\dot{w}_{\alpha_i}$, where $w_{\alpha_i} \in W$ is the corresponding reflection and $\dot{w} \in N$ is a preimage of $w \in W$. Let

$$g = \begin{cases} \dot{w}_r u, \ \ u \in U, \ \text{ if } \ G \ \text{ is of type } \ A_r, \ r > 8; \\ \dot{w}_{r-1} u, \ u \in U, \ \text{if } G \text{ is of type } B_r, C_r, r > 8, \ \text{or} \ \ D_{2l}, \ 2l = r > 8; \\ \dot{w}_{r-2} u, \ \ u \in U, \ \text{ if } \ G \ \text{ is of type } \ D_{2l+1}, \ 2l+1 = r > 8. \end{cases} \tag{3.12}$$

If we take elements that are conjugate to such g, we will have big elements as well if we add to this set all regular unipotent elements (see [11], Proof of Theorem 6). Below, by a big element in the case of a group of Lie type of rank > 8 we will mean either regular unipotent elements or elements that are conjugate to elements g of the form (3.12). For other groups, a big element is a non-central element. Note that sets of elements which we call big are invariant under automorphisms of G (for groups of Lie type this follows from the description of automorphisms [18], for others it is an obvious statement).

Lemma 2. *Let G be a finite quasisimple group, let $q \in G$ be a fixed element, and let $f \in \mathrm{Aut}(G)$ be a fixed automorphism. Then there exist two big elements $y_1, y_2 \in G$ such that $y_1 f(y_2^{-1})$ and $y_2 q$ both are big.*

Proof. We may assume $q \notin Z(G)$. For sporadic groups and for groups of small Lie rank the statement is trivial. Let $G/Z(G)$ be an alternating group, then there exists $y_2 \in G$ such that $y_2 (\operatorname{mod} Z(G))$ and $y_2 q (\operatorname{mod} Z(G))$ are cycles of maximal length (see [7]). Now $f(y_2^{-1})(\operatorname{mod} Z(G))$ is again a cycle of maximal length. Thus we can find an element $y_1 \in G$ such that $y_1(\operatorname{mod} Z(G))$ and $y_1 f(y_2^{-1})(\operatorname{mod} Z(G))$ are both cycles of maximal length.

Now let G be a group of Lie type with $\mathrm{rank}(G) > 8$. If $q \in Z(G)$, we can take $f(y_2^{-1}) = u \in U$ a regular unipotent element and $y_1 = \dot{w}_a$, where $a = r, r-1$, or $r-2$ with respect to the cases (3.12). Suppose $q \notin Z(G)$. Then there exists an element $\gamma \in G$ such that $\gamma^{-1}q\gamma = vu$, where $v \in U^-, u \in U$ (here U^- and U are groups generated by negative and positive root subgroups, respectively)[8]. Put $y_2 = \gamma\dot{w}_a v^{-1}\gamma^{-1}$, where $a = r, r-1, r-2$ is defined according to (3.12). Let w_0 be the longest element in the corresponding Weyl group and let $\dot{w}_0$ be a preimage of w_0 in N. Then $\dot{w}_0\dot{w}_a v^{-1}\dot{w}_0 = \dot{w}'_a u'$ for some element w'_a which is also defined according to (3.12) and for some $u' \in U$ (this follows from the definitions in (3.12)). Hence y_2 is also a big element. The element $y_2 q = \gamma\dot{w}_a u\gamma^{-1}$ is also big. Further, let $v' \in U^-$ be an element such that $v'v$ is a regular unipotent element. Put $y_1 = f(\gamma\dot{w}_a v'\gamma^{-1})$. Then this element is also big as well as the element

$$y_1 f(y_2^{-1}) = f((\gamma\dot{w}_a v'\gamma^{-1})(\gamma v\dot{w}_a^{-1}\gamma^{-1})) = f(\gamma\dot{w}_a(v'v)\dot{w}_a^{-1}\gamma^{-1})$$

because $v'v$ is a regular unipotent element. □

Lemma 3. *Let $n \geq \mathrm{ecn}(\Gamma/Q)$ and let $C_1, \ldots, C_n$ be generating conjugacy classes of Γ. Then there exists an element $\gamma \in C_1C_2\cdots C_n$ such that the permutation $\pi(\gamma)$ has no fixed point.*

Proof. This follows from the transitivity of π. □

Lemma 4. *Let $n \geq 2\,\mathrm{ecn}(\Gamma/Q)$ and let $C_1, \ldots, C_n$ be generating conjugacy classes of Γ. Then in the product $C_1C_2\cdots C_n$ one can find an element*

$$q = \prod_{j=1}^{j=m} q_j, \quad q_j \in Q_j,$$

where each q_j is a big element of Q_j.

Proof. Let $\gamma \in \Gamma$ be an element such that the permutation $\pi(\gamma)$ has no fixed point and let $\pi(\gamma) = s_1 s_2 \cdots s_k$ be the decomposition into independent cycles. We may assume that the cycles $s_1, \ldots, s_k$ correspond to the natural order on $[1, m]$, i.e., $s_1 = (12\ldots r), s_2 = ((r+1)(r+2)\ldots)\ldots$. Consider the cycle $s_1 = (12\ldots r)$. Note, all groups Q_j are isomorphic to each other. If we put $Q_0 = Q_1$ and fix the isomorphisms $\phi_j : Q_0 \longrightarrow Q_j$ defined by the formula $\phi(Q_0) = \gamma^{j-1}Q_0\gamma^{1-j}$, then we can consider elements of the groups Q_j also as elements of the group Q_0. By

$f_1 : Q_0 \longrightarrow Q_0$ we denote the automorphism defined by the formula $f_1(q) = \gamma^r q \gamma^{-r}$ for $q \in Q_0 = Q_1$. Further, if

$$y = \prod_{j=1}^{j=r} y_j, \quad y_j \in Q_j, \tag{3.13}$$

then

$$[y, \gamma] = (y_1 f_1(y_r^{-1})) \times (y_2 y_1^{-1}) \times \cdots \times (y_r y_{r-1}^{-1}), \tag{3.14}$$

where $(y_1 f_1(y_r^{-1})) \in Q_1, (y_j y_{j-1}^{-1}) \in Q_j$ for every $2 \leq j \leq r$.

Put in (3.13) $y_j = 1$ for even j. Then from (3.14)

$$[y, \gamma] = y_1 \times y_1^{-1} \times y_3 \times y_3^{-1} \times \cdots \times y_{r-1} \times y_{r-1}^{-1}, \tag{3.15}$$

where $y_j \in Q_j, \ y_j^{-1} \in Q_{j+1}$ if r is even and

$$[y, \gamma] = (y_1 f_1(y_r^{-1})) \times y_1^{-1} \times y_3 \times y_3^{-1} \times \cdots \times y_r, \tag{3.16}$$

where $(y_1 f_1(y_r^{-1})) \in Q_1, \ y_j \in Q_j, \ y_j^{-1} \in Q_{j+1}$ if r is odd.

Below we consider an element $y \in Q$ constructed according to some rule for a cycle s_1. If we construct elements $y(l)$ for every cycle s_l that is contained in $\pi(\gamma)$ according to the same rule and if we take the product of the elements $y(l)$ and denote it by y, we get from (3.15) and (3.16) a system of independent parameters $y_1, \ldots, y_s, \ s \leq m/2 + 1$, such that

$$[y, \gamma] = \prod_{j=1}^{j=m} v_j, \quad v_j \ \in Q_j, \tag{3.17}$$

where, under an appropriate renumeration of $y_1, \ldots, y_s$, we have

$$v_j = \begin{cases} y_j \quad \text{or} \\ y_{j-1}^{-1} \quad \text{or} \\ y_j f_l(y_r^{-1}) \quad \text{for some} \quad j < r \quad \text{and some} \quad l \leqslant k. \end{cases} \tag{3.18}$$

(In (3.18) f_l is the automorphism corresponding to the cycle s_l which is constructed by the same rule as f_1 for s_1.) Moreover, the same parameter y_j of (3.18) occurs for exactly two components in (3.17), namely, either $v_j = y_j, \ v_{j+1} = y_j^{-1}$, or $v_j = y_j f_l(y_r^{-1})), \ v_{j+1} = y_j^{-1}$ for some y_r and in the latter case $v_r = y_r$.

Now let $\delta \in \Gamma$ be an element satisfying the condition

$$\delta = \gamma^{-1} q', \quad q' \in Q, \tag{3.19}$$

and let

$$q' = \prod_{j=1}^{j=m} q'_j, \quad q'_j \in Q_j. \tag{3.20}$$

Further, consider again the decomposition $\pi(\gamma) = s_1 s_2 \cdots s_k$ and a cycle $s_1 = (12\ldots r)$. We have $\pi(\gamma^{-1}) = s_1^{-1}s_2^{-1}\cdots s_k^{-1}$ and $s_1^{-1} = (r(r-1)\ldots 1)$. Put

$$n_1 = \prod_{j=1}^{j=r} q'_j. \tag{3.21}$$

Further, consider the sequence of elements of the group Q_0

$$u_1 = q'_1, u_2 = q'_1q'_2, \ldots, u_{r-1} = q'_1q'_2\cdots q'_{r-1}, u_r = 1 \tag{3.22}$$

and consider each element u_j of this sequence also as an element of Q_j. Put

$$w_1 = \prod_{j=1}^{j=r} u_j. \tag{3.23}$$

From (3.13), (3.14) and (3.21)–(3.23) we get

$$[\gamma, w_1] = q_1'^{-1} \times (q'_1q_2'^{-1}q_1'^{-1}) \times \cdots \times (q'_1q'_2\cdots q'_{r-1})$$

and therefore

$$[\gamma, w_1](w_1n_1w_1^{-1}) = \prod_{j=1}^{j=r} q''_j, \quad q''_1 = \cdots = q''_{r-1} = 1, \quad q''_r = q'_1q'_2\cdots q'_r. \tag{3.24}$$

Now let us construct in the same way elements $w_2, \ldots, w_k$ and put $w = w_1 \times w_2 \times \cdots \times w_k$. We have

$$w\gamma^{-1}q'w^{-1} = \gamma^{-1}[\gamma, w](wq'w^{-1}) = \gamma^{-1}\prod_{l=1}^{l=k}[\gamma, w_l](w_ln_lw_l^{-1}). \tag{3.25}$$

By Lemma 3 we can find in a product of $e = \mathrm{ecn}(\Gamma/Q)$ generating conjugacy classes of Γ an element γ such that $\pi(\gamma)$ has no fixed point. In any other product of e generating conjugacy classes of Γ we can find an element δ of the form (3.19). Moreover, (3.24) and (3.25) imply that the element δ can be chosen such that the factor q' in (3.19) has trivial components except for one index in each cycle $\pi(\gamma)$ (like in (3.24)). Thus, in the product of $2e$ generating conjugacy classes of Γ we can find an element of the form

$$y\gamma y^{-1}\gamma^{-1}q' = [y,\gamma]q' = \prod_{j=1}^{j=m} v_j q'_j,$$

where each v_j has the form (3.18). Let j be the last number of a cycle $s_l = (j_1 j_2 \ldots j{-}1 j)$. Hence $v_j = y_j$ if s_l is an odd cycle, or $v_j = y_{j-1}^{-1}$ if it is an even cycle. Let s_l be an odd cycle. By Lemma 2 we can take parameters y_{j_1}, y_j such that both parameters as well as $y_j q'_j$ and $y_{j_1} f_l(y_j^{-1})$ are big. Also, we can take the parameters $y_{j_3}, \ldots, y_{j-2}$ to be big. Let s_l be an even cycle. By Lemma 2, we can take the parameters $y_{j_1}, \ldots, y_{j-1}$ to be big and such that $y_{j-1}^{-1} q'_j$ is also big. The construction of the element q' implies (see (3.24)) that $q'_{j_1} = q'_{j_1+1} = \cdots = q'_{j-1} = 1$ and therefore all elements

$$v_{j_1} q'_{j_1} = v_{j_1}, \ldots, v_{j-1} q'_{j-1} = v_{j-1}, v_j q'_j$$

are big. Since the same can be done for every cycle, we have our statement. □

Now we can prove Theorem 2.

Let c be the constant defined above and let

$$e_c = \min\{c,\ \operatorname{ecn}(Q_0)\}.$$

Let $\sigma \in \Gamma$, then in a product of $e = \operatorname{ecn}(\Gamma/Q)$ generating conjugacy classes of Γ we can find an element of the form σq for some $q \in Q$. Further, in a product of $2e$ generating conjugacy classes of Γ we can find an element $q' \in Q$ which has big components (Lemma 4) and hence we can find in a product of $2ee_c$ generating conjugacy classes every element of the group Q, in particular, the element q^{-1}. Thus, in a product of $e + 2ee_c$ generating conjugacy classes we can find every element σ of Γ.□

4. Extended covering numbers for some finite perfect groups

4.1. Firm groups.

Let Γ be a finite perfect group. Then there exists a series of normal subgroups of Γ :

$$1 = \Gamma_n \lhd \Gamma_{n-1} \lhd \cdots \lhd \Gamma_1 \lhd \Gamma_0 = \Gamma \qquad (4.1)$$

satisfying the following conditions:

a. Γ_1 is a minimal normal subgroup of Γ such that $\Gamma_0/\Gamma_1 = \circ\prod_j S_j$, where $\{S_j\}$ are quasisimple groups;

b. every factor Γ_i/Γ_{i+1} for $i > 1$ is either nilpotent or is isomorphic to $\circ\prod_{j=1}^{j=m} Q_j$, where $\{Q_j\}$ are isomorphic quasisimple groups, $m > 1$, and the natural action of Γ/Γ_i (by conjugation) on $\{Q_j\}$ is transitive.

Indeed, we can construct such series in the following way. Let $\Gamma_1 \lhd \Gamma$ be a minimal normal subgroup of Γ satisfying the condition a.. Obviously, we can find a decreasing series of normal subgroups $\{\Gamma_i\}_{i\geq 2}$ of Γ which are contained in Γ_1 and satisfying conditions b., but with $m \geq 1$. Let us show that in each factor isomorphic to a semisimple group we have $m > 1$. Suppose $m = 1$ for a factor i, i.e., $\Gamma_i/\Gamma_{i+1} \cong Q$ for a quasisimple finite group Q. Since $\mathrm{Aut}(Q)$ is a soluble group (this follows from the Classification) and Γ is a perfect group, the natural homomorphism $\Gamma/\Gamma_i \longrightarrow \mathrm{Aut}(Q)$ is trivial. Hence $\Gamma/Z(Q)\Gamma_{i+1} = Q/Z(Q) \times \Gamma/\Gamma_i$ and therefore there exists a normal subgroup $N \lhd \Gamma$ such that $\Gamma/N = Q\circ(\circ\prod_j S_j)$. This contradicts the choice of Γ_1. Thus the condition b. also holds.

Below we assume that n is the smallest number such that a. and b. hold. Let

$$n - 1 = n_1 + n_2, \tag{4.2}$$

where n_1 and n_2 are the numbers of the nilpotent and the semisimple factors Γ_i/Γ_{i+1}, $i > 1$, respectively.

Let d be a general constant such that

$$\mathrm{ecn}(S) \leq d \cdot \mathrm{rank}\ S \tag{4.3}$$

for every quasisimple perfect group S (where $\mathrm{rank}\, S$ is the Lie rank of S if S is a group of Lie type, $\mathrm{rank}\, S = n$ if $S/Z(S) = A_n$, and $\mathrm{rank}\, S = 1$ if S is a sporadic group (see [11]).

Proposition 2. *Let Γ be a firm finite group. Then*

$$\mathrm{ecn}(\Gamma) \leq (3\,\mathrm{gen}(\Gamma))^{n_1}(1+2c)^{n_2} \cdot \max\{d\,\mathrm{rank}\, S_j\},$$

where c is the general constant of Theorem 2.

Proof. Let $N \lhd G$ be a nilpotent normal subgroup of the finite group G and let every factor N_i/N_{i+1} be an augmentative $\mathbb{Z}[G]$-module. Then $\mathrm{ecn}(G) \leq 3\,\mathrm{gen}(\Gamma)\,\mathrm{ecn}(G/N)$ ([11, Proposition 3]). Now our estimate follows from Theorem 2 and (4.3). □

4.2. Two step groups. Let Γ be a finite perfect group that has a normal nilpotent subgroup $N \lhd \Gamma$ such that the corresponding factor $G = \Gamma/N = \circ\prod_{i=1}^{i=l} S_i$, where S_i is a quasisimple group.

Let $V = N_1/N_2$ where $N_1 = N$, $N_2 = [N, N]$. We may assume that V is an augmentative $\mathbb{Z}[G]$-module. Indeed, if it is not, there exists a normal subgroup $\tilde{N}$ of Γ which is contained in N such that $\Gamma/\tilde{N}$ is a central extension $\tilde{G}$ of G. Hence $\tilde{G} = \circ\prod_{i=1}^{i=l} \tilde{S}_i$, where $\tilde{S}_i$ is a central extension of S_i. Thus, we can consider $\tilde{N}$ instead of N. If we take the smallest normal nilpotent subgroup of Γ such that the corresponding factor is a central product of quasisimple groups, we will have our assumption. Now we can apply Corollary 2 to Γ:

$$\mathrm{ecn}(\Gamma) \le (\mathrm{gen}(G) + 1)(\mathrm{gen}(N) + 1)\,\mathrm{ecn}(G). \tag{4.4}$$

Using [11, Theorem 3] we get

$$\mathrm{ecn}(G) \le d \cdot \log_2 |G|. \tag{4.5}$$

Using $\mathrm{gen}(S_i) = 2$ we can simply get $\mathrm{gen}(G)+1 \le l+2$. Further, $\mathrm{gen}(N) \le \mathrm{gen}(N/[N, N]) \le \log_2 |(N/[N, N])| \le \log_2 |N|$. Now we have from (4.4) and (4.5)

Proposition 3. *Let Γ be a group satisfying the condition* **4.2**. *Then*

$$\mathrm{ecn}(\Gamma) \le d(l + 2)(\log_2 |N| + 1)\log_2 |G|.$$

4.3. Some linear groups. Let Γ be a finite perfect group satisfying the conditions **4.2**.

Proposition 4. *Let $\rho : \Gamma \longrightarrow GL(U)$ be a faithful irreducible representation. Suppose G is a quasisimple group of Lie type. Then*

$$\mathrm{ecn}(\Gamma) \le c \cdot \dim U \log_2 \dim U$$

for a general constant c which does not depend on Γ or ρ.

Proof. We will use

Lemma 5. *Let N be a nilpotent subgroup and let $A \lhd N$ be an abelian normal subgroup of N. Then*

$$c_g(N) \le \mathrm{gen}(N/A).$$

Proof. Let $L = L(N)$ be the graded Lie ring of N (see Section 3) and let $L_A \le L$ be the subalgebra generated by the elements

$(A \cap N_i)(\mathrm{mod}(N_{i+1}))$ for every i. Since $A \lhd N$, the algebra L_A is an abelian ideal of L. Let $n_1, \dots, n_r \in N$, $a_1, \dots, a_s \in A$ be elements such that $\{n_1(\mathrm{mod}(A)), \dots, n_r(\mathrm{mod}(A))\}$ is a minimal set of generators of N/A and $\{n_1, \dots, n_r, a_1, \dots, a_s\}$ is a set of generators of N. Since $[a_i, L_A] = 0$ for every $i = 1, \dots, s$ we have

$$[L, L_A] = [n_1, L_A] + \dots + [n_r, L_A]. \tag{4.6}$$

Let $\tilde{N} = \langle n_1, \dots, n_r\rangle$ and let $\tilde{L} \leq L$ be the Lie subring generated by the elements $(\tilde{N} \cap N_i)(\mathrm{mod}(N_{i+1}))$ for every i. Obviously,

$$L = \tilde{L} + L_A. \tag{4.7}$$

Now (4.6) , (4.7) and Proposition 1 imply that $\{n_1, \dots, n_r\}$ is a commutator generating set for N and hence we have our statement. □

Lemma 6. *Let $n = p^k$ and let $N \leq S_n$ be a transitive p-subgroup. Then*

$$\mathrm{gen}(N) \leq k.$$

Proof. Let

$$X = [1, n] = \bigcup_{i=1}^{i=p} X_i, \ \ X_i = [(i-1)p^{k-1} + 1, ip^{k-1}].$$

We consider S_n as a group acting on X and let P_i be a Sylow p-subgroup of the permutation group of X_i which is isomorphic to $S_{p^{k-1}}$. Further, let $\Sigma = \langle\sigma\rangle \leq S_n$ be the subgroup interchanging $\{X_i\}$ (we assume $\sigma^p = 1$). Let $P \leq S_n$ be a Sylow p-subgroup. Then we may assume

$$P = Q \cdot \Sigma, \quad Q = P_1 \times P_2 \times \dots \times P_p \lhd P, \quad N \leq P.$$

Let

$$\phi : P \longrightarrow P/Q = \Sigma, \ \ p_i : Q \longrightarrow P_i$$

be the natural homomorphisms. Since N is a transitive subgroup of S_n, we have $\phi(N) = \Sigma$. Let $n_0 \in N$ be a fixed preimage of σ. Further, let

$$O = \mathrm{Ker}(N \xrightarrow{\phi} \Sigma).$$

Then $N = \langle n_0, O\rangle$.

Further, the map p_i induces the surjective map

$$\bar{p}_i : V = O/[O,O]O^p \longrightarrow V_i = p_i(O)/[p_i(O), p_i(O)]p_i(O)^p.$$

The action of n_0 by conjugation on O induces the action on V. Moreover, $n_0^p(v) = v$ for every $v \in V$ because $n_0^p \in O$. Note, all subspaces V_i are isomorphic to each other and the element n_0 acts on the set $\{V_i\}$ as a p-cycle. We identify V_i with a vector space $n_0^{i-1}(V_1)$ and consider the $F_p[\langle n_0\rangle]$-module $\prod_i V_i$. Put $y \in \prod_i V_i$, then

$$y = (v_1, v_2, \dots, v_p), \quad v_1, v_2, \dots v_p \in V_1, \tag{4.8}$$

and

$$n_0(y) = (v_p, v_1, \dots, v_{p-1}). \tag{4.9}$$

Consider the action of n_0 on the vector space V over the field $\mathbb{F}_p$. Let $V = U_1 \oplus \cdots \oplus U_s$ be the decomposition of V into indecomposable $\langle n_0\rangle$-modules, where each U_j corresponds to a Jordan block of n_0. We have

$$s = \dim_{\mathbb{F}_p} V^{\langle n_0\rangle}. \tag{4.10}$$

If $x \in V^{\langle n_0\rangle}$ then (4.8) and (4.9) imply

$$x = (v, v, \dots, v), \; v \in V_1. \tag{4.11}$$

From (4.10), (4.11)

$$s \leq \dim_{\mathbb{F}_p} V_1. \tag{4.12}$$

Further, since N is a transitive group, the group $p_1(O)$ is also a transitive subgroup of $S_{p^{k-1}}$. We can make the induction hypothesis (with respect to k):

$$\operatorname{gen} p_1(O) \leq k - 1. \tag{4.13}$$

But $\operatorname{gen} p_1(O) = \dim_{\mathbb{F}_p} V_1$ and therefore (4.12) and (4.13) imply

$$s \leq k - 1. \tag{4.14}$$

Now for every $j \leq s$ choose one element $u_j \in U_j \setminus (n_0 - 1)U_j$. Then

$$\sum_{j=1}^{j=s} \mathbb{F}_p[\langle n_0\rangle] u_j = V. \tag{4.15}$$

Let $\{n_j\}_{j=1}^{j=s}$ be the set of preimages of elements u_j in the group O. Then (4.15) shows that $N = \langle n_0, n_1, \dots, n_s\rangle$. Now our statement follows from (4.14). □

Lemma 7. *Let $N \leq \mathrm{GL}(U)$ be an irreducible finite nilpotent group. Then*

$$c_g(N) \leq \log_2 \dim U.$$

Proof. Let $N = \prod_p N_p$ be the decomposition of N into the product of its Sylow p-groups. Since N is irreducible, the N-module U is decomposed into the tensor product $\otimes_p U_p$ of irreducible N_p-modules. Hence

$$\dim U = \prod_p \dim U_p. \tag{4.16}$$

Further, the union of commutator generating sets of all components $\{N_p\}$ is a commutator generating set of N. Hence

$$c_g(N) \leq \sum_p c_g(N_p). \tag{4.17}$$

Now (4.16) and (4.17) reduce our statement to the case of p-groups. Now we assume that N is a p-group. Since N is irreducible, N is monomial of dimension p^k for some k and there exists a transitive irreducible representation $\phi : N \longrightarrow S_{p^k}$. We have $A = \mathrm{Ker}\,\phi$ is an abelian normal subgroup of N. By Lemmas 5 and 6 we obtain

$$c_g(N) \leq \mathrm{gen}(\phi(N)) \leq k = \log_p \dim U \leq \log_2 \dim U.$$

□

Now we can prove Proposition 4.

Since ρ is irreducible, the Clifford Theorem implies that the restriction of ρ to N is a sum of irreducible representations $\rho_i : N \longrightarrow \mathrm{GL}(U_i)$, $i = 1, \dots, m$, and

$$U = U_1 + \cdots + U_m.$$

Moreover, the group $G = \Gamma/N$ acts on the set $\{U_i\}$ transitively. Hence there exists a transitive permutation representation

$$\phi : G \longrightarrow S_m.$$

Put $m_1 = \dim U_1$. Then

$$\dim U = m m_1. \tag{4.18}$$

Consider $N_i = \rho_i(N)$ for every $i = 1, \dots, m$. By Lemma 7,

$$c_g(N_i) \leq \log_2 m_i. \tag{4.19}$$

Let $X \subset N$ be a set such that $\rho_i(X)$ is a commutator generating set of N_i for every k. Since ρ is faithful, the set X is a commutator generating set of N. Now from (4.19),

$$c_g(N) \leq m \log_2 m_1. \tag{4.20}$$

From (4.3),(4.20) and Theorem 1,

$$\operatorname{ecn}(\Gamma) \leq 3d \cdot (m \log_2 m_1 + 1) \operatorname{rank} G. \tag{4.21}$$

(Here d is a general constant as in (4.3) and k as in Theorem 1 is equal to two because $\operatorname{gen}(G) = 2$.) Further, since G is a group of Lie type and the permutation representation ϕ is transitive, we have $m = 1$ or $m \geq c_0 \cdot 2^{\operatorname{rank} G}$ for a general constant c_0. Hence, if $m > 1$,

$$\operatorname{rank} G \leq c_1 \log_2 m \tag{4.22}$$

for a general constant c_1. Now, if $m > 1$, then from (4.18),(4.21), (4.22)

$$\operatorname{ecn}(\Gamma) \leq c \cdot m \log_2 m_1 \log_2 m \leq c \cdot \dim U \cdot \log_2 \dim U \tag{4.23}$$

for a general constant c.

Let ρ be an irreducible representation. Then $m = 1$ and $\dim U = m_1$. The kernel of the natural homomorphism $\theta : G \longrightarrow N/[N, N]$ (θ is induced by conjugation) is contained in the center of G (because G is quasisimple). Hence $\operatorname{gen}(N) \geq c_2 \operatorname{rank} G$ for some general constant c_2. Now let r, s be numbers from the proof of Lemma 5. We have $r \leq \log_2 m_1$. Since s is less or equal to the number of generators of a subgroup of the diagonal group of GL_{m_1}, we have $s \leq m_1$. Hence $\operatorname{gen}(N) = r + s \leq 2m_1$ and therefore $m_1 \geq c_3 \operatorname{rank} G$ for some general constant c_3. Now from (4.21) we have again (4.23). □

5. Perfect Linear Algebraic Groups

5.1. The extended covering number for firm groups. Let Γ be a linear perfect algebraic group, let $R_u(\Gamma)$ be its unipotent radical and $G = \Gamma/R_u(\Gamma)$. Since Γ is perfect, $G = G_1 \circ \cdots \circ G_k$ where G_i is a simple algebraic group corresponding to the root system R_i. Let r_i be the Lie rank of G_i. Put

$$l_i = \begin{cases} r_i + 1 & \text{if all roots of} \quad R_i \quad \text{have the same length,} \\ 2r_i & \text{if roots of } R_i \quad \text{have different length,} \end{cases}$$

and put

$$l(\Gamma) = \max\{l_i \mid 1 \leq i \leq k\}.$$

Theorem 3. *Let Γ be a connected linear algebraic group. Suppose Γ is a firm perfect group. Then*

$$\overline{\mathrm{ecn}}(\Gamma) \leq \overline{\mathrm{ecn}}(G) + 1 \ \leq \ l(\Gamma) + 1.$$

Proof. Put $e = \overline{\mathrm{ecn}}(G)$. Let $Q_1, \ldots, Q_e, Q_{e+1}$ be generating conjugacy classes of Γ and let $C_1, \ldots, C_e, C_{e+1}$ be their images in G. Further, let H be a maximal torus of G. There is only a finite number of maximal proper closed subgroups of G that contain the maximal torus H (see [2]). Hence there exists an open subset $X \subset C_{e+1}$ such that

$$\overline{\langle H, x\rangle} = G \quad \text{for every} \quad x \in X. \tag{5.1}$$

Let

$$R_1 = [R_u(\Gamma), R_u(\Gamma)], \ldots, R_k = [R_{k-1}, R_u(\Gamma)], R_{k+1} = [R_k, R_u(\Gamma)] = 1.$$

Further, let $V_i = R_i/R_{i+1}$. Then V_i is an augmentative $\mathbb{Z}[G]$-module for every i (because Γ is a firm perfect group). Let $[V_i, H] = \langle [v, h] \ \mid \ v \in V_i, h \in H\rangle$. From (5.1) we have for every i

$$[V_i, H] + [V_i, x] = V_i \quad \text{for every} \quad x \in X. \tag{5.2}$$

Let $\chi_{i1}, \ldots, \chi_{is_i}$ be the non-trivial weights corresponding to the natural linear representation of the torus H on V_i. Then

$$\{h \in H \ \mid \ [V_i, h] \neq [V_i, H] \quad \text{for some} \quad i\} = \bigcup_{i,j} \mathrm{Ker}(\chi_{ij}) \neq H. \tag{5.3}$$

Now (5.2) and (5.3) imply that there exists an open subset $Y \subset H$ such that for every i

$$V_i \quad \text{is an augmentative} \quad \mathbb{Z}[\langle y, x\rangle]\text{-module for every} \quad x \in X,\ y \in Y. \tag{5.4}$$

Since

$$\overline{C_1 C_2 \ldots C_e} = G,$$

there exists a dense open subset $Y_1 = Y \cap C_1 C_2 \cdots C_e \cap H$ in H such that for every $y_1 \in Y_1$ and for every $x \in X$ the $\mathbb{Z}[\langle y_1, x\rangle]$-module V_i is augmentative for every i (this follows from (5.4)). Since the set $\{gHg^{-1} \ \mid \ g \in G\}$ contains a dense open subset of G [2], we have a dense subset $M = \{(y, x)\}$ of $G \times \overline{C_{e+1}}$ such that V_i is an augmentative $\mathbb{Z}[\langle y, x\rangle]$-module for every $(y, x) \in M$ and every i and, moreover, $\{y \mid (y, x) \in M\}$ is a dense open subset of G. Obviously,

$$\overline{\{yx \ \mid \ (y, x) \in M\}} = G. \tag{5.5}$$

Now if $g = yx$, $(y, x) \in M$, we have for $u_1, u_2 \in R_u(\Gamma)$

$$
\begin{aligned}
(u_1 y u_1^{-1})(u_2 x u_2^{-1}) &= yx(x^{-1}y^{-1})(u_1 y u_1^{-1})(u_2 x u_2^{-1}) \\
&= g(x^{-1}[y^{-1}, u_1]x[x^{-1}, u_2]). \qquad (5.6)
\end{aligned}
$$

For every $u \in R_u(\Gamma)$ there exist $u_1, u_2 \in R_u(\Gamma)$ such that

$$u = x[y^{-1}, u_1]x^{-1}[x^{-1}, u_2] \qquad (5.7)$$

(because V_i is an augmentative $\mathbb{Z}[\langle y, x\rangle]$-module for every i ([11], proof of Proposition 2)). Now (5.5), (5.6), and (5.7) imply

$$\Gamma = \overline{Q_1 Q_2 \cdots Q_e Q_{e+1}}. \qquad (5.8)$$

Now our statement follows from (5.8) and [11, Theorem 1]. □

Theorem 3 implies $\operatorname{ecn}(\Gamma) \leq 2l(\Gamma) + 2$. However, the proof of Theorem 3 supplies a better estimate.

Corollary 3. *Let Γ be a connected linear algebraic group that is also a firm perfect group, then*

$$\operatorname{ecn}(\Gamma) \leq 2l(\Gamma) + 1.$$

Proof. The proof of Theorem 3 implies that there is a dense open subset $S \subset G$ such that $\{gu \mid g \in S, u \in R_u(\Gamma)\} \leq Q_1 Q_2 \cdots Q_{e+1}$ for all generating conjugacy classes $Q_1, Q_2, \dots, Q_{e+1}$, where $e = l(\Gamma)$. Further, let $P_1, \dots, P_e$ be generating conjugacy classes of Γ and let $D_1, \dots, D_e$ be their images in G. Since $D_1 D_2 \cdots D_e$ contains a dense open subset S_1 of G and $S_1 S = G$, we have

$$\Gamma = P_1 P_2 \cdots P_e Q_1 Q_2 \cdots Q_{e+1}.$$

□

Example 5.1. Let $G = \mathrm{SL}_n(C)$ and let $V = U \oplus \cdots \oplus U = nU$, $n > 2$, where U is an irreducible 3-dimensional G-module. Further, let $\Gamma = V \cdot G$ be a semidirect product. Then Γ is a connected algebraic group that is also a firm perfect group. Here $l(G) = r(G) + 1 = 2$. Let us show

$$\overline{\operatorname{ecn}}(\Gamma) = 3.$$

Indeed, let Q be a conjugacy class of Γ generated by a semisimple element $g \in G$ of order > 2. Consider the set

$$M = V \cap Q^2 = V \cap \{(v_1 \sigma_1 g \sigma_1^{-1} v_1^{-1})(v_2 \sigma_2 g \sigma_2^{-1} v_2^{-1})$$

$$\mid\ \sigma_1, \sigma_2 \in G,\ v_1, v_2 \in V\}.$$

Put $g_1 = \sigma_1 g \sigma_1^{-1}, g_2 = \sigma_2 g \sigma_2^{-1}$. The element $(v_1 g_1 v_1^{-1})(v_2 g_1 v_2^{-1})$ belongs to V if and only if $g_2 = g_1^{-1}$. Put $g_1 = \gamma$. Then $(v_1 g_1 v_1^{-1})(v_2 g_2 v_2^{-1}) = [v_1, \gamma][\gamma, v_2] \in V_\gamma$, where V_γ is the direct γ-invariant complement of V^γ. Since $\dim V_\gamma = 2n$, we have $\dim M \leq \dim V_\gamma + 2 < \dim V$ if $n > 2$. Hence $\overline{Q^2} \neq \Gamma$ and therefore $\overline{\mathrm{ecn}}(\Gamma) > 2$. According to the Theorem 3, $\overline{\mathrm{ecn}}(\Gamma) \leq 3$. Hence $\overline{\mathrm{ecn}}(\Gamma) = 3$.

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