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NON–DEGENERATE SYSTEMS AND GENERIC PROPERTIES OF THE INTEGRABLE HAMILTONIAN SYSTEMS

1. Introduction

Among all integrable hamiltonian systems of special interest are so-called non-degenerate systems. The class of non-degenerate systems was defined by A.T. Fomenko in 1986 for needs of theory of topological classification of such systems (see [4, 3]). But this class was in use before implicitly, since numerous elucidations of the topology of integrable systems showed that most of known integrable systems arised from classical mechanics and physics are non-degenerate for almost all energy levels (see, for instance, [9, 11]). The purpose of this note is to clarify whether the non-degenerate systems are generic among all integrable. This question is important, because for non-degenerate systems there was discovered Fomenko-Ziechang isoenergetic topological invariant that describes the topology of such systems up to the topological equavalence. Also non-degenerate systems are now better understood (see [2, 4, 5, 7, 10]).

Let $(M^4\,,\,\omega)$ be a symplectic manifold. Consider a smooth Hamilton function H and its vector field sgrad H. This vector field is characterised by equality $\omega(\xi\,,\,\mathrm{sgrad}\,H) = \xi(H)$ that ought to be held for all $\xi \in T_x M$. Here $\omega(\cdot\,,\cdot)$ is the symplectic form and $\xi(\cdot)$ is the differentiating along ξ. We suppose this system to be integrable by an additional integral F. Let $Q_h = H^{-1}(h)$ where h is a regular value of H.

Definition 1. *We say that integral F is non-degenerate on Q_h if the critical points of $F\,|_{Q_h}$ comprise non-degenerate manifolds, i.e., the Hessian of $F\,|_{Q_h}$ does not vanish on the transversal planes. We call system $v = \mathrm{sgrad}\,H$ non-degenerate if it admits non-degenerate integral. Otherwise, call this system degenerate.*

For an integrable system $v = \mathrm{sgrad}\,H$ and its fixed integral F introduce the momentum mapping $\mu : M^4 \to R^2$ by the formula

$$\mu(x) = \big(H(x)\,,\,F(x)\big) \quad x \in M^4$$

Definition 2. *Given*

$$K = \{x \in M^4 \mid \operatorname{rank}(dH(x)\,,\, dF(x)) < 2\},$$

the set $\Sigma = \mu(K)$ *is called bifurcational diagram.*

Introduce two metrics on the set of all integrable systems on $(M^4\,,\,\omega)$. We would consider the set of non-degenerate systems relatively these metrics. Our aim is to obtain the characteristics of this set in terms of the general topology.

The set of all Hamilton functions for which the systems are integrable inherits the natural C^r-metric from the space of all C^r-smooth functions on M^4. So, the systems are close if their Hamilton functions are close in C^r-metric. We shall call this metric *weak* . It should be noted that for a given hamiltonian system the Hamilton function is determined up to the addition of a constant. So, we shall work with the functions and differ them even if they are equal up to the constant.

The second metric will be defined analogously. Consider the set of all pairs $(H\,,\,F)$ where H is a Hamilton function and F is an integral such that Poisson bracket $\{H\,,\,F\}_\omega$ vanishes. This set inherits C^r-metric from the space of all pairs of C^r smooth functions on M^4. We shall call this metric *strong*. It should be noted that the perturbation in the strong metric is the same that perturbation of the Poisson action on M^4.

2. Integrable systems and weak metric

Theorem 1. *(on the density of degenerate systems)*

a) *Every integrable (non-degenerate) system* $v = \operatorname{sgrad} H$ *can be made integrable degenerate on the given level* $\{H = h\}$ *by a small perturbation in the weak mertic.*

b) *We can destroy the topological structure (= Fomenko-Zieschang invariant) of an arbitrary integrable system and add the arbitrary number of critical circles preserving the integrability and non-degeneracy of this system.*

Proof. Let T_1 be a regular torus of the system $v = \operatorname{sgrad} H$. Then there are the action-angle coordinates I_1, I_2 ϕ_1, ϕ_2 in a neighborhood of T_1, in which H is a function of I_1, I_2. In this coordinates this system looks as following

$$\frac{dI_j}{dt} = 0 \quad ; \quad \frac{d\phi_j}{dt} = \omega_j = -\frac{\partial H}{\partial I_j}$$

We can change H inside this neighborhood in such a way that it remains the function of I_1, I_2 and then the system $v = \operatorname{sgrad} H$ remains integrable. Therefore, we can assume the following conditions to be held:

$$\det\left(\frac{\partial^2 H}{\partial I_i\, \partial I_j}\right) \neq 0 \qquad \text{and} \qquad \frac{\partial^2 H}{\partial I_i\, \partial I_j} \neq 0.$$

Then, there is exists a torus T for which $\frac{\omega_1}{\omega_2}$ is rational. This means that the trajectories of the system $v = \operatorname{sgrad} H$ are closed on T. Then we can choose I_1 in such a way that the vector fields $\operatorname{sgrad} H$ and $\operatorname{sgrad} I_1$ are linearly dependent on T. If we consider I_1 as an integral of the system $v = \operatorname{sgrad} H$ then the torus T is a critical submanifold relatively the I_1. Assume that I_i are zeros on T. Consider the following set

$$\mathbf{H} = \{H \mid \{H, I_1\} = 0 \text{ in a neighborhood of } T\}.$$

Clearly, it is the set of functions that does not depend on ϕ_1 in a neighborhood of T. H belongs to $\mathbf{H}$. We will perturbe H in the class of $\mathbf{H}$. Let $h = H\mid_{\phi_1=0}$. Note that h is a regular smooth function of I_1, I_2, ϕ_2. Show that $I_1\mid_{h=\alpha,\phi_1=0}$ has degenerate points (in the sence of Morse). Indeed, T is a critical torus. Hence these points lie on T and have $\phi_1 = 0$, so they are not isolated. Perturbe h as follows. $h_\varepsilon = h + \varepsilon g(I_1, I_2) sin(k\phi_2)$, where g is a smooth function such that $g = 1$ inside a small enough neigborhood of the origin and $g = 0$ outside a larger one. Taking ε small we obtain a small perturbation. Now it is easy to check that $I_1\mid_{h_\varepsilon=\alpha,\phi_1=0}$ has non-degenerate points only (in the sence of Morse). New system is integrable and has critical circles in a small neighborhood of T. Obviously, we destroyed the topological structure by the adding the critical circles. We can make degenerate singularities by the same way. We should take another function w instead of sinus, such that $w = \text{const}$ in a small interval. Theorem is proved.

Definition 3. *Non-trivial integral F is called periodic if all trajectories of system $u = \operatorname{sgrad} F$ are closed.*

Lemma 1. *Let O be a non-degenerate saddle circle of the system $v = \operatorname{sgrad} H$ relatively periodic integral F. Suppose that the deformed system $v = \operatorname{sgrad} H_\varepsilon$ is integrable. Then, O will, slightly deformed, remain a critical circle of the new system provided that H and H_ε are close with respect to C^r metric ($r > 2$).*

Proof. One can perturb symplectic structure ω instead of H, because H is a regular function in a neighborhood of O.

Let $Q_h = \{x \in M \mid H(x) = h\}$ and O lies in Q_0. Let Π^2 be a two dimensional transversal to O in Q_0. Poincare mapping $\pi : \Pi^2 \to \Pi^2$ is defined in a small neighborhood of $x_0 = O \cap \Pi^2$. Note that x_0 is a fixed point of diffeomorphism π. Thus the differential $d\pi$ is well defined at x_0. The circle O is non-degenerate relatively periodic integral F. This implies that the eigenvalues of $d\pi$ are not equal to 1. Therefore, the fixed point of a close diffeomorphism exists and is close to x_0. We used there that H and H_ε are close with respect to C^r metric ($r > 2$).

Till now we have not used the integrability of perturbed system. We showed that the new system has a closed trajectory near O. We need to show that this trajectory is critical. If it is not the case, then this trajectory lies in a regular torus. Hence, one of eigenvalues of $d\pi_\varepsilon$ should be equal to 1. Lemma is proved.

One can see, that the main reason of the such stability is that the eigenvalues of the differential of Poincare map are not trivial. Hence, this lemma can be extended on the case of minimal and maximal circles.

The proof of the previous lemma implies also, that if O satisfies the condition in this lemma, then O is an essential critical circle. This means that we can not choose another integral G such, that O is not a critical circle of the system $v = \operatorname{sgrad} H$ relatively G.

Definition 4. *We say that a trajectory $\gamma(t)$ of an integrable system $v = \operatorname{sgrad} H$ is an essential critical one if there is no additional integral G such that $\gamma(t)$ lies in a regular torus.*

Obviously, the saddle circles are essential provided that they are hyperbolic. Note that we are able to create a new hyperbolic circle "at arbitrary place" by a small deformation of H. This follows from the proof of theorem 1.

Definition 5. *The topological space T is called a space of the first category if it can be presented as at most countable union of nowhere dense sets.*

Using the similar technique as in the proof of theorem 1 we obtain

Theorem 2. *The space of all (C^r smooth) integrable systems on a compact symplectic manifold (M, ω) is of the first category (with respect to weak metric C^r, $r > 2$).*

Proof. The manifold M has a countable basis of open sets $\{S_n\}_{n=1}^{\infty}$. Consider an integrable system on (M, ω) and its Hamiltonian function H. A given S_k can intersect with essential critical trajectories of this system. Let $\mathbf{H}_k$ be the set of all Hamiltonian functions satisfying the following conditions:

1) The corresponding system $v = \operatorname{sgrad} H$ is integrable.

2) The open set S_k does not intersect with essential critical trajectories of the system $v = \operatorname{sgrad} H$.

Show that $\cup_{k=1}^{\infty}\mathbf{H}_k$ is the set of all integrable Hamiltonians on M. Indeed, a given integrable system has an open set D filled with regular tori. Then there exists a number n such that $S_n \subset D$.

Show that $\mathbf{H}_k$ is nowhere dense set in the set of all integrable Hamiltonians. Let $H \subset \mathbf{H}_k$. Perturb H in such a way that the new system has an essential critical trajectory intersecting S_k. One can do this in the same way as in the proof of the theorem 1. Denote the new Hamiltonian by H_ε. The previous lemma implies that we can not eliminate this trajectory by a small deformation of H_ε. This completes the proof.

Remark. In fact, we do not need in the compactness of M. When M is not a compact manifold then we should consider integrable systems $v = \operatorname{sgrad} H$ for which the regular Liouville tori comprise an open set.

Corollary. *Every subset of the set of all integrable systems on a symplectic manifold (M^4, ω) is of the first category (in weak metric).*

Now, let us study whether the non-degenerate systems are dense in some sense. One can show that non-degenerate systems are dense within a subclass of integrable systems which admit periodic integral even locally.

Theorem 3. *Let U be an open set in M^4 such that there is defined a periodic integral F on U of the system $v = \operatorname{sgrad} H$. Then, for given compact domain U_1 such that $U_1 \subset U$ and a number h one can perturb H inside U in such a way that*

1) *H_ε coincides with H outside U.*

2) *H_ε commutes with F inside U.*

3) *F is non-degenerate integral of the system $v = \operatorname{sgrad} H_\varepsilon$ in $U_1 \cap \{H_\varepsilon = h\}$.*

Remark. One can see, that h must be such that $U_1 \cap \{H = h\}$ is not an empty set. Otherwise, this theorem is trivial.

Proof. of this theorem uses the similar technique as above. Periodic integral F allows us to perform deformations of H using the standart transversal theorems. That is why it is worth to show the brief draft of this proof.

Consider a transversal Π^3 to the trajectories of the system $u = \text{sgrad}\, F$. We can restrict H on Π^3, and consider the family of two-dimensional surfaces $\Pi^2_\alpha = \Pi^3 \cap \{H = \alpha\}$. Note, that $H\, |_{\Pi^3}$ is a regular function. Thus, Π^2_α is a smooth submanifold.

We aim to perturb H in such a way, that $F\, |_{\Pi^3 \cap H = \text{const}}$ has non-degenerate critical points only. Then F would be non-degenerate integral for perturbed system. Consider the Poincare mapping $\mathbf{F} : \Pi^2_\alpha \to \Pi^2_\alpha$ of system $u = \text{sgrad}\, F$. This mappins is defined on Π^2_α for every α, so we will use one symbol $\mathbf{F}$ to denote it. Obviously, $\mathbf{F}$ preserves H. H_ε should be invariant under $\mathbf{F}$ also, since they should commute. F is a periodic integral. Thus, there is a number N such that $\mathbf{F}^N = id$. Call positive number n the period of a point $x \in \Pi^2_\alpha$, if $\mathbf{F}^n(x) = x$ and $\mathbf{F}^k(x) \neq x$ for every k, $0 < k < n$. This is the case of the main theorem from [12]. This theorem implies that all points $x \in \Pi^2_\alpha$ have the same period n except may be the finite number of them.

We would like to make all critical points of the restriction $F\, |_{\Pi^2_h}$ non-degenerate in the sense of Morse, by a small perturbation of H. Show, that it is possible in a neighborhood of the special points, for which the period is not equal to periods of other points. It is easy to see that in these neighborhoods we can uniformly choose the coordinates (x, y, α) in which the diffeomorfism $\mathbf{F}$ is a rotation trough an angle $2\pi s/n$ with axis $\{x = 0, y = 0\}$. Here s is an integer number, such that $(s, n) = 1$. Thus, we can use the equivariant theory of singularities to perform our perturbation inside the neighborhoods of special points. Then, cut off these neighborhoods and consider the space of orbits under the action of diffeomorfism $\mathbf{F}$. This set is obvoiusly a smooth manifold. We can think that H and F are smooth functions on it, since they are equivariant. Then, apply the standart theorem of transversality to the quotient functions. Theorem is proved.

This theorem supposes the existing of a periodic integral. In fact, periodic integral does not exist always. There is an example when periodic integral can not be defined even in a small neighborhood of a certain closed trajectory. There is a purely topological obstruction. This example can be found in [6].

Let us consider when this integral can exist. Remember how the action coordinate can be defined. One should take one-form κ such that $d\kappa = \omega$. In a small neighborhood of a regular torus κ can be defined always. Then one should integrate κ along a generator of tori that lie in a this neighborhood.

This technique works when we consider regular tori. It is an open question when it can be extended onto singular fibres. It is possible to do, when one can choose a common generator of all fibres. In example that is mentioned above these fibres have not any common generator. One can show (see [6]) that the assumption that a periodic integral exists in theorem can be replaced by geometrical conditions. Namely, we could assume that

1) the critical manifolds are circles only, and the critical level sets of $F|_{Q^3}$ are glued of Liouville cylinders and these circles, and

2) the bifurcational diagram can be presented as a finite union of continue curves with equations $F(H) = w_i(H)$.

Let us summary the properties of integrable systems in weak metric. The systems that admit non-degenerate integral at fixed isoenergy submanifold are dense within the systems with "not so comlex" topological structure. But integrable systems are topologically unstable under the perturbations in weak metric.

To summarise the propositions from above, consider how can change the Fomenko invariant after a small deformation in weak metric. We shall suppose that initial and deformed systems are both non-degenerate, i.e., they admit non-degenerate integrals for certain level of hamilton functions. Let us formulate results in terms of the Fomenko inveriant. Recall, that this is a graph Γ. To each vertex of Γ there is assigned a letter-atom which codes a neighborhood of connected component of critical level set of $F|_{Q^3}$. Each edge is related to the family of Liouville tori, which connects two critical value sets. More accurately definition of this invariant can be found in [3].

According to lemma 1, critical circles remain, if the initial circles meet a certain condition, which is satisfied most often. But a letter-atom can dissociate into more simple atoms. This is the case when two or more critical circles left a common level of F. In terms of invariant this means that a vertex is replaced by a certain subgraph. The simplest letter-atoms, that correspond to critical level sets with one critical circle, can not dissociate. There are three types of such letter-atoms. They were denoted by A, A^* and B in [3].

Theorem implies, that new essential critical circles can be borned. In terms of the Fomenko invariant this means that new vertices can be placed onto an edge of Γ.

Let us now consider the following situation. Given an integrable system $v = \operatorname{sgrad} H$ and its fixed integral F. Denote by H_ε a perturbed hamiltonian, such that the system $v = \operatorname{sgrad} H_\varepsilon$ is integrable. It may happen that this system has not an integral F_ε, which is close to F.

Such is the case more often than not. And this is the chief reason of structural unstability.

The formalism of weak metric seems to be natural, since one can think that we get a "random" hamiltonian and then try to find an integral. Under this phylosophy the structural unstability is not a significant factor.

3. Integrable systems and Strong Metric

Frequently, integrable systems from classical mechanics are, in fact, the systems with parameters. Both Hamilton function and the additional integral depend smoothly on parameters. This situation should be studied using the formalism of strong metric. We will see below, that non-degenerate systems are topologically stable when small perturbations in strong metric. The following theorem is very simple, but has an important consequences.

Theorem 4. *Let O be non-degenerate circle of the system $v = \operatorname{sgrad} H$ relatively integral f. Then after a small perturbation in the strong metric the circle O will be slightly deformed thought it remain non-degenerate.*

Proof. Let Π^3 be a transversal to O. Obviously, $H \mid_{\Pi^3}$ has not critical points in a neighborhood of $\Pi^3 \cap O$. Denote $\tilde{f} = f \mid_{\Pi^3}$ and $\tilde{H} = H \mid_{\Pi^3}$. $\tilde{f} \mid_{\tilde{H}=\text{const}}$ has non-degenerate critical points only. This property preserves if one perturb $\tilde{H}$ and $\tilde{f}$. Theorem is proved.

Remark. Theorem 5 is not valid for non-degenerate tori. This fact can be easily obtained from the proof of theorem 1.

Corollary. *Let system $v = \operatorname{sgrad} H$ has non-degenerate integral f on $Q_h = \{H = h\}$. Suppose, that all critical manifolds of $f \mid_{Q_h}$ are circles only and there is only one circle on every critical level of f. Then the Fomenko-Zieschang invariant of the system on Q_h does not change under small perturbation in the strong metric.*

It would be interesting to investigate can we approximate a given (degenerate) system by non-degenerate ones in the strong metric. It is more complicated problem, since we can not offer another integral. We have to deform a given one. Suppose that a periodic integral exists.

It is clear from the proof that the perturbations in the theorem 3 can be made locally. In most cases one can show that f can be approximated by another integral f_1 such that $\{f_1, H\}_\omega = 0$ and $f_1 = g(H, F)$ in a neighborhood of the singular set. Then perturbing H in this neighborhood and let $f_2 = f_1$ outside and $f_2 = g(H_\varepsilon , f)$ into this neighborhood one obtain the *strong* version of the theorem 3.

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