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MATHEMATICAL PROBLEMS OF NONLINEARITY

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Oscillations in Dynamic Systems with an Entropy Operator

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This paper considers dynamic systems with an entropy operator described by a perturbed constrained optimization problem. Oscillatory processes are studied for periodic systems with the following property: the entire system has the same period as the process generated by its linear part. Existence and uniqueness conditions are established for such oscillatory processes, and a method is developed to determine their form and parameters. Also, the general case of noncoincident periods is analyzed, and a method is proposed to determine the form, parameters, and the period of such oscillations. Almost periodic processes are investigated, and existence and uniqueness conditions are proved for them as well.

Keywords: entropy, dynamic systems, optimization, oscillatory process

1. Introduction

Dynamic systems with an entropy operator (DSEOs) form a class of nonlinear systems [1] in which the nonlinearity is described by a perturbed mathematical programming problem with an entropy objective function. DSEOs turned out to be a useful framework for studying a fairly wide range of applied problems: modeling of demoeconomic systems [2], traffic flows [3, 4], the labor market [5], and urban agglomerations [6]; the development of diagnostics algorithms for atmospheric waves [7] and oil-bearing strata [8]; analysis of stationary modes in dynamic image reconstruction procedures from projections in computerized tomography [9]; identification of dynamic systems [10], etc.

The mathematical class of DSEOs is based on a physical model of a dynamic system with self-reproduction of matter, energy, or information exchanged stochastically between the subsystems [14]. Moreover, self-reproduction processes have an evolutionary nature ("slow"), whereas

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exchange processes occur much more intensively (“fast”). A large amount of work has been devoted to the study of such two-velocity systems [11, 12].

The fast processes relax in rather short periods (in the time scale of “slow” processes) to local equilibrium states, and we can describe it by a locally stationary state [13]. Since exchange processes are assumed to be stochastic, their locally stationary states can be described by the corresponding entropy operator [14].

Research into DSEOs has some history and problems examined in related areas. Apparently, the monograph [15], devoted to studying processes in urban and regional systems, was the first publication on models with an entropy operator.

A feature of DSEOs is a specific nonlinearity description by the *argmax* function. In [16], forced periodic modes in a dynamic system with the *argmin* function without constraints were investigated. The paper [17] considered the convergence of iterative algorithms for finding the *argmin* of a continuous function.

The entropy operator in DSEOs is also described by the *argmax* function, but the constrained maximum of the entropy function. Constraints can be specified as a system of equalities or inequalities. In the latter case, the mathematical model of the entropy operator reduces to a perturbed mathematical programming problem, which significantly complicates the study of the operator itself and the corresponding dynamic system.

The general properties of an entropy operator with equality constraints (existence, continuity, boundedness, differentiability, and the Lipschitz condition) were studied in [19, 20].

This paper completes the cycle of research works on the qualitative properties of DSEOs. Oscillatory processes in this class of dynamic systems are investigated. The mathematical model of DSEOs with the τ_0-oscillatory linear part is considered, and existence conditions for τ_0-periodic oscillations are obtained using the directing function of the differential equation of DSEOs. (In some publications, directing functions are also called *guiding*.) An asymptotic method is developed to determine the form and parameters of periodic oscillations for the parameterized family of DSEOs. This method involves the representation of τ_0- and τ-periodic solutions in the form of a corresponding functional power series. A system of recursive equations is derived to determine the functional coefficients of the series. Nonautonomous DSEOs are considered, and existence conditions are established for their almost-periodic oscillations based on an appropriate modification of the fixed point method.

2. The mathematical model of autonomous DSEOs

Consider a DSEO of the form

$$\frac{d\mathbf{x}}{dt} = W\mathbf{x} + S\boldsymbol{\theta}[\mathbf{x}], \quad \mathbf{x} \in R^n, \quad \mathbf{x}(0) = \mathbf{x}^0, \tag{2.1}$$

with the following notation:

- $$\boldsymbol{\theta}[\mathbf{x}] = \arg\max_{\mathbf{y}\in R^m_+} \{H(\mathbf{y}) \mid T\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})\} \in R^m_+ \tag{2.2}$$

  is an entropy operator.

- W is an $n \times n$ matrix whose spectrum contains two eigenvalues $\lambda_1 = \imath\omega_0$ and $\lambda_2 = -\imath\omega_0$ ($\omega_0 > 0$); the other $(n - 2)$ eigenvalues have a negative real part.

- S is an $n \times m$ matrix that has full rank.
- T is a nonnegative $r \times m$ matrix that has full rank and normalized columns:

$$\mathbf{e}_{(r)}^{\mathsf{T}} T = \mathbf{e}_{(m)}^{\mathsf{T}}, \tag{2.3}$$

  where $\mathbf{e}_{(r)}$ and $\mathbf{e}_{(m)}$ denote unit column vectors of dimensions r and m, respectively.

- $\boldsymbol{\varphi}(\mathbf{x}) \in R^r$ is a p-differentiable and bounded vector function:

$$\mathbf{0} < \boldsymbol{\varphi}^- \leqslant \boldsymbol{\varphi}(\mathbf{x}) \leqslant \boldsymbol{\varphi}^+ \quad \text{for } \mathbf{x} \neq \mathbf{0}. \tag{2.4}$$

In the expression (2.2), the entropy has the form

$$H(\mathbf{y}) = -\left\langle \mathbf{y}, \ln \frac{\mathbf{y}}{\mathbf{a}} \right\rangle, \quad \mathbf{a} \in \mathcal{A} = [\mathbf{0}, \mathbf{1}]. \tag{2.5}$$

The entropy operator is a vector $\boldsymbol{\theta}[\mathbf{x}]$ with the components

$$\begin{gathered} \theta_j[\mathbf{x}] = a_j \prod_{l=1}^{r} [z_l(\mathbf{x})]^{t_{lj}}, \quad j = \overline{1, m}, \\ \Phi_k(\mathbf{z}(\mathbf{x})) = \sum_{j=1}^{m} t_{kj} a_j \prod_{l=1}^{r} [z_l(\mathbf{x})]^{t_{lj}} = \varphi_k(\mathbf{x}), \quad k = \overline{1, r}, \end{gathered} \tag{2.6}$$

where the vector $\mathbf{z}$ consists of the exponential Lagrange multipliers for problem (2.2).

The mathematical model (2.1), (2.2) was introduced in order to study the spatio-temporal dynamics of the population, where the first term in this equation is modeled as the process of biological reproduction of the population, and the second as migration [2]. The properties of the entropy operators in this equation were studied in [20, 26].

3. The existence of τ_0-periodic oscillations

The matrix W in Eqs. (2.1) has pure imaginary conjugate eigenvalues in the spectrum. They correspond to the τ_0-periodic function $\widetilde{\mathbf{u}}(t, \mathbf{x})$:

$$\widetilde{\mathbf{u}}(\tau_0 + t, \mathbf{x}) = \widetilde{\mathbf{u}}(t, \mathbf{x}), \quad \tau_0 = \frac{2\pi}{\omega_0}. \tag{3.1}$$

Considering the existence of the τ_0-periodic function (3.1), we transform Eqs. (2.1) to

$$\frac{d\mathbf{x}}{dt} = S\boldsymbol{\theta}[\mathbf{x}] + \widetilde{\mathbf{u}}(t, \mathbf{x}). \tag{3.2}$$

Recall some useful properties of the entropy operator $\boldsymbol{\theta}[\mathbf{x}]$ and related definitions.

Let G be a convex domain in the space R^m. The following properties were established in [20, 26].

Theorem 1. *Assume that the entropy operator* $\boldsymbol{\theta}[\mathbf{x}]$ (2.2) *satisfies conditions* (2.2)–(2.6) *in a domain* G. *Then the operator* $\boldsymbol{\theta}[\mathbf{x}]$ *satisfies the Lipschitz condition in the domain* G *with a constant* L_G:

$$\left\|\boldsymbol{\theta}\left[\mathbf{x}^{(1)}\right]-\boldsymbol{\theta}\left[\mathbf{x}^{(2)}\right]\right\| \leqslant L_G\left\|\mathbf{x}^{(1)}-\mathbf{x}^{(2)}\right\|, \quad \left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) \in G. \tag{3.3}$$

Theorem 2. *In addition to the assumptions of Theorem* 1, *let the function* $\boldsymbol{\varphi}(\mathbf{x})$ *in* (2.2) *be p-differentiable.*

Then the entropy operator $\boldsymbol{\theta}[\mathbf{x}]$ (2.2) *is Fréchet p-differentiable.*

To present further results, we employ some concepts of geometrical nonlinear analysis [16, 22, 25].

Consider a system with the τ_0-periodic right-hand side:

$$\frac{d\mathbf{v}}{dt}=\mathbf{g}(t, \mathbf{v}), \quad t \in R, \quad \mathbf{v} \in R^n, \quad \mathbf{g}(t+\tau_0, \mathbf{v})=\mathbf{g}(t, \mathbf{v}). \tag{3.4}$$

A continuously differentiable function $U(\mathbf{v})$, $\mathbf{v} \in R^n$ is said to be *directing* for equation (3.4) if, for some $r>0$,

$$\frac{dU(\mathbf{v})}{dt}=\langle\nabla U(\mathbf{v}), \mathbf{g}(t, \mathbf{v})\rangle>0, \quad 0 \leqslant t \leqslant \tau_0, \quad \|\mathbf{v}\| \geqslant r. \tag{3.5}$$

Hence, the vector field $\nabla U(\mathbf{v})$ is nondegenerate on the spheres $K(\rho)=\{\|\mathbf{v}\|=\rho\}$ for $\rho>r$, and its *rotation* $\gamma(\nabla U(\mathbf{v}))$ is well-defined and independent of ρ. In this case, the rotation $\gamma(\nabla U(\mathbf{v}))$ is called *the topological index* of the directing function $U(\mathbf{v})$ and denoted by $\operatorname{ind} U(\mathbf{v})$.

The following result was proved in [22].

Theorem 3. *Assume that system* (3.4) *has the directing function with a nonzero topological index. Then system* (3.4) *has at least one* τ_0*-periodic solution.*

Reverting to equation (3.2), we set $n=m$.

Theorem 4. *Assume that the entropy operator* (2.2)–(2.6) *satisfies the Lipschitz condition with a constant* L_G *on the set* G, *and for some* $r>0$,

$$\langle\mathbf{x}, \widetilde{\mathbf{u}}(t, \mathbf{x})\rangle \geqslant 0, \quad 0 \leqslant t \leqslant \tau_0, \quad \|\mathbf{x}\| \geqslant r. \tag{3.6}$$

Then system (3.1) *has at least one* τ_0*-periodic solution.*

Proof. Consider the set $\|\mathbf{x}\| \geqslant r$ and the scalar product

$$\langle S\boldsymbol{\theta}, \mathbf{x}\rangle=\langle S\boldsymbol{\theta}-\mathbf{x}+\mathbf{x}, \mathbf{x}\rangle=\|\mathbf{x}\|^2+\langle S\boldsymbol{\theta}-\mathbf{x}, \mathbf{x}\rangle \geqslant\|\mathbf{x}\|^2-\|S\boldsymbol{\theta}-\mathbf{x}\|\,\|\mathbf{x}\| \geqslant\|\mathbf{x}\|(\|\mathbf{x}\|-r) \geqslant 0.$$

We introduce the function

$$U(\mathbf{x})=\frac{1}{2}\|\mathbf{x}\|^2.$$

Obviously,

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} U(\mathbf{x})=\infty.$$

Due to condition (3.6), for $\|\mathbf{x}\| \geqslant r$ and $0 \leqslant t \leqslant \tau_0$ we obtain

$$\langle\nabla_{\mathbf{x}} U(\mathbf{x}), S\boldsymbol{\theta}+\widetilde{\mathbf{u}}(t, \mathbf{x})\rangle=\langle\mathbf{x}, S\boldsymbol{\theta}\rangle+\langle\mathbf{x}, \widetilde{\mathbf{u}}(t, \mathbf{x})\rangle>0.$$

Here, $\nabla_{\mathbf{x}} U(\mathbf{x})=\mathbf{x}$, and therefore $\operatorname{ind} U(\mathbf{x})=1$; for details, see [16, 25]. The conclusion follows by Theorem 3. □

4. An asymptotic method for determining the form and parameters of τ_0-periodic oscillations

The conditions of Theorem 3 in principle guarantee the existence of a periodic solution. However, how should the form of a τ_0-periodic solution be determined? The main idea is to study periodic modes by asymptotic methods.

Consider a parameterized family of homogeneous DSEOs:

$$\frac{d\mathbf{x}}{dt} = W\mathbf{x} + \varepsilon S\boldsymbol{\theta}[\mathbf{x}], \quad \mathbf{x} \in R^n, \quad \mathbf{x}(0) = \mathbf{x}^0, \quad \varepsilon \in [0,\, 1]. \tag{4.1}$$

For $\varepsilon = 1$, we have the basic system (2.1); for $\varepsilon = 0$, the so-called *generating* system

$$\frac{d\mathbf{u}}{dt} = W\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}^0 \in R^n. \tag{4.2}$$

It produces the τ_0-periodic solution $\widetilde{\mathbf{u}}(t)$. Due to the properties of the matrix W (2.1), the general solution of system (4.2) contains the τ_0-periodic components $\left(\tau_0 = \frac{2\pi}{\omega}\right)$ corresponding to the eigenvalues λ_1 and λ_2 and the components corresponding to the eigenvalues $\lambda_3, \ldots, \lambda_n$. The latter components vanish as $t \to \infty$.

The τ_0-periodic solution corresponding to $\varepsilon = 0$ can be written as

$$\widetilde{\mathbf{u}}(t) = \mathbf{A}\sin(\omega t) + \mathbf{B}\cos(\omega t), \tag{4.3}$$

where the vectors $\mathbf{A}$ and $\mathbf{B}$ have components A_i and B_i, $i = \overline{1,\, n}$, calculated through the elements of the matrix W and the initial conditions $\mathbf{u}^0$.

Now, we address the family (4.1). As shown in [18], the operator $\boldsymbol{\theta}[\mathbf{x}]$ is analytic, i. e., can be represented by a segment of a multidimensional power series around the point $\mathbf{x} = \mathbf{0}$. Due to condition (2.4), we have

$$\boldsymbol{\theta}[\mathbf{x}] = A^{(1)}\mathbf{x} + A^{(2)}\mathbf{x}^{(2)} + \cdots, \tag{4.4}$$

where $A^{(1)}$ is an $n \times n$ matrix, $A^{(2)}$ is an $n \times p^{(2)}$ matrix, $\mathbf{x}^{(2)}$ is a vector of dimension $p^{(2)}$ with all possible lexigographically ordered paired products of the components of the vector $\mathbf{x}$, …, and so on. A method for calculating the matrix coefficients of the power series (4.4) was proposed in [18]. This method is based on the properties of operator equations with a smooth right-hand side [21].

Theorem 5. *Assume that the matrix T satisfies conditions* (2.3)*, the function $\boldsymbol{\varphi}(\mathbf{x})$ is p-analytic, and conditions* (2.4) *hold.*

Then the τ_0-periodic solution of equation (2.1) *is represented by the power series*

$$\widetilde{\mathbf{x}}(t) = \widetilde{\mathbf{x}}_{(0)}(t) + \mathbf{x}_{(1)}(t)\varepsilon + \mathbf{x}_{(2)}(t)\varepsilon^2 + \ldots, \tag{4.5}$$

where $\mathbf{x}_{(1)}(t)$, $\mathbf{x}_{(2)}(t)$, … are its functional coefficients. They are different-order corrections (0, 1, 2, …) *of the τ_0-periodic solution given by the following chain of recursive equations*:

- *for the zero correction* $\left(\varepsilon^0\right)$,

$$\frac{d\widetilde{\mathbf{x}}_{(0)}}{dt} = W\widetilde{\mathbf{x}}_{(0)}(t); \tag{4.6}$$

- *for the first correction* (ε^1),

$$\frac{d\mathbf{x}_{(1)}}{dt} = W\mathbf{x}_{(1)}(t) + S\left(A^{(1)}\widetilde{\mathbf{x}}_{(0)}(t) + A^{(2)}\widetilde{\mathbf{x}}^{(2)}_{(0)}\right); \tag{4.7}$$

- *for the second correction* (ε^2),

$$\frac{d\mathbf{x}_{(2)}}{dt} = W\mathbf{x}_{(2)}(t) + S\left(A^{(1)}\mathbf{x}_{(1)}(t) + A^{(2)}\mathbf{x}^{(2)}_{(1)}(t)\right); \tag{4.8}$$

- ...

Here, the vectors $\widetilde{\mathbf{x}}^{(2)}_{(0)}$ and $\mathbf{x}^{(2)}_{(1)}$ are composed of all possible lexigographically ordered paired products of the components of the vectors $\widetilde{\mathbf{x}}_{(0)}$ and $\mathbf{x}_{(1)}$, respectively.

Proof. Consider Eq. (2.1). Under the assumptions of the theorem, the entropy operator $\boldsymbol{\theta}[\mathbf{x}]$ is analytic and can be represented by the power series (4.4). Then Eq. (2.1) takes the form

$$\frac{d\mathbf{x}}{dt} = W\mathbf{x} + \varepsilon S\left(A^{(1)}\mathbf{x} + A^{(2)}\mathbf{x}^{(2)} + \cdots\right), \quad \mathbf{x} \in R^n, \quad \mathbf{x}(0) = \mathbf{x}^0, \quad \varepsilon \in [0,\, 1]. \tag{4.9}$$

Substituting the expression (4.5) into both sides and equating the terms with identical powers of ε, we arrive at the system of recursive equations (4.6)–(4.8). The proof of the convergence of such procedures is given in [30]. □

According to this theorem, all corrections, including the zero correction, have the same structure: for the next correction, the input of the linear system with the matrix W is the periodic influences representing linear transformations of the previous correction. In practice, they can be calculated using the Laplace transform [27].

5. An asymptotic method for determining the form and parameters of $\boldsymbol{\tau}$-periodic oscillations

The period τ of the periodic solution of system (4.1) may differ from the τ_0-periodic solution of the generating system (4.2).

We return to Eq. (4.1) and the generating system (4.2) producing the τ_0-periodic solution of (4.3). Let system (4.1) have a periodic mode $\widetilde{\mathbf{x}}(t+\tau) = \widetilde{\mathbf{x}}(t)$ with an unknown period $\tau \neq \tau_0$.

Periodicity means that on the interval $[0,\, \tau]$ there exists a value $t = t^*$ such that $\left\|\frac{d\widetilde{\mathbf{x}}}{dt}\right\|_{t^*} = \mathbf{0}$. Taking this time instant as initial, we obtain

$$\left\|\frac{d\widetilde{\mathbf{x}}}{dt}\right\|_0 = \mathbf{0}. \tag{5.1}$$

According to [30], the periodic solutions of systems (4.1), including their period τ, are analytic functions of the parameter ε:

$$\tau(\varepsilon) = \tau_0(1 + \alpha(\varepsilon)), \quad \alpha(\varepsilon) = \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots, \tag{5.2}$$

where α_1, α_2, ... are constants to be determined.

We introduce a time transformation similar to (5.2):

$$t = \eta \left(1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots\right). \tag{5.3}$$

Then Eq. (4.1) takes the form

$$\frac{d\mathbf{x}(\eta)}{d\eta} = \left(1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots\right)(W\mathbf{x}(\eta) + \varepsilon S\boldsymbol{\theta}[\mathbf{x}(\eta)]). \tag{5.4}$$

All possible periodic solutions of this equation have period 2π in the time scale η.

We construct the periodic solution $\widetilde{\mathbf{x}}(\eta)$ of this equation as a power series in the parameter ε:

$$\widetilde{\mathbf{x}}(\eta) = \widetilde{\mathbf{x}}_{(0)}(\eta) + \mathbf{x}_{(1)}(\eta)\varepsilon + \mathbf{x}_{(2)}(\eta)\varepsilon^2 + \ldots, \tag{5.5}$$

where $\widetilde{\mathbf{x}}_{(0)}(\eta)$ is the τ_0-periodic solution of the generating system, and $\mathbf{x}_{(1)}(\eta)$, $\mathbf{x}_{(2)}(\eta)$, $\ldots$ are functional coefficients of the series.

By analogy with the previous section, we substitute this series into both sides of (5.4) and equate the terms with identical powers of ε. As a result, we derive the following system of recursive equations:

- for the zero correction $\left(\varepsilon^0\right)$,

$$\frac{d\widetilde{\mathbf{x}}_{(0)}}{d\eta} = W\widetilde{\mathbf{x}}_{(0)}(\eta); \tag{5.6}$$

- for the first correction $\left(\varepsilon^1\right)$,

$$\frac{d\mathbf{x}_{(1)}}{d\eta} = W\mathbf{x}_{(1)}(\eta) + \alpha_1 W\widetilde{\mathbf{x}}_{(0)}(\eta) + S\left(A^{(1)}\widetilde{\mathbf{x}}_{(0)}(\eta) + A^{(2)}\widetilde{\mathbf{x}}_{(0)}^{(2)}(\eta)\right); \tag{5.7}$$

- for the second correction $\left(\varepsilon^2\right)$,

$$\frac{d\mathbf{x}_{(2)}}{d\eta} = W\mathbf{x}_{(2)}(\eta) + \alpha_1 W\mathbf{x}_{(1)}(\eta) + \alpha_2 W\widetilde{\mathbf{x}}_{(0)}(\eta) + S\left(A^{(1)}\mathbf{x}_{(1)}(\eta) + A^{(2)}\mathbf{x}_{(1)}^{(2)}(\eta)\right); \tag{5.8}$$

- ...

Therefore, the zero correction is the τ_0-periodic solution of the linear autonomous system (5.6) with the matrix W containing a pair of the pure imaginary eigenvalues $\pm\imath\omega_0$. All other corrections are forced periodic solutions generated by the linear system subjected to periodic perturbations:

- for the first correction,

$$\mathbf{f}_1(\eta) = \alpha_1 W\widetilde{\mathbf{x}}_{(0)}(\eta) + S\left(A^{(1)}\widetilde{\mathbf{x}}_{(0)}(\eta) + A^{(2)}\widetilde{\mathbf{x}}_{(0)}^{(2)}(\eta)\right); \tag{5.9}$$

- for the second correction,

$$\mathbf{f}_2(\eta) = \alpha_1 W\mathbf{x}_{(1)}(\eta) + \alpha_2 W\widetilde{\mathbf{x}}_{(0)}(\eta) + S\left(A^{(1)}\mathbf{x}_{(1)}(\eta) + A^{(2)}\mathbf{x}_{(1)}^{(2)}(\eta)\right); \tag{5.10}$$

- ...

Using condition (5.1), we finally get the following equations for the parameters α_1, α_2, ...:

- for the first correction,

$$\left\|\alpha_1 W \widetilde{\mathbf{x}}_{(0)}(\eta) + S\left(A^{(1)}\widetilde{\mathbf{x}}_{(0)}(\eta) + A^{(2)}\widetilde{\mathbf{x}}^{(2)}_{(0)}(\eta)\right)\right\|_{\eta=0} = 0; \tag{5.11}$$

- for the second correction,

$$\left\|\alpha_1 W \mathbf{x}_{(1)}(\eta) + \alpha_2 W \widetilde{\mathbf{x}}_{(0)}(\eta) + S\left(A^{(1)}\mathbf{x}_{(1)}(\eta) + A^{(2)}\mathbf{x}^{(2)}_{(1)}(\eta)\right)\right\|_{\eta=0} = 0; \tag{5.12}$$

- ...

Clearly, each equation presented is quadratic in one variable (α_1, α_2, ...).

6. Almost periodic oscillations in nonautonomous DSEOs

Almost periodic functions are a broader class that includes, among other members, periodic functions. For example, the function

$$f(t) = A_1 \sin\omega_1 t + A_2 \sin\omega_2 t$$

is not periodic if the frequencies ω_1 and ω_2 are incommensurable.

We adopt the definition of an almost periodic function introduced in [23] and repeated in [24].

A continuous function $f(t, x)$, $x \in X$, is said to be almost periodic on the interval $-\infty < t < \infty$ if for an arbitrarily small positive number $\epsilon(x)$ there exists a positive number $\mu(\epsilon(x))$ such that within each interval of length $\mu(\epsilon(x))$ it is possible to find at least one number $\tau(\epsilon)$ with the property

$$|f(t + \tau(\epsilon(x)), x) - f(t, x)| < \epsilon(x), \quad x \in X. \tag{6.1}$$

Consider a nonautonomous DSEO of the form

$$\frac{d\mathbf{x}}{dt} = W\mathbf{x} + S\boldsymbol{\theta}[\mathbf{x}] + \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x} \in R^n, \quad \mathbf{x}(0) = \mathbf{x}^0, \quad \mathbf{f} \in R^n, \tag{6.2}$$

where $\mathbf{f}(t, \mathbf{x})$ is an almost periodic vector function and $\boldsymbol{\theta}[\mathbf{x}]$ is the entropy operator (2.2) with an analytic function $\boldsymbol{\varphi}(\mathbf{x})$. The matrices W and S are Hurwitzian (their eigenvalues are real and negative) and have full rank n.

Following [28], we introduce the matrizer of Eq. (6.2):

$$M^t_s(W) = \exp[W(t - s)]. \tag{6.3}$$

Let w_{min} denote the maximum eigenvalue. Then the matrizer norm satisfies the upper bound

$$\left\|M^t_s(W)\right\| \leqslant \exp(-w_{min}(t - s)). \tag{6.4}$$

Using the matrizer (6.3), we pass to the integral equation with the almost periodic function $\mathbf{f}(s, \mathbf{x}(s))$:

$$\mathbf{x}(t) = \mathbf{x}^{(0)} + \int\limits_0^t M^t_s(W)[S\boldsymbol{\theta}[\mathbf{x}(s)] + \mathbf{f}(s, \mathbf{x}(s))]\, ds. \tag{6.5}$$

Theorem 6. *Assume that in the space R^n there exists a ball S_r: $\|\mathbf{x}\|_{(\nu)} = \max\limits_{0\leqslant t\leqslant\nu} \|\mathbf{x}(t)\| \leqslant r$ of values of almost periodic functions $\mathbf{x}(t)$ such that*:

- *The almost periodic function* $\mathbf{f}(t, \mathbf{x}(t))$ *satisfies the Lipschitz condition in the variable* $\mathbf{x}$ *with a constant* $L_1(r)$:

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\|_{(\nu)} \leqslant L_1(r)\|\mathbf{x} - \mathbf{y}\|_{(\nu)}, \quad -\infty < t < \infty, \quad (\mathbf{x}, \mathbf{y}) \in S_r. \tag{6.6}$$

- *The entropy operator* $\boldsymbol{\theta}[\mathbf{x}(t)]$ *satisfies the Lipschitz condition with a constant* $L_2(r)$:

$$\|\boldsymbol{\theta}[\mathbf{x}(t)] - \boldsymbol{\theta}[\mathbf{y}(t)]\|_{(\nu)} \leqslant L_2\|\mathbf{x} - \mathbf{y}\|_{(\nu)}. \tag{6.7}$$

- $$q = \frac{1}{w_{min}}(L_2 s_{max} + L_1)\|\mathbf{x} - \mathbf{z}\|_{(\nu)} \leqslant 1. \tag{6.8}$$

Then Eq. (6.5) *has a unique almost periodic solution* $\check{\mathbf{x}}(t)$ *in the ball* S_r.

Proof. Consider the integral operator

$$\Pi[\mathbf{x}(t)] = \int_0^t M_s^t(W)[S\boldsymbol{\theta}[\mathbf{x}(s)] + \mathbf{f}(s, \mathbf{x}(s))]\, ds$$

acting in the space C_ν $(0 \leqslant \nu \leqslant T)$ of all almost periodic vector functions with the norm

$$\|\mathbf{x}\|_{(\nu)} = \max_{0 \leqslant t \leqslant \nu} \|\mathbf{x}(t)\|_E,$$

where E is Euclidean space. Let $(\mathbf{x}(t), \mathbf{z}(t)) \in C_\nu$, and

$$\|\Pi[\mathbf{x}(t)] - \Pi[\mathbf{z}(t)]\| = \left\| \int_0^t M_s^t(W)\{S(\boldsymbol{\theta}[\mathbf{x}(s)] - \boldsymbol{\theta}[\mathbf{z}(s)]) + (\mathbf{f}(s, \mathbf{x}(s)) - \mathbf{f}(s, \mathbf{z}(s)))\}\, ds \right\|.$$

Due to conditions (6.6) and (6.7) and the upper bound (6.4), we have the inequality

$$\|\Pi[\mathbf{x}(t)] - \Pi[\mathbf{z}(t)]\|_\nu \leqslant \frac{1}{w_{min}}(L_2 s_{max} + L_1)\|\mathbf{x} - \mathbf{z}\|_{(\nu)},$$

where s_{max} is the maximum element of the matrix S. □

7. Conclusions

This paper has considered oscillatory processes in DSEOs. They can arise in autonomous systems whose linear part contains pure imaginary eigenvalues.

Existence conditions have been established for periodic oscillations in two cases: when their period coincides with that generated by the linear part and when it does not. In both cases, a functional power series-based method has been proposed to determine their form, parameters, and period.

The problem of almost periodic oscillations in nonautonomous DSEOs has been investigated. An existence and uniqueness theorem has been proved for almost periodic solutions in DSEOs.

Conflict of interest

The author declares that he has no conflicts of interest.

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