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P-ADIC DIOPHANTINE APPROXIMATION ON THE VERONESE CURVE WITH A NON-MONOTONIC ERROR

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This paper is devoted to a generalization of a result of Beresnevich [1] for the field of p-adic numbers $\mathbb{Q}_p$. The main problem in this area is when the p-adic evaluation of the derivative at the root of a polynomial is small. Beresnevich solved this problem for the real case using a lemma of Kleinbock and Margulis [2]. In the present paper a direct proof is given based on a result regarding the resultant of two polynomials without common roots in $\mathbb{Q}_p$.

In 1967 Sprindžuk [3] proved Mahler's conjecture of 1932 that for any $\varepsilon > 0$ the inequality

$$|P(x)| < H^{-n-\varepsilon} \tag{1}$$

has infinitely many solutions in integer polynomials

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

$a_j \in \mathbb{Z},\ 0 \le j \le n,\ H = H(P) = \max_{1\le j\le n} |a_j|$.

Sprindžuk [3] also proved an analogue of Mahler's hypothesis for other fields, in particular, the field of p-adic numbers $\mathbb{Q}_p$. Stronger version of these results can be found in [4–7].

One of the assumptions in Sprindžuk's proof [3] is that the error function is monotonically decreasing, although for $n = 1$ this condition is not necessary. In the real case, Beresnevich [1] solved a similar problem to (1) for integer polynomials of arbitrary degree. In this paper a slight strengthening of Sprindžuk's result is proved which generalizes [6] in $\mathbb{Q}_p$.

Theorem 1. *Let $\Psi(x)$ be a function of the real variable x such that $\Psi(x) \to 0$ as $x \to \infty$ and $\sum_{H=1}^{\infty} H^n \Psi(H)$ converges. Then for any prime p and almost all $w \in \mathbb{Q}_p$ (in the sense of Haar measure) the inequality*

$$|P(w)|_p < \Psi(H(P)) \tag{2}$$

has only finitely many solutions $P \in \mathbb{Z}[w]$.

1. Auxiliary Statements. We denote by $\mathcal{P}_n(H)$ the set of irreducible primitive polynomials $P \in \mathbb{Z}[w]$ of degree n and height $H(P) = H$ such that

$$|a_n| = H(P), \quad |a_n|_p > p^{-n}.$$

Also define $\mathcal{P}_n = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H)$. Let

$$A(\Psi) = \{w \in \mathbb{Q}_p :\ |P(w)|_p < \Psi(H) \quad \text{for infinitely many} \quad P \in \mathcal{P}_n\}.$$

Following the reduction to irreducible primitive leading polynomials as in [1, 2] it is sufficient to show that $\mu(A(\Psi)) = 0$ to prove the theorem.

2. Preliminary remarks. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the roots of P in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers.

Lemma 1. *If* $P \in \mathcal{P}_n(H)$ *then* $\max\limits_{1 \le i \le n} |\gamma_i|_p < p^n$.

For the proof see [3]. For $1 \le i \le n$ consider the sets

$$S(\gamma_i) = \{w \in \mathbb{Q}_p : \ |w - \gamma_i|_p = \min_{1 \le j \le n} |w - \gamma_j|_p\}.$$

Fix a root γ_1 of $P \in \mathcal{P}_n(H)$ and order the other roots so that

$$|\gamma_1 - \gamma_2|_p \le \ldots \le |\gamma_1 - \gamma_n|_p.$$

Lemma 2. *Let* $P \in \mathcal{P}_n(H)$ *and* $w \in S(\gamma_1)$. *Then*

$$|w - \gamma_1|_p < |P(w)|_p |P'(\gamma_1)|_p^{-1} \tag{3}$$

and

$$|w - \gamma_1|_p < (|P(w)|_p |P'(\gamma_1)|_p^{-1} |\gamma_1 - \gamma_2|_p)^{1/2}. \tag{4}$$

Let $\varepsilon > 0$ be sufficiently small, $d > 0$ be a large fixed number and let $\varepsilon_1 = \varepsilon/d$, $T = [\varepsilon_1^{-1}]$. For a polynomial P define the real numbers ρ_j and the integers l_j by

$$|\gamma_1 - \gamma_j| = H^{-\rho_j}, \quad \frac{l_j - 1}{T} \le \rho_j < \frac{l_j}{T}, \quad 2 \le j \le n.$$

Further define the numbers $r_i = \dfrac{l_{i+1} + \ldots + l_n}{T}$, $1 \le i \le n-1$. All polynomials $P \in \mathcal{P}_n(H)$ corresponding to the same vector $\mathbf{l} = (l_2, \ldots, l_n)$ are grouped together into a class $\mathcal{P}_n(H, \mathbf{l})$. We define $\mathcal{P}_n(\mathbf{l}) = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H, \mathbf{l})$. It is readily verified that the number of classes $\mathcal{P}_n(H, \mathbf{l})$ is finite [3].

Let $K_0 = \{w \in \mathbb{Q}_p : \ |w|_p < p^n\}$ be the disc of radius p^n centered at 0. By lemma 1 $A(\Psi) \subseteq K_0$. Hence, it is sufficient to prove the theorem for points $w \in S(\gamma_i) \cap K_0$, $1 \le i \le n$.

Lemma 3. *Let* $w \in S(\gamma_1)$ *and* $P \in \mathcal{P}_n(H)$. *Then*

$$H^{-r_1} \ll |P'(\gamma_1)|_p \ll H^{-r_1 + (n-1)\varepsilon_1},$$

$$|P^{(j)}(\gamma_1)|_p \ll H^{-r_j + (n-j)\varepsilon_1},$$

for $2 \le j \le n$.

For the proof see Lemma 5 in [6].

Lemma 4. *Let* $\delta > 0$, $\sigma > 0$ *be real numbers,* $n \ge 2$ *a natural number and* $H = H(\delta, n)$ *a sufficiently large real number. Further, let* $P, Q \in \mathbb{Z}[w]$ *be two relatively prime polynomials of degree at most* n *with* $\max\{H(P), H(Q)\} \le H$. *Let* $K(\alpha, p^{-t})$ *be a disc of radius* p^{-t} *centered at* α *where* t *is defined by the inequalities* $p^{-t} \le H^{-\sigma} < p^{-t+1}$. *If there exists a number* $\tau > 0$ *such that for all* $w \in K(\alpha, p^{-t})$,

$$\max\{|P(w)|_p, |Q(w)|_p\} < H^{-\tau}$$

then $\tau + 2\max(\tau - \sigma, 0) < 2n + \delta$.

Lemma 4 is proved in [8].

The notation $X \ll Y$ will mean $X = O(Y)$; if $X \ll Y$ and $Y \ll X$ then $X \asymp Y$ will be used.

3. Proof of Theorem 1. Let $B(P)$ denote the set of $w \in K_0 \cap S(\gamma_1)$ such that the inequality $|P(w)|_p < \Psi(H)$ has a solution for $P \in \mathcal{P}_n(H, \mathbf{l})$.

Since the sum

$$\sum_{H=1}^{\infty} H^n \Psi(H) = \sum_{t=0}^{\infty} \sum_{2^t \le H < 2^{t+1}} H^n \Psi(H)$$

converges then $H^n\Psi(H)$ and $\sum_{2^t \le H < 2^{t+1}} H^n \Psi(H)$ tends to 0 as $H \to \infty$ and $t \to \infty$. Therefore,

$$\Psi(H) = o(H^{-n}) \quad \text{as} \quad H \to \infty$$

and

$$\sum_{2^t \le H < 2^{t+1}} \Psi(H) H^n = o(1) \ll 1. \tag{5}$$

For $\Psi(H) \ll H^{-n-1}$ the theorem was proved in [6]. Hence, from now on it is assumed that

$$H^{-n-1} \ll \Psi(H) \ll H^{-n}.$$

We take $M > M_0(\varepsilon)$ and divide the range of $\Psi(H)$ into M parts of the form

$$H^{-n-1+i/M} \ll \Psi(H) \ll H^{-n-1+(i+1)/M}, \quad i = 0, \ldots, M-1. \tag{6}$$

Let the number of H for which this inequality holds be greater then H^{θ}. Fix a sufficiently large number t and consider $2^t \le H < 2^{t+1}$; then we obtain

$$\sum_{2^t \le H < 2^{t+1}} \Psi(H) H^n > H^{-n-1+i/M+n+\theta} > 2^{t(\theta-1+i/M)}.$$

If $\theta - 1 + i/M > 0$ then we have a contradiction with (5). Hence, from now on we assume that

$$0 \le \theta < 1 - i/M. \tag{7}$$

From Lemma 2

$$\begin{aligned} |w - \gamma_1|_p &\ll \Psi(H) H^{r_1} \ll 2^{t(r_1-n-1+(i+1)/M)}, \\ |w - \gamma_1|_p &\ll (\Psi(H) H^{r_1} H^{-\rho_2})^{1/2} \ll 2^{t/2(r_2-n-1+(i+1)/M)}. \end{aligned} \tag{8}$$

If the second estimate here is less than the first estimate then

$$r_1 + \frac{l_2}{T} > n + 1 - \frac{i+1}{M}. \tag{9}$$

Again, for t sufficiently large, define the set $M_t(\mathbf{l}) = \bigcup_{2^t \le H < 2^{t+1}} \mathcal{P}_n(H, \mathbf{l})$. Divide the set $K_0 \cap S(\gamma_1)$ into discs K_j of radius $2^{-\sigma_1 t}$ for $\sigma_1 > 0$. Firstly, consider discs K_j for which there is at most one $P \in M_t(\mathbf{l})$ where (2) is true. The number of these discs and hence the number of these polynomials is at most $p^n 2^{\sigma_1 t}$. Summing the estimate of (8) and (9) for P in this class leads to

$$\sum_{P \in M_t(\mathbf{l})} \mu(B(P)) \ll 2^{t(2\sigma_1 + r_2/2 - n/2 - 1/2 + (i+1)/(2M))}.$$

If $\sigma_1 < \dfrac{n}{2} + \dfrac{1}{2} - \dfrac{r_2}{2} - \dfrac{i+1}{2M}$, then this series is convergent and, by the Borel–Cantelli lemma, the proof is complete in this case.

Now, let $\sigma_2 = \dfrac{n}{2} + \dfrac{1}{2} - \dfrac{r_2}{2} - \dfrac{i+1}{2M} - \beta_1$, $\beta_1 > 0$ and consider the other discs. Let P and Q be distinct polynomials in $M_t(\mathbf{l})$ such that $B(P)$ and $B(Q)$ intersect in a disc K_j of radius $2^{-t\sigma_2}$. By Taylor's formula for P in the neighborhood of the root γ_1 we estimate $|P(w)|_p$ by

$$|P(w)|_p \ll 2^{t(-r_2+(n-2)\varepsilon_1-n-1+(i+1)/M+r_2+2\beta_1)} = 2^{t(-n-1+(i+1)/M+2\beta_1+(n-2)\varepsilon_1)}.$$

Using lemma 4 with $\tau = n+1-(i+1)/M - 2\beta_1 - (n-2)\varepsilon_1$, $\sigma = \sigma_2$ and

$$\tau - \sigma_2 = n+1-\frac{i+1}{M} - 2\beta_1 - (n-2)\varepsilon_1 - \frac{n}{2} - \frac{1}{2} + \frac{r_2}{2} + \frac{i+1}{2M} + \beta_1$$

so that

$$2(\tau - \sigma_2) = n+1-\frac{i+1}{M} - 2\beta_1 - 2(n-2)\varepsilon_1 + r_2$$

we obtain

$$2n+2+r_2-4\beta_1 - \frac{2(i+1)}{M} - 3(n-2)\varepsilon_1 < 2n+\delta.$$

If $4\beta_1 + \dfrac{2(i+1)}{M} + \delta + 3(n-2)\varepsilon_1 \le 2$ or $\dfrac{i}{M} \le 1 - 2\beta_1 - \dfrac{1}{M} - \dfrac{\delta}{2} - \dfrac{3(n-2)\varepsilon_1}{2}$, then this contradicts lemma 4. Hence, we must assume that

$$0 \le \theta \le \theta_1, \quad \theta_1 = \frac{1}{M} + 2\beta_1 + \frac{\delta}{2} + \frac{3(n-2)\varepsilon_1}{2}.$$

Fix H, with $2^t \le H < 2^{t+1}$. The number of such H satisfying (6) is $\le 2^{t\theta_1}$. For θ_1 apply the same arguments as above and say that the disc K contains $P \in M_t(\mathbf{l})$ if there exists a point $w_0 \in K$ such that $|P(w_0)|_p < \Psi(H)$. If discs of radius $2^{-t\sigma_3}$, $\sigma_3 > 0$, contain $< 3 \cdot 2^{t\theta_1}$ polynomials with the same $H = a_n$ then we get the following estimate for the measure of the set $B(P)$:

$$\sum_{P \in M_t(\mathbf{l})} \mu(B(P)) \ll 2^{t(\sigma_3 + r_2/2 - n/2 - 1/2 + (i+1)/(2M) + \theta_1)}. \tag{10}$$

This gives a convergent series if $\sigma_3 < \dfrac{n}{2} + \dfrac{1}{2} - \dfrac{r_2}{2} - \dfrac{i+1}{2M} - \theta_1$. Let

$$\sigma_4 = \frac{n}{2} + \frac{1}{2} - \frac{r_2}{2} - \frac{i+1}{2M} - \theta_1 - \beta_2, \quad \beta_2 > 0,$$

and consider discs of radius $2^{-t\sigma_4}$ which contain at least $3 \cdot 2^{t\theta_1}$ polynomials. By Taylor's formula and the previous estimates

$$|P(w)|_p \ll 2^{t(-r_2+(n-2)\varepsilon_1-n-1+(i+1)/M+r_2+2\theta_1+2\beta_2)} =$$

$$= 2^{t(-n-1+(i+1)/M+2\theta_1+2\beta_2+(n-2)\varepsilon_1)} = 2^{-t\sigma_5}. \tag{11}$$

Consider new polynomials $R_i = P_{i+1} - P_1$, where $i = 2, 3$. Lemma 4 can be used on R_2 and R_3 with $\deg R_i \le n-1$ $(i = 2, 3)$ and $|R_i(w)| \ll 2^{-t\sigma_5}$. It follows from (11) and Lemma 4 that

$$\tau = n+1-\frac{i+1}{M} - 2\theta_1 - 2\beta_2 - (n-2)\varepsilon_1,$$

$$\tau - \sigma_4 = \frac{n+1}{2} - \frac{i+1}{2M} + \frac{r_2}{2} - \theta_1 - \beta_2 - (n-2)\varepsilon_1.$$

These imply that

$$\tau + 2(\tau - \sigma_4) = 2n + 2 - \frac{2(i+1)}{M} - 4\theta_1 - 4\beta_2 - 3(n-2)\varepsilon_1 + r_2 < 2(n-1) + \delta.$$

Choose constants $\beta_1,\ \beta_2,\ \varepsilon_1,\ \delta,\ M$ such that the inequality $8\beta_1 + 4/M + 4\beta_2 + 9(n-2)\varepsilon_1 + 3\delta < 2$ holds. Then $\tau + 2(\tau - \sigma_4) > 2(n-1) + \delta$ and this contradicts Lemma 4. This completes the proof for case (9).

Now assume that the opposite inequality (9) holds, i.e. that $r_1 + l_2 T^{-1} \le n + 1 - (i+1)/M$. Then, we estimate $|w - \gamma_1|_p$ using (3), (this is more precise that (4)). First we consider the case when

$$n - \frac{1}{2} < r_1 + \frac{l_2}{T} \le n + 1 - \frac{i+1}{M}.$$

Divide the set $K_0 \cap S(\gamma_1)$ into discs K_j of radius $2^{-t\sigma_6}$ with $\sigma_6 > 0$ and consider only those discs that contain at most one polynomial $P \in M_t(\mathbf{l})$. Having used the first estimate of (8) and summed it over all chosen discs and over t we obtain

$$c(n) \sum_{t=1}^{\infty} 2^{t(-n-1+(i+1)/M+r_1+\sigma_6)}.$$

This series converges if $\sigma_6 < n + 1 - \dfrac{i+1}{M} - r_1$.

For $\beta_3 > 0$ take

$$\sigma_7 = n + 1 - \frac{i+1}{M} - r_1 - \beta_3$$

and consider discs that contain at least two polynomials. Choose two of these, P and Q say and use Taylor's formula for to estimate $|P(w)|_p$ and $|Q(w)|_p$ on a disc of radius $2^{-t\sigma_7}$ so that

$$\max(|P(w)|_p, |Q(w)|_p) \ll 2^{t(-n-1+(i+1)/M+\beta_3+(n-1)\varepsilon_1)}.$$

Now, use Lemma 4 with $\tau = n + 1 - \dfrac{i+1}{M} + \beta_3 - (n-1)\varepsilon_1$ and $\tau - \sigma_7 = r_1 - (n-1)\varepsilon_1$ to obtain

$$\tau + 2(\tau - \sigma_7) = n + 1 + 2r_1 - \frac{i+1}{M} - \beta_3 - 3(n-1)\varepsilon_1 < 2n + \delta.$$

If $2r_1 \ge l_2 T^{-1} + r_1 > n - 0.5$ then we have a contradiction if

$$\frac{i+1}{M} + \gamma_1 + 3(n-1)\varepsilon_1 + \delta < \frac{1}{2}. \tag{12}$$

As $1/M + \beta_3 + 3(n-1)\varepsilon_1 + \delta$ is arbitrarily small, than to satisfy (12) we need $i/M < 0.4$. If $i/M \ge 0.4$, then from (7)

$$0 \le \theta < 0.6. \tag{13}$$

Fix such a polynomial of height H with $2^t \le H < 2^{t+1}$ satisfying (6). From (13), the number of such H does not exceed $2^{3t/5}$. An analogous argument to that used to obtain (10) is used and we divide the set $K_0 \cap S(\gamma_1)$ into discs K_j of radius $2^{-t\sigma_8}$ with $\sigma_8 > 0$. Assume that the disc K_j contains at most $3 \cdot 2^{3t/5}$ polynomials. Then the upper estimate for the measure of the set $K_0 \cap S(\gamma_1)$ will be similar to that in (10), i.e.

$$c(n) 2^{t(-n-1+r_1+(i+1)/M+0.6+\sigma_8)}.$$

The series with this summand converges if $\sigma_8 < n + 1 - (i+1)/M - 0.6 - r_1$.

Assume that

$$\sigma_9 = n + 1 - r_1 - \frac{i+1}{M} - 0.6 - \beta_4, \quad \beta_4 > 0,$$

and consider discs K_j which contain at least $3 \cdot 2^{0.6t}$ polynomials $P_i(w)$, $1 \le i \le L$, $L > 3 \cdot 2^{0.6t}$. Apply Taylor's formula to each of them over K_j to obtain an upper estimate for $|P_i(w)|_p$, namely

$$|P_i(w)|_p \ll 2^{t(-r_1+(n-1)\varepsilon_1-n-1+r_1+(i+1)/M+0.6+\beta_4)} = 2^{t(-n-1+(i+1)/M+0.6+\beta_4+(n-1)\varepsilon_1)}.$$

As at least three polynomials $P_{1i}(w)$, $P_{2i}(w)$ and $P_{3i}(w)$ have the same height (i.e. $a_n = H$), the polynomials $R_1(w) = P_{2i}(w) - P_{1i}(w)$ and $R_2(w) = P_{3i}(w) - P_{1i}(w)$ satisfy the following system of inequalities:

$$|R_i(w)|_p \ll 2^{t(-n-1+(i+1)/M+0.6+\beta_4+(n-1)\varepsilon_1)},$$

$$\deg R_i(w) \le n-1, \quad i = 1, 2.$$

Apply lemma 4 with $\tau = n+1-(i+1)/M-0.6-\beta_4-(n-1)\varepsilon_1$, $2(\tau-\sigma_9) = 2r_1-2(n-1)\varepsilon_1$ and

$$\tau + 2(\tau - \sigma_9) = n + 1 + 2r_1 - \frac{i+1}{M} - 0.6 - \beta_4 - 3(n-1)\varepsilon_1.$$

As $(i+1)/M \le 1$ and $2r_1 \ge l_2T^{-1} + r_1 \ge n - 0.5$ then the RHS in the above equation is greater than $2n - 1.1 - \beta_4 - 3(n-1)\varepsilon_1$. Choose $\beta_4 + 3(n-1)\varepsilon_1$ less than 0.1. Then, for $\delta < 0.8$ this contradicts the inequality $\tau + 2(\tau - \sigma_9) < 2(n-1) + \delta$ from Lemma 4.

Next, if

$$2 - \frac{\varepsilon}{2} < r_1 + \frac{l_2}{T} \le n - \frac{1}{2} \qquad (14)$$

then fix any sufficiently large integer H and divide the set K_0 into discs K_j of radius $H^{-l_2/T}$. The number of these discs is $\ll H^{l_2/T}$. Let the disc K_j contain at most H^{θ_2} polynomials. Then

$$\sum_P \mu(B(P)) \ll \sum_H H^{\theta_2} H^{l_2/T} H^{r_1} \Psi(H) \ll \sum_H H^n \Psi(H) H^{\theta_2 - n + l_2/T + r_1}. \qquad (15)$$

If this is the summand in a series then the series converges if $\theta_2 < n - l_2/T - r_1$. Hence, by the Borel–Cantelli lemma, the set of those w, which belong to infinitely many $B(P)$ has zero measure. From condition (14) we have that $\theta_2 \ge 1/2$ and we need to consider the case when $\theta_2 \ge n - l_2/T - r_1$. In this case the proof of theorem is analogous to Proposition 6 from [2]. Finally, if $\varepsilon \le r_1 + l_2T^{-1} \le 2 - \varepsilon/2$, then again the proof will be the same as for Proposition 7 from [2].

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References

1. *Beresnevich V.V.* On a theorem of V. Bernik in the metric theory of Diophantine approximation // Acta Arith. 2005. V. 117. № 1. P. 71–80.
2. *Kleinbock D.Y., Margulis G.A.* Flows on homogeneous spaces and Diophantine approximation on manifolds // Ann. Math. 1998. V. 148. P. 339–360.
3. *Sprindžuk V.* Mahler's problem in the Metric Theory of Numbers // Transl. Math. Monographs. Providence, R.I.: Amer. Math. Soc. 1969. V. 25.
4. *Baker A.A.* On a theorem of Sprindžuk // Proc. Roy. Soc., London Ser. A. 1966. V. 292. P. 92–104.
5. *Beresnevich V.V.* On approximation of real numbers by real algebraic numbers // Acta Arith. 1999. V. 90. P. 97–112.

6. *Beresnevich V., Bernik V., Kovalevskaya E.* On approximation of p-adic numbers by p-adic algebraic numbers // J. of Number Theory. 2005. V. 111. P. 33–56.
7. *Bernik V.I.* On the exact order of approximation of zero by values of integral polynomials // Acta Arith. 1989. V. 53. P. 17–28.
8. *Bernik V.I., Melnichuk Y.I.* On integer polynomials of a p-adic variable with small norm in the disc // Proc. Lvov Polytech. Inst. 1985. V. 182. P. 63–64.

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P-adic Diophantine approximation on the Veronese curve with a non-monotonic error

Summary

A p-adic analogue of the convergence part of Khintchine's Theorem for polynomials is proved with a non-monotonic error function. This is a small strengthening of Sprindžuk's theorem and a generalization of a result of Beresnevich.