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An Asymptotic Relation for Conformal Radii of Two Nonoverlapping Domains

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We consider a family of continuously varying closed Jordan curves given by a polar equation, such that the interiors of the curves form an increasing or decreasing chain of domains. Such chains can be described by the Löwner – Kufarev differential equation. We deduce an integral representation of a driving function in the equation. Using this representation we obtain an asymptotic formula, which establishes a connection between conformal radii of bounded and unbounded components of the complement of the Jordan curve when the bounded component is close to the unit disk.

Key words: Löwner – Kufarev equation, conformal radius, asymptotic expansion, nonoverlapping domains.

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INTRODUCTION

We denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ the open disk of the radius $r > 0$ and centered at the origin, $\mathbb{D} = \mathbb{D}_1$. Let Ω be a simply connected domain which is a proper subset of the complex plane and $w_0 \in \Omega$. According to the Riemann mapping theorem there are a unique number $r > 0$ and a unique function g conformally mapping Ω onto the disk \mathbb{D}_r and such that $g(w_0) = 0$, $g'(w_0) = 1$. This r is called a conformal radius of the domain Ω with respect to the point w_0 . Let now Ω be a domain in the extended complex plane with at least two boundary points and $w_0 = \infty \in \Omega$. There are a unique $r > 0$ and a unique function g , which is analytic in Ω except ∞ , where it has the expansion $g(w) = w + c_0 + c_1 w^{-1} + \dots$, and maps Ω one-to-one onto $\{|z| > \frac{1}{r}\}$. This r is called a conformal radius of the domain Ω with respect to ∞ . In both cases we denote the conformal radius of Ω with respect to the point w_0 by $r(\Omega, w_0)$. Note that if f maps conformally the unit disk \mathbb{D} onto $\Omega \subset \mathbb{C}$ and $f(0) = w_0$, $f'(0) > 0$, then $r(\Omega, w_0) = f'(0)$.

Let Ω_1, Ω_2 be disjoint simply connected domains in the extended complex plane, $0 \in \Omega_1$, $\infty \in \Omega_2$. Then $r(\Omega_1, 0)r(\Omega_2, \infty) \leq 1$, where the equality sign holds if and only if Ω_1 and Ω_2 are the bounded and unbounded components of the complement of a circle with the center at the origin. This results in a corollary of the theorem about nonoverlapping domains obtained by N. A. Lebedev using the area principle [1]. We will consider the case when Ω_1 and Ω_2 are the bounded and unbounded components of a closed Jordan curve, respectively. Let $f : \mathbb{D} \rightarrow \Omega_1$ and $F : \{|z| > 1\} \rightarrow \Omega_2$ be conformal maps. The composition $F^{-1} \circ f$ determines a homeomorphism of the unit circle which is called a conformal welding. We refer the reader to the works [2–7].

In the article, we use the Löwner – Kufarev parametric method to establish an asymptotic relation for conformal radii of two nonoverlapping domains. The Löwner equation



is a differential equation describing a continuously increasing sequence of simply connected domains of a special type, i.e. the so called slit domains [8]. Kufarev [9] and Pommerenke [10] generalized the Löwner equation to a wider class of domains.

Given a chain of simply connected domains $\Omega(t)$, $t \in [0, T)$, such that $0 \in \Omega(t_1) \subset \subset \Omega(t_2)$, $0 \leq t_1 < t_2 < T$, the function $f(z, t) = e^t z + \dots$, conformally mapping \mathbb{D} onto $\Omega(t)$ for each fixed $t \in [0, T)$, a.e. satisfies the (Löwner – Kufarev) equation [9, 11]

$$\frac{\partial f(z, t)}{\partial t} = z \frac{\partial f(z, t)}{\partial z} p(z, t), \quad z \in \mathbb{D}, \quad t \in [0, T), \tag{1}$$

where, for all $t \in [0, T)$, $p(z, t)$ is analytic in \mathbb{D} with respect to z , $p(0, t) = 1$, $\operatorname{Re} p(z, t) > 0$ and $p(z, t)$ is measurable with respect to t for any $z \in \mathbb{D}$. A similar statement can be formulated for a decreasing chain of domains.

We consider a chain of bounded domains $\Omega(t)$, $0 \in \Omega(t)$, $\Omega(0) = \mathbb{D}$ with a boundary $\Gamma(t)$ and a chain $\Omega^*(t)$, $\infty \in \Omega^*(t)$, of unbounded domains with the same boundary $\Gamma(t)$. The method of Löwner-Kufarev evolution can be used to establish a connection between conformal radii of these domains. In [6], it is shown that if $\Omega(t)$ is decreasing, $p(\cdot, t) \in C^2(\overline{\mathbb{D}})$ for $t \in [0, T)$, $p(z, \cdot)$ is continuous in $[0, T)$ for $z \in \overline{\mathbb{D}}$ and $p(z, t)$, $p'(z, t)$ and $p''(z, t)$ are bounded in $\overline{\mathbb{D}} \times [0, T)$, then $\ln(r(\Omega^*(0), \infty)) = t + o(t)$, $t \rightarrow +0$.

We suppose now, that $\Gamma(t)$ is given by the polar equation $r = \gamma(\psi, t)$. Let $G(t)$, $0 \in G(t)$ be a chain of domains bounded by a curve with the polar equation $r = \gamma^{-1}(\psi, t)$. Let $f(z, t) = a(t)z + \dots$ and $g(z, t) = b(t)z + \dots$ conformally map \mathbb{D} onto $\Omega(t)$ and $G(t)$, respectively, where $a(t) = r(\Omega(t), 0)$, $b(t) = r(G(t), 0)$ are positive, strictly monotone and continuous functions, $a(0) = b(0) = 1$. We can always choose the parameter t so that $a(t) = e^t$ ($a(t) = e^{-t}$ in the case of a decreasing chain of domains). The following theorem gives the asymptotic expansion for $b(t)$ in a neighbourhood of $t = 0$.

Theorem 1. *Let $\Omega(t)$, $t \in [0, T)$, be a chain of domains (increasing or decreasing), $0 \in \Omega(t)$, $r(\Omega(t), 0) = e^{\pm t}$ for each $t \in [0, T)$, $\Omega(0) = \mathbb{D}$, and the boundary $\Gamma(t)$ for each $t \in [0, T)$ given by the polar equation $r = \gamma(\psi, t)$, $\psi \in [0, 2\pi]$, where $\gamma \in C^{3+\alpha}$, $\alpha \in (0, 1)$. Let $G(t)$ be a chain of domains bounded by the family of curves with the polar equation $r = \gamma_1(\psi, t) = (\gamma(\psi, t))^{-1}$. Then*

$$\log r(G(t), 0) = \mp t + \left(\frac{1}{2\pi} \int_0^{2\pi} (\dot{\gamma}(\varphi, 0))^2 - \ddot{\gamma}(\varphi, 0) d\varphi \right) t^2 + o(t^2), \quad t \rightarrow +0. \tag{2}$$

By $\dot{\gamma}$, $\ddot{\gamma}$ we denote the first and second derivatives with respect to the parameter t , respectively. In general, we use the following convention. If f is a function of a real or complex variable and t is a parameter, then \dot{f} denotes the derivative with respect to t , while f' denotes the derivative with respect to another variable.

It is not difficult to see that $r(\Omega^*(t), \infty) = r(G(t), 0)$ for $t \in [0, T)$. So, we have the following corollary of Theorem 1, which is the main result of the article.

Corollary 1. *Let a chain of domains $\Omega(t)$ and their boundaries $\Gamma(t)$ be the same as in Theorem 1, and $\Omega^*(t)$ be a chain of the unbounded components of the complement*



of $\Gamma(t)$. Then

$$\log r(\Omega^*(t), \infty) = \mp t + \left(\frac{1}{2\pi} \int_0^{2\pi} (\dot{\gamma}(\varphi, 0))^2 - \ddot{\gamma}(\varphi, 0) d\varphi \right) t^2 + o(t^2), \quad t \rightarrow +0. \quad (3)$$

In Sect. 1 we deduce the integral representation for a driving function in the Löwner – Kufarev equation. We use it in Sect. 2, where Theorem 1 is proved.

1. LÖWNER – KUFAREV EQUATION

The following theorem gives the integral representation for the driving function $p(z, t)$ in the Löwner – Kufarev equation (1). Note that we do not suppose that $f'(0, t) = e^{\pm t}$, as it is usually done.

Theorem 2. Let $\Omega(t)$, $t \in [0, T)$ be a chain of domains (increasing or decreasing), $\Omega(0) = \mathbb{D}$, $0 \in \Omega(t)$, with a boundary $\partial\Omega(t)$ given by the polar equation $r = \gamma(\psi, t) = 1 + \delta(\psi, t)$, $\psi \in [0, 2\pi]$, where $\delta \in C^{3+\alpha}$, $\alpha \in (0, 1)$. Let $f(z, t) = a(t)z + \dots$ conformally map \mathbb{D} onto $\Omega(t)$, $a(t) > 0$. Then f is differentiable with respect to t for $t \in [0, T)$, $z \in \mathbb{D}$, and satisfies the equation (1) where

$$p(z, t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\varphi}, t)|} \dot{\delta}(\psi(\varphi, t), t) \cos(\beta(\psi(\varphi, t), t)) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad (4)$$

with $\psi(\varphi, t) = \arg f(e^{i\varphi}, t)$ and $\beta(\psi, t) = -\arctan\left(\frac{\gamma'(\psi, t)}{\gamma(\psi, t)}\right)$.

Remark 1. Note that $\beta(\psi, t)$ is an angle between a normal to the boundary $\partial\Omega(t)$ at the point $\gamma(\psi, t)e^{i\psi}$ and a radius vector of this point.

Remark 2. Here, the function $p(z, t)$ is analytic in \mathbb{D} with respect to z , $\operatorname{Re} p(z, t) > 0$ if $\Omega(t)$ is increasing and $\operatorname{Re} p(z, t) < 0$ if $\Omega(t)$ is decreasing, $p(0, t) = \pm 1$ if $a(t) = e^{\pm t}$.

Remark 3. Differentiating (1) with respect to z and putting $z = 0$, we obtain an equation for the conformal radius $r(\Omega(t), 0) = a(t)$

$$\frac{d}{dt} \log a(t) = p(0, t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\varphi}, t)|} \dot{\delta}(\psi(\varphi, t), t) \cos(\beta(\psi(\varphi, t), t)) d\varphi.$$

First, we prove the following lemma.

Lemma 1. Let $\Omega_1 \subset \Omega$ be domains bounded by simple closed curves Γ , Γ_1 given by the polar equations $r = \gamma(\psi)$, $r = \gamma_1(\psi)$, $\psi \in [0, 2\pi]$, $\gamma, \gamma_1 \in C^{3+\alpha}$, $\alpha \in (0, 1)$. Let f and f_1 conformally map \mathbb{D} onto Ω and Ω_1 , respectively, $f(0) = f_1(0) = 0$, $f'(0) > 0$, $f_1'(0) > 0$. Let $\delta(\psi) = \gamma(\psi) - \gamma_1(\psi)$ satisfy $|\delta(\psi)| < \varepsilon$, $|\delta'(\psi)| < \varepsilon$, $|\delta''(\psi)| < \varepsilon$. Then

$$f_1(z) = f \left(z \left(1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\varphi})|} \delta(\psi(\varphi)) \cos \beta(\psi(\varphi)) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi \right) \right) + O(\varepsilon^2), \quad z \in \mathbb{D}, \quad \varepsilon \rightarrow +0, \quad (5)$$

with $\psi(\varphi) = \arg f(e^{i\varphi})$ and $\beta(\psi) = -\arctan\left(\frac{\gamma'(\psi)}{\gamma(\psi)}\right)$.



We need the following theorem obtained by Siryk [12] (see also [13, p. 371]). It provides the asymptotic representation for functions conformally mapping \mathbb{D} onto domains close to \mathbb{D} .

Theorem 3. [12] *Let Ω be a domain that contains 0 and is bounded by a curve given by the polar equation $r = 1 - \delta(\psi)$, $0 \leq \psi \leq 2\pi$, where ψ is twice differentiable and satisfies the conditions*

$$|\delta(\psi)| < \varepsilon, \quad |\delta'(\psi)| < \varepsilon, \quad |\delta''(\psi)| < \varepsilon.$$

Then a function $f : \mathbb{D} \rightarrow \Omega$, $f(0) = 0$, $f'(0) > 0$, mapping \mathbb{D} conformally onto Ω has the asymptotic representation

$$f(z) = z \left(1 - \frac{1}{2\pi} \int_0^{2\pi} \delta(\psi) \frac{e^{i\psi} + z}{e^{i\psi} - z} d\psi \right) + O(\varepsilon^2), \quad \varepsilon \rightarrow +0. \quad (6)$$

Proof of Lemma 1. We denote the inverse function by $g = f^{-1}$. Since $\gamma \in C^{3+\alpha}$, $f, f', f'', f^{(3)}$ can be continuously extended to $\overline{\mathbb{D}}$ [14, p. 49] and f' does not vanish there [14, p. 48]. Hence $g, g', g'', g^{(3)}$ can be continuously extended to $\overline{\mathbb{D}}$.

The function g maps the curve Γ_1 onto a simple closed curve in \mathbb{D} , which has the following equation

$$\omega(\varphi) = g(\gamma_1(\psi(\varphi))e^{i\psi(\varphi)}) = g(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)}), \quad 0 \leq \varphi \leq 2\pi. \quad (7)$$

We have

$$\begin{aligned} \omega(\varphi) - e^{i\varphi} &= g(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)}) - g(f(e^{i\varphi})) = \\ &= -g'(f(e^{i\varphi}))\delta(\psi(\varphi))e^{i\psi(\varphi)} + O(\varepsilon^2) = -\frac{1}{f'(e^{i\varphi})}\delta(\psi(\varphi))e^{i\psi(\varphi)} + O(\varepsilon^2) = \\ &= -\frac{1}{|f'(e^{i\varphi})|}\delta(\psi(\varphi))e^{i(\varphi - \beta(\psi(\varphi)))} + O(\varepsilon^2), \quad \varepsilon \rightarrow +0, \end{aligned} \quad (8)$$

where $\beta(\psi(\varphi)) = \arg \frac{f'(e^{i\varphi})e^{i\varphi}}{f'(e^{i\psi(\varphi)})} = -\arctan \frac{\gamma'(\psi(\varphi))}{\gamma(\psi(\varphi))}$. Differentiating (7), we obtain

$$\begin{aligned} \omega'(\varphi) &= g'(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})[f'(e^{i\varphi})ie^{i\varphi} - \psi'(\varphi)e^{i\psi(\varphi)}(\delta'(\psi(\varphi)) + i\delta(\psi(\varphi)))], \\ \omega''(\varphi) &= g''(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})[f'(e^{i\varphi})ie^{i\varphi} - \psi'(\varphi)e^{i\psi(\varphi)}(\delta'(\psi(\varphi)) + i\delta(\psi(\varphi)))]^2 + \\ &\quad + g'(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})[-f''(e^{i\varphi})e^{2i\varphi} - f'(e^{i\varphi})e^{i\varphi} - \\ &\quad - \psi''(\varphi)e^{i\psi(\varphi)}(\delta'(\psi(\varphi)) + i\delta(\psi(\varphi))) - (\psi'(\varphi))^2e^{i\psi(\varphi)}(\delta''(\psi(\varphi)) + 2i\delta'(\psi(\varphi)) - \delta(\psi(\varphi)))]. \end{aligned}$$

Since, $g'', g^{(3)}$ can be continuously extended to $\overline{\mathbb{D}}$ and $|\delta(\psi)| < \varepsilon$, $|\delta'(\psi)| < \varepsilon$, $|\delta''(\psi)| < \varepsilon$ we obtain the following estimates

$$\omega'(\varphi) = [g'(f(e^{i\varphi})) + O(\varepsilon)][f'(e^{i\varphi})ie^{i\varphi} + O(\varepsilon)] = ie^{i\varphi} + O(\varepsilon), \quad \varepsilon \rightarrow +0, \quad (9)$$

$$\omega''(\varphi) = [g''(f(e^{i\varphi})) + O(\varepsilon)][f'(e^{i\varphi})ie^{i\varphi} + O(\varepsilon)]^2 + [g'(f(e^{i\varphi})) + O(\varepsilon)][-f''(e^{i\varphi})e^{2i\varphi} -$$



$$\begin{aligned}
 -f'(e^{i\varphi})e^{i\varphi} + O(\varepsilon) &= \left[-\frac{f''(e^{i\varphi})}{(f'(e^{i\varphi}))^3} + O(\varepsilon) \right] [-(f'(e^{i\varphi}))^2 e^{2i\varphi} + O(\varepsilon)] + \\
 &+ \left[\frac{1}{f'(e^{i\varphi})} + O(\varepsilon) \right] [-f''(e^{i\varphi})e^{2i\varphi} - f'(e^{i\varphi})e^{i\varphi} + O(\varepsilon)] = \\
 &= \frac{f''(e^{i\varphi})}{f'(e^{i\varphi})} e^{2i\varphi} - \frac{f''(e^{i\varphi})}{f'(e^{i\varphi})} e^{2i\varphi} - e^{i\varphi} + O(\varepsilon) = -e^{i\varphi} + O(\varepsilon), \quad \varepsilon \rightarrow +0. \quad (10)
 \end{aligned}$$

Dividing (9) by $\omega(\varphi)$ gives

$$\frac{\partial}{\partial \varphi} \log \omega(\varphi) = \frac{ie^{i\varphi} + O(\varepsilon)}{e^{i\varphi} + O(\varepsilon)} = i + O(\varepsilon), \quad \varepsilon \rightarrow +0, \quad (11)$$

therefore

$$\frac{\partial}{\partial \varphi} \arg \omega(\varphi) = 1 + O(\varepsilon), \quad \varepsilon \rightarrow +0, \quad (12)$$

Hence, the curve $g(\Gamma_1)$ can be given by a polar equation for ε small enough. Denote $\mu(\varphi) = \arg \omega(\varphi) - \varphi$, $0 \leq \varphi \leq 2\pi$. Using (8) we obtain

$$\begin{aligned}
 \mu(\varphi) &= \arg \frac{g(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})}{e^{i\varphi}} = \arg \left(1 + \frac{g(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)}) - e^{i\varphi}}{e^{i\varphi}} \right) = \\
 &= \arg \left(1 - \frac{1}{|f'(e^{i\varphi})|} \delta(\psi(\varphi))e^{-i\beta(\psi(\varphi))} + O(\varepsilon^2) \right), \quad \varepsilon \rightarrow +0,
 \end{aligned}$$

Therefore, it is not difficult to see that $\mu(\varphi) = O(\varepsilon^2)$. From (12) we conclude that $\mu'(\varphi) = O(\varepsilon)$. Let $r = 1 - \Delta(\varphi)$ be the polar equation of $g(\Gamma_1)$. Then $\Delta(\varphi) = 1 - |\omega(\varphi_1)|$, where φ_1 is a unique solution of $\arg \omega(\varphi_1) = \varphi$. Hence $\mu(\varphi_1) + \varphi_1 = \varphi$ and $\varphi_1 = \varphi + O(\varepsilon^2)$. Applying (8) gives

$$\begin{aligned}
 \Delta(\varphi) &= 1 - |\omega(\varphi_1)| = 1 - |\omega(\varphi + O(\varepsilon^2))| = 1 - |\omega(\varphi) + O(\varepsilon^2)| = \\
 &= 1 - |e^{i\varphi} + (\omega(\varphi) - e^{i\varphi}) + O(\varepsilon^2)| = 1 - |e^{i\varphi} - \frac{1}{|f'(e^{i\varphi})|} \delta(\psi(\varphi))e^{i(\varphi - \beta(\psi(\varphi)))} + O(\varepsilon^2)| = \\
 &= 1 - |1 - \frac{\delta(\psi(\varphi))}{|f'(e^{i\varphi})|} e^{-i\beta(\psi(\varphi))} + O(\varepsilon^2)| = \\
 &= \frac{\delta(\psi(\varphi))}{|f'(e^{i\varphi})|} \cos \beta(\psi(\varphi)) + O(\varepsilon^2), \quad 0 \leq \varphi \leq 2\pi, \quad \varepsilon \rightarrow +0. \quad (13)
 \end{aligned}$$

From (13) it easily follows that $|\Delta(\varphi)| = O(\varepsilon)$. Now we want to deduce similar estimates for $|\Delta'(\varphi)|$, $|\Delta''(\varphi)|$. It follows from (8)–(10) that $|\omega(\varphi)'| = O(\varepsilon)$, $|\omega(\varphi)''| = O(\varepsilon)$. Differentiating the equation $\Delta(\varphi + \mu(\varphi)) = 1 - |\omega(\varphi)|$ we obtain

$$\Delta'(\varphi + \mu(\varphi))(1 + \mu'(\varphi)) = -|\omega(\varphi)|',$$

$$\Delta''(\varphi + \mu(\varphi))(1 + \mu'(\varphi))^2 + \Delta'(\varphi + \mu(\varphi))\mu''(\varphi) = -|\omega(\varphi)|''$$

Hence $|\Delta'(\varphi)| = O(\varepsilon)$, $|\Delta''(\varphi)| = O(\varepsilon)$.

Thus, the curve $g(\Gamma_1)$ has the polar equation $r = 1 - \Delta(\varphi)$ where $|\Delta(\varphi)| = O(\varepsilon)$, $|\Delta'(\varphi)| = O(\varepsilon)$, $|\Delta''(\varphi)| = O(\varepsilon)$, $\varepsilon \rightarrow +0$. Therefore, we can apply Theorem 3 for



$h = g \circ f_1$, $h(0) = 0$, $h'(0) > 0$, conformally mapping \mathbb{D} onto the domain bounded by $g(\Gamma_1)$. Hence, (6) gives

$$h(z) = z \left(1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta(\psi(\varphi))}{|f'(e^{i\varphi})|} \cos \beta(\psi(\varphi)) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi \right) + O(\varepsilon^2), \quad z \in \mathbb{D}, \quad \varepsilon \rightarrow +0.$$

Since $f_1 = f \circ h$, we obtain (5). □

Proof of Theorem 2. Let $\Omega(t)$ be an increasing chain. Fix $t \in [0, T)$, h such that $t + h \in [0, T)$. First, let h be negative, so $\Omega(t + h) \in \Omega(t)$. Denote $\lambda(\psi, h) = \gamma(\psi, t) - \gamma(\psi, t + h)$. Since $\gamma \in C^{3+\alpha}$, we conclude that $|\lambda(\psi, h)| = O(h)$, $|\lambda'(\psi, h)| = O(h)$, $|\lambda''(\psi, h)| = O(h)$. So we can apply Lemma 1. By (5), we obtain

$$\begin{aligned} f(z, t + h) &= f \left(z \left(1 - \int_0^{2\pi} s(\varphi, z, t) \lambda(\psi(\varphi, t), h) d\varphi \right), t \right) + o(h) = \\ &= f \left(z \left(1 + h \int_0^{2\pi} s(\varphi, z, t) \dot{\delta}(\psi(\varphi, t), t) d\varphi \right), t \right) + o(h), \end{aligned}$$

where

$$s(\varphi, z, t) = \frac{1}{2\pi} \frac{1}{|f'(e^{i\varphi}, t)|} \cos(\beta(\psi(\varphi, t), t)) \frac{e^{i\varphi} + z}{e^{i\varphi} - z}.$$

Therefore,

$$\begin{aligned} f(z, t + h) - f(z, t) &= f(z + zh \int_0^{2\pi} s(\varphi, z, t) \dot{\delta}(\psi(\varphi, t), t) d\varphi, t) - f(z, t) + o(h) = \\ &= f'(z, t) zh \int_0^{2\pi} s(\varphi, z, t) \dot{\delta}(\psi(\varphi, t), t) d\varphi + o(h), \quad h \rightarrow 0. \end{aligned}$$

Let now h be positive. Then $\Omega(t) \in \Omega(t + h)$, and $\lambda(\psi, h) = \gamma(\psi, t + h) - \gamma(\psi, t)$ satisfies all the conditions of Lemma 1. So we have

$$\begin{aligned} f(z, t) &= f \left(z \left(1 - \int_0^{2\pi} s(\varphi, z, t + h) \lambda(\psi(\varphi, t + h), h) d\varphi \right), t + h \right) + o(h) = \\ &= f \left(z \left(1 - h \int_0^{2\pi} s(\varphi, z, t + h) \dot{\delta}(\psi(\varphi, t + h), t) d\varphi \right), t + h \right) + o(h), \end{aligned}$$

So we obtain

$$\begin{aligned} f(z, t + h) - f(z, t) &= f(z, t + h) - \\ &- f(z - zh \int_0^{2\pi} s(\varphi, z, t + h) \dot{\delta}(\psi(\varphi, t + h), t) d\varphi, t + h) + o(h) = \end{aligned}$$



$$= f'(z, t+h)zh \int_0^{2\pi} s(\varphi, z, t+h)\dot{\delta}(\psi(\varphi, t+h), t)d\varphi + o(h), \quad h \rightarrow 0,$$

Thus, we have shown that f is differentiable with respect to t and satisfies (1). One can similarly repeat the proof for a decreasing chain of domains. \square

2. PROOF OF THEOREM 1

Let, for each $t \in [0, T)$, $f(\cdot, t)$ and $g(\cdot, t)$ conformally map \mathbb{D} onto $\Omega(t)$ and $G(t)$, respectively, $f(0, t) = g(0, t) = 0$, $f'(0, t) > 0$, $g'(0, t) > 0$. Denote $\delta(\psi, t) = \gamma(\psi, t) - 1$, $\delta_1(\psi, t) = \gamma_1(\psi, t) - 1$. By Theorem 2 and Remark 3, conformal radii satisfy the equations

$$\frac{d}{dt} \log r(\Omega(t), 0) = p(0, t), \quad \frac{d}{dt} \log r(G(t), 0) = q(0, t), \quad (14)$$

where $p(z, t)$, $q(z, t)$ are given by

$$p(z, t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\varphi}, t)|} \dot{\delta}(\psi(\varphi, t), t) \cos(\beta(\psi(\varphi, t), t)) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad (15)$$

$$q(z, t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|g'(e^{i\varphi}, t)|} \dot{\delta}_1(\psi_1(\varphi, t), t) \cos(\beta_1(\psi_1(\varphi, t), t)) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad (16)$$

with $\psi(\varphi, t) = \arg f(e^{i\varphi}, t)$, $\beta(\psi, t) = -\arctan(\frac{\gamma'(\psi, t)}{\gamma(\psi, t)})$, $\psi_1(\varphi, t) = \arg g(e^{i\varphi}, t)$, $\beta_1(\psi, t) = -\arctan(\frac{\gamma_1'(\psi, t)}{\gamma_1(\psi, t)})$. First, we want to prove the following equalities

$$p(z, 0) = -q(z, 0) = \frac{1}{2\pi} \int_0^{2\pi} \dot{\delta}(\varphi, 0) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad z \in \mathbb{D}, \quad (17)$$

$$\frac{\partial q(z, 0)}{\partial t} - \frac{\partial p(z, 0)}{\partial t} = \frac{1}{\pi} \int_0^{2\pi} (\dot{\delta}(\varphi, 0))^2 - \ddot{\delta}(\varphi, 0) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad z \in \mathbb{D}. \quad (18)$$

Elementary calculations lead us to the formulas

$$\dot{\delta}_1(\psi, 0) = -\dot{\delta}(\psi, 0), \quad (19)$$

$$\dot{\delta}'_1(\psi, 0) = -\dot{\delta}'(\psi, 0), \quad (20)$$

$$\ddot{\delta}_1(\psi, 0) = -\ddot{\delta}(\psi, 0) + 2(\dot{\delta}(\psi, 0))^2. \quad (21)$$

Since $f'(e^{i\varphi}, 0) = 1$, $\psi(\varphi, 0) = \varphi$, $\beta(\psi, 0) = 0$, representation (15) gives

$$p(z, 0) = \frac{1}{2\pi} \int_0^{2\pi} \dot{\delta}(\varphi, 0) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad z \in \mathbb{D}.$$

Similarly we obtain

$$q(z, 0) = \frac{1}{2\pi} \int_0^{2\pi} \dot{\delta}_1(\varphi, 0) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad z \in \mathbb{D}.$$

Thus, (19) gives (17).



Denote by $P(t)$ the expression under the integral in (15). Elementary calculations yield the following result

$$\dot{P}(0) = -\frac{\partial}{\partial t}(|f'(e^{i\varphi}, t)|) \Big|_{t=0} \delta(\varphi, 0) + \dot{\delta}'(\varphi, 0)\dot{\psi}(\varphi, 0) + \ddot{\delta}(\varphi, 0). \tag{22}$$

Similarly, we denote by $Q(t)$ the expression under the integral in (16) and obtain

$$\dot{Q}(0) = -\frac{\partial}{\partial t}(|g'(e^{i\varphi}, t)|) \Big|_{t=0} \delta_1(\varphi, 0) + \dot{\delta}'_1(\varphi, 0)\dot{\psi}_1(\varphi, 0) + \ddot{\delta}_1(\varphi, 0). \tag{23}$$

It easily follows from (1) that $\dot{\psi}(\varphi, 0) = \text{Im } p(e^{i\varphi}, 0)$, $\dot{\psi}_1(\varphi, 0) = \text{Im } q(e^{i\varphi}, 0)$. Therefore, (17) gives

$$\dot{\psi}_1(\varphi, 0) = -\dot{\psi}(\varphi, 0), \quad 0 \leq \varphi \leq 2\pi. \tag{24}$$

Differentiating (1) with respect to z and putting $t = 0$ we conclude that $\frac{\partial^2 f(z, t)}{\partial z \partial t} \Big|_{t=0} = p(z, 0) + zp'(z, 0)$. Hence

$$\dot{f}'(e^{i\varphi}, 0) = \frac{\partial}{\partial t} f'(e^{i\varphi}, t) \Big|_{t=0} = p(e^{i\varphi}, 0) + e^{i\varphi} p'(e^{i\varphi}, 0).$$

Since

$$f'(e^{i\varphi}, t) = f'(e^{i\varphi}, 0) + \dot{f}'(e^{i\varphi}, 0)t + o(t), \quad t \rightarrow +0,$$

we see that

$$\begin{aligned} |f'(e^{i\varphi}, t)| &= |e^{i\varphi} + \dot{f}'(e^{i\varphi}, 0)t + o(t)| = \\ &= 1 + |\dot{f}'(e^{i\varphi}, 0)| \cos(\arg(\dot{f}'(e^{i\varphi}, 0)) - \varphi)t + o(t), \quad t \rightarrow +0. \end{aligned}$$

Thus,

$$\frac{\partial}{\partial t} |f'(e^{i\varphi}, t)| \Big|_{t=0} = |\dot{f}'(e^{i\varphi}, 0)| \cos(\arg(\dot{f}'(e^{i\varphi}, 0)) - \varphi), \quad 0 \leq \varphi \leq 2\pi. \tag{25}$$

From (25) and (17) we deduce

$$\frac{\partial}{\partial t} |g'(e^{i\varphi}, t)| \Big|_{t=0} = -\frac{\partial}{\partial t} |f'(e^{i\varphi}, t)| \Big|_{t=0}, \quad 0 \leq \varphi \leq 2\pi. \tag{26}$$

Formulas (19)–(24) and (26) show that $\dot{Q}(0) - \dot{P}(0) = -2\ddot{\delta}(\psi, 0) + 2(\dot{\delta}(\psi, 0))^2$, which leads to (18). One can deduce (2) from (14) and (17), (18). Indeed, let, for example, $\Omega(t)$ be an increasing chain of domains. Using (14) and (17) we obtain

$$\frac{d}{dt} \log r(G(t), 0) \Big|_{t=0} = q(0, 0) = -p(0, 0) = -1,$$

Similarly, using (18) we obtain

$$\frac{d^2}{dt^2} \log r(G(t), 0) \Big|_{t=0} = \frac{d}{dt} q(0, t) \Big|_{t=0} = \frac{1}{\pi} \int_0^{2\pi} (\dot{\delta}(\varphi, 0))^2 - \ddot{\delta}(\varphi, 0) d\varphi.$$

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АСИМПТОТИЧЕСКОЕ СООТНОШЕНИЕ ДЛЯ КОНФОРМНЫХ РАДИУСОВ ДВУХ НЕНАЛЕГАЮЩИХ ОБЛАСТЕЙ

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В статье рассматривается семейство замкнутых жордановых кривых, заданных в полярной системе координат и непрерывно зависящих от параметра, и такое, что области, ограниченные этими кривыми, образуют возрастающее или убывающее семейство. Такое семейство областей описывается дифференциальным уравнением Левнера – Куфарева. Для рассмотренного случая получено интегральное представление для управляющей функции в этом уравнении. Используя это представление, получено асимптотическое соотношение, связывающее конформные радиусы ограниченной и неограниченной компоненты дополнения к жордановой кривой, когда ограниченная компонента близка к единичному кругу.

Ключевые слова: уравнение Левнера – Куфарева, конформный радиус, асимптотическое разложение, неналегающие области.

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