Алгебра и анализ
Том 34 (2022), №3

To N. K. Nikolski on the occasion of his 80th birthday

ON THE MAXIMAL IDEAL SPACES OF H^∞ ON COVERINGS OF BORDERED RIEMANN SURFACES

© A. BRUDNYI

The paper describes the topological structure of the maximal ideal space of the algebra of bounded holomorphic functions on a covering of a bordered Riemann surface. Some applications of the obtained results to the theory of bounded operator-valued holomorphic functions on Riemann surfaces are presented.

§1. Introduction

1.1. Let $r\colon \widetilde{R} \to R$ be an unbranched covering of a bordered Riemann surface R. Our main goal in this paper is to give a topological description of the maximal ideal space $\mathfrak{M}(H^\infty(\widetilde{R}))$ of the Banach algebra $H^\infty(\widetilde{R})$ of bounded holomorphic functions on $\widetilde{R}$ equipped with pointwise multiplication and the supremum norm. Most of the present results were established in the author's earlier papers. In this paper, we provide a brief overview of these results.

Recall that $\mathfrak{M}(H^\infty(\widetilde{R}))$ is the set of nonzero homomorphisms

$$\varphi\colon H^\infty(\widetilde{R}) \to \mathbb{C}$$

endowed with the *Gelfand topology*, the weak* topology induced by $(H^\infty(\widetilde{R}))^*$. It is a compact subspace of $(H^\infty(\widetilde{R}))^*$. The algebra $H^\infty(\widetilde{R})$ separates points of $\widetilde{R}$ and each evaluation functional δ_x of functions in $H^\infty(\widetilde{R})$ at $x \in \widetilde{R}$ is an element of $\mathfrak{M}(H^\infty(\widetilde{R}))$. Hence, the correspondence $x \mapsto \delta_x$ embeds $\widetilde{R}$ into $\mathfrak{M}(H^\infty(\widetilde{R}))$ as an open subset. The corona theorem proved in [4] states that $\widetilde{R}$ is dense in $\mathfrak{M}(H^\infty(\widetilde{R}))$. This result is derived from the classical Carleson corona theorem [15] for $H^\infty(\mathbb{D})$, where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, by

Ключевые слова: maximal ideal space, interpolating sequence, Blaschke product, Gleason part, analytic disk, covering dimension, cohomology, Freudenthal compactification.

Research is supported in part by NSERC.

means of a Forelli-type projection constructed in [4] applied to functions in $H^\infty(\mathbb{D})$. The functions in $H^\infty(\widetilde{R})$ are continuously extended to $\mathfrak{M}(H^\infty(\widetilde{R}))$ by means of the *Gelfand transform* $\widehat{\ }\colon H^\infty(\widetilde{R}) \to C(\mathfrak{M}(H^\infty(\widetilde{R})))$, $\widehat{a}(\varphi) := \varphi(a)$. Here $C(\mathfrak{M}(H^\infty(\widetilde{R})))$ is the algebra of continuous complex-valued functions on $\mathfrak{M}(H^\infty(\widetilde{R}))$ equipped with the supremum norm.

1.2. The topological structure of the maximal ideal space $\mathfrak{M}(H^\infty(\mathbb{D}))$ is well understood. In the current paper, we focus on properties of $\mathfrak{M}(H^\infty(\widetilde{R}))$ analogous to the following properties of $\mathfrak{M}(H^\infty(\mathbb{D}))$.

Recall that for a commutative unital complex Banach algebra A the *Gleason parts* are the equivalence classes in its maximal ideal space defined by the relation $\|\varphi - \xi\|_{A^*} < 2$.

(a) Each Gleason part of $\mathfrak{M}(H^\infty(\mathbb{D}))$ is either an analytic disk or a point. The former case means that there is a continuous one-to-one map $L_\xi\colon \mathbb{D} \to P(\xi)$ onto the Gleason part $P(\xi)$ containing ξ such that $\widehat{f} \circ L_\xi \in H^\infty(\mathbb{D})$ for every $f \in H^\infty(\mathbb{D})$. Moreover, $P(\xi)$ is an analytic disk if and only if ξ belongs to the closure of an interpolating sequence for $H^\infty(\mathbb{D})$. In particular, the set of nontrivial Gleason parts is an open dense subset of $\mathfrak{M}(H^\infty(\mathbb{D}))$. These classical results are due to Hoffman [19].

(b) The covering dimension of $\mathfrak{M}(H^\infty(\mathbb{D}))$ is 2, the set of trivial Gleason parts of $\mathfrak{M}(H^\infty(\mathbb{D}))$ is totally disconnected, and the Čech cohomology group $H^2(\mathfrak{M}(H^\infty(\mathbb{D})), \mathbb{Z})$ is trivial. These important results are established by Suárez [27, 28] (see also [7] for alternative proofs).

(c) $\mathfrak{M}(H^\infty(\mathbb{D}))$ is the Freudenthal (end) compactification of the set of its nontrivial Gleason parts, see [7].

In a forthcoming paper, we will use the present results to develop a Stein-like theory and the Oka–Grauert principle on $\mathfrak{M}(H^\infty(\widetilde{R}))$ analogous to those on $\mathfrak{M}(H^\infty(\mathbb{D}))$, see [8, 9].

1.3. It is well known (see, e.g., [29, Section 8]) that R can be regarded as a domain in a compact Riemann surface S such that

$$S \setminus \bar{R} = \bigsqcup_{i=1}^{k} D_i, \tag{1}$$

where the D_i are open disks with analytic boundaries ∂D_i and $\bar{R} \subset S$ is the closure of R.

Let $A(R)$ be the Banach algebra of bounded holomorphic functions on R continuous up to the boundary equipped with the supremum norm. Since an open proper neighborhood of $\bar{R}$ in S is Stein, $A(R)$ separates points of $\bar{R}$. The

pullback by r (hereafter denoted as r^*) maps $A(R)$ isometrically into $H^\infty(\widetilde{R})$; hence, the transpose of r^* induces a continuous surjective map

$$\widehat{r}\colon \mathfrak{M}(H^\infty(\widetilde{R})) \to \bar{R}$$

such that $\widehat{r}|_{\widetilde{R}} = r$. Thus,

$$\mathfrak{M}(H^\infty(\widetilde{R})) = \widehat{r}^{-1}(R) \sqcup \bigsqcup_{i=1}^{k} \widehat{r}^{-1}(\partial D_i). \tag{2}$$

The paper is organized as follows.

In §§2 and 3, we describe the structure of $\widehat{r}^{-1}(R)$ and $\widehat{r}^{-1}(\partial D_i)$.

In §4, we formulate some results on dimension, cohomology and the end compactification for $\mathfrak{M}(H^\infty(\widetilde{R}))$.

§5 contains some applications of the obtained results in the theory of bounded operator-valued holomorphic functions on Riemann surfaces.

In §6, we collect some results on interpolating sequences for $H^\infty(\widetilde{R})$.

Finally, §7 contains proofs of new results of the paper.

§2. Structure of $\widehat{\mathbf{r}}^{-\mathbf{1}}(\mathbf{R})$

For proofs of results of this section, see, e.g., [3, Subsection 2.1], [5, Subsection 2.2]. Also, for basic properties of fiber bundles, see, e.g., [21].

We consider the covering $r\colon \widetilde{R} \to R$ as a fiber bundle over R with a discrete fiber F, the quotient of the fundamental group $\pi_1(R)$ by the action of its subgroup $\pi_1(\widetilde{R})$ by left multiplications. Let $E(R, \beta F)$ be the space obtained from $\widetilde{R}$ by taking the Stone-Čech compactification of fibers under r. Then the following holds:

(2.A) $E(R, \beta F)$ is a locally compact Hausdorff space of covering dimension 2;

(2.B) r extends to a continuous map

$$r_E\colon E(R, \beta F) \to R$$

such that $\big(E(R, \beta F), R, r_E, \beta F\big)$ is a fiber bundle over R with fiber βF;

(2.C) $\widetilde{R}$ embeds in $E(R, \beta F)$ as an open dense subbundle.

Recall that for a normal space X, the covering dimension $\dim X$ is at most n if every open cover of X can be refined by an open cover whose order is at most $n+1$. If $\dim X \leqslant n$ and the statement $\dim X \leqslant n-1$ is false, then $\dim X = n$.

If $K \subset S$ is a compact set and $\widetilde{K} = r^{-1}(K)$, $K_E = r_E^{-1}(K)$, then $\widetilde{K}$ is dense in K_E and a bounded continuous function on $\widetilde{K}$ admits a continuous extension to the compact set K_E if and only if it is uniformly continuous with respect

to the path metric on $\widetilde{R}$ induced by the pullback by r of a Riemannian metric on S. In particular, this is valid for restrictions to $\widetilde{K}$ of bounded holomorphic functions defined on $\widetilde{U} = r^{-1}(U)$, where U is an open neighborhood of K. Hence,

(2.D) every function $f \in H^\infty(\widetilde{R})$ admits an extension $\widehat{f} \in C(E(R,\beta F))$; moreover, the algebra formed by these extensions separates points of $E(R,\beta F)$.

Thus, there is an injective continuous map ι of $E(R,\beta F)$ in $\mathfrak{M}(H^\infty(\widetilde{R}))$ sending $x \in E(R,\beta F)$ to the evaluation functionals δ_x of functions $\widehat{f}$ at x. In fact,

(2.E) $\iota\colon E(R,\beta F) \to \mathfrak{M}(H^\infty(\widetilde{R}))$ is a *topological embedding* whose image coincides with the open set $\widehat{r}^{-1}(R)$.

Therefore, without loss of generality we may identify $\widehat{r}^{-1}(R)$ and $E(R,\beta F)$; hence, $r_E = \widehat{r}|_{E(R,\beta F)}$.

Next, a *path-component* of a topological space X is an equivalence class of X under the equivalence relation: $x \sim y$ if there is a path from x to y, i.e., a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

The space $E(R,\beta F)$ is the disjoint union of nontrivial path-components. Moreover, if C is a path-component of $E(R,\beta F)$, then $r_E|_C\colon C \to R$ is a subbundle of $r_E\colon E(R,\beta F) \to R$ with a fiber $F_C \subset \beta F$, the orbit of a point under the right action of $\pi_1(R)$ on βF extending by continuity the right action of $\pi_1(R)$ on F: $(g, \pi_1(\widetilde{R}) \cdot h) \mapsto \pi_1(\widetilde{R}) \cdot (h \cdot g)$; here $\pi_1(\widetilde{R}) \cdot h \in F$ is the orbit of $h \in \pi_1(R)$ under the left action of $\pi_1(\widetilde{R})$. In particular, if F is infinite, then the cardinality of the set of path-components of $E(R,\beta F)$ is the same as that of $\beta F \cong \beta\mathbb{N}$, i.e., $2^{\mathfrak{c}}$. By the definition, we also have

(2.F) for every path-component C there is a connected unbranched covering $r_C\colon C' \to R$ with a discrete fiber $F_{C'}$ and a continuous injective map $i_C\colon C' \to E(R,\beta F)$ such that $i_C(C') = C$ and $r \circ i_C = r_C$.

The map i_C is not necessarily a topological embedding (but clearly it is a local embedding), see [3]. In addition, if $\pi_1(\widetilde{R})$ is a normal subgroup of $\pi_1(R)$ (in this case the covering $r\colon \widetilde{R} \to R$ is said to be *regular*), then all C' are biholomorphic to $\widetilde{R}$.

Let $U \subset \mathfrak{M}(H^\infty(\widetilde{R}))$ be an open set. A continuous function f on U is said to be *holomorphic* if its restriction to $U \cap \widetilde{R}$ (which is an open dense subset of U by the corona theorem) is holomorphic in the standard sense. The space of functions holomorphic on U is denoted by $\mathcal{O}(U)$. Next, if X is a complex analytic space, then a continuous map $F\colon X \to \mathfrak{M}(H^\infty(\widetilde{R}))$ is said

to be holomorphic if for every open $U \subset \mathfrak{M}(H^\infty(\widetilde{R}))$ and every $f \in \mathcal{O}(U)$ the function $F^*f = f \circ F$ is holomorphic on the open subset $F^{-1}(U) \subset X$.

In accordance with this definition, every map

$$i_C \colon C' \to E(R, \beta F) \subset \mathfrak{M}(H^\infty(\widetilde{R}))$$

in (2.F) is a holomorphic injection (see, e.g., [3, Lemma 3.2] for the proof). In particular, if $r \colon \widetilde{R} \to R$ is the universal covering of R, i.e., if $\widetilde{R}$ is biholomorphic to $\mathbb{D}$, then every C' is biholomorphic to $\mathbb{D}$ and $\iota \colon C' \to \mathfrak{M}(H^\infty(\widetilde{R})) \cong \mathfrak{M}(H^\infty(\mathbb{D}))$ is a maximal analytic disk in the sense of Hoffman.[1] In this case, the image of C' is a Gleason part of $\mathfrak{M}(H^\infty(\widetilde{R}))$. In general, every Gleason part of $\mathfrak{M}(H^\infty(\widetilde{R}))$ intersecting $E(R, \beta G)$ is the union of some path-components of $E(R, \beta G)$. However, it is still unknown *whether there are Gleason parts of* $\mathfrak{M}(H^\infty(\widetilde{R}))$ *containing more than one path-component of* $E(R, \beta G)$.

§3. Structure of $\widehat{\mathbf{r}}^{-1}(\partial \mathbf{D_i})$

Let us choose some open disks $D_i' \subset S$ containing D_i such that each $A_i = D_i' \setminus D_i \subset S$ is biholomorphic up to the boundary to an annulus

$$A := \{z \in \mathbb{C} \,:\, c < |z| < 1\}$$

with ∂D_i homeomorphic to the outer boundary circle, see (1). We set

$$\widetilde{A}_i = r^{-1}(A_i), \quad 1 \leqslant i \leqslant k.$$

By definition, each connected component of $\widetilde{A}_i$ is biholomorphic either to an annulus or to $\mathbb{D}$. If the covering is not regular, then these components are not necessarily biholomorphic. We cover A_i by two open sets A_{i1}, A_{i2} biholomorphic (by the above chosen biholomorphism between A_i and A) to

$$A_1 = \left\{z = re^{i\theta} \in A : \, -\pi < \theta < \frac{\pi}{2}\right\} \text{ and } A_2 = \left\{z = re^{i\theta} \in A : 0 < \theta < \frac{3\pi}{2}\right\},$$

respectively, and set

$$\widetilde{A}_{ij} = r^{-1}(A_{ij}).$$

Then, each connected component of $\widetilde{A}_{ij}$ is biholomorphic by means of r to A_{ij}; hence, $\widetilde{A}_{ij}$ is biholomorphic to $A_{ij} \times F$.

We denote by $\partial_o A_{ij} = \overline{A}_{ij} \cap \partial D_i$ the part of the 'outer' boundary of A_{ij}.

Now, [10, Proposition 3.2] implies that

(3.A) every function in $H^\infty(\widetilde{A}_{ij})$ has a continuous extension to $\widehat{r}^{-1}(A_{ij} \cup \partial_o A_{ij})$.

[1] This follows from (4.A) of §4.

Let $q_j\colon \mathfrak{M}(H^\infty(\widetilde{A}_{ij})) \to \mathfrak{M}(H^\infty(\widetilde{R}))$ be continuous maps transposed to the isometric embeddings $H^\infty(\widetilde{R}) \hookrightarrow H^\infty(\widetilde{A}_{ij})$ defined by restrictions of functions on $\widetilde{R}$ to $\widetilde{A}_{ij}$, $j = 1, 2$. Here, $q_j|_{\widetilde{A}_{ij}}$ is the natural inclusion of $\widetilde{A}_{ij}$ into $\mathfrak{M}(H^\infty(\widetilde{R}))$. Then, due to the corona theorem for $H^\infty(\widetilde{R})$, the image of q_j coincides with $\widehat{r}^{-1}(\overline{A}_{ij})$. Now, as a consequence of (3.A) one obtains that q_j *maps* $(\widehat{r}\circ q_j)^{-1}(A_{ij}\cup\partial_o A_{ij})$ *bijectively onto* $\widehat{r}^{-1}(A_{ij}\cup\partial_o A_{ij})$, see [10, Corollary 3.4]. Hence, q_j maps every compact subset of $(\widehat{r}\circ q_j)^{-1}(A_{ij}\cup\partial_o A_{ij})$ homeomorphically onto its image, and since $(\widehat{r}\circ q_j)^{-1}(A_{ij}\cup\partial_o A_{ij})$ is σ-compact as it is the union of countably many compact subspaces $(\widehat{r}\circ q_j)^{-1}(K)$, where $K\subset A_{ij}\cup\partial_o A_{ij}$ is compact,

(3.B) $q_j\colon (\widehat{r}\circ q_j)^{-1}(A_{ij}\cup\partial_o A_{ij}) \to \widehat{r}^{-1}(A_{ij}\cup\partial_o A_{ij})$ is a homeomorphism.

Next, by the Caratheodory theorem, A_{ij} is biholomorphic up to the boundary to $\mathbb{D}$ and this biholomorphism maps $\partial_o A_{ij}$ homeomorphically onto an open arc $\gamma\subset\partial\mathbb{D}$. Hence, there is a map $\varphi_{ij}\colon \mathbb{D}\times F\to\widetilde{A}_{ij}$ biholomorphic up to the boundary that can be extended to a homeomorphism

$$\widehat{\varphi}_{ij}\colon \mathfrak{M}(H^\infty(\mathbb{D}\times F)) \to \mathfrak{M}(H^\infty(\widetilde{A}_{ij}))$$

such that

$$\widehat{\varphi}_{ij}^{-1}\big((\widehat{r}\circ q_j)^{-1}(A_{ij}\cup\partial_o A_{ij})\big) = \widehat{p}^{-1}(\mathbb{D}\cup\gamma). \tag{3}$$

Here $\widehat{p}\colon \mathfrak{M}(H^\infty(\mathbb{D}\times F)) \to \bar{\mathbb{D}}$ is the continuous extension of the projection onto the first coordinate $p\colon \mathbb{D}\times F\to\mathbb{D}$.

Thus,

(3.C) $q_j\circ\widehat{\varphi}_{ij}$ maps $\widehat{p}^{-1}(\mathbb{D}\cup\gamma)$ biholomorphically onto $\widehat{r}^{-1}(A_{ij}\cup\partial_o A_{ij})$.

This means that the pullback $(q_j\circ\widehat{\varphi}_{ij})^*$ of the homeomorphism $q_j\circ\widehat{\varphi}_{ij}$ sends holomorphic functions on open subsets of $\widehat{r}^{-1}(A_{ij}\cup\partial_o A_{ij})$ to holomorphic functions on their preimages in $\widehat{p}^{-1}(\mathbb{D}\cup\gamma)$.

The structure of $\mathfrak{M}(H^\infty(\mathbb{D}\times F))$ was described in [11]. Specifically, since the algebra $H^\infty(\mathbb{D}\times F)$ is logmodular on its Shilov boundary (see [2, Lemma 2]), the general result of Hoffman [20] implies that the Gleason parts of $\mathfrak{M}(H^\infty(\mathbb{D}\times F))$ are either points or analytic disks. Moreover, a point belongs to an analytic disk if and only if it is in the closure of an interpolating sequence for $H^\infty(\mathbb{D}\times F)$, and the set of trivial (i.e., one-point) Gleason parts is compact and totally disconnected, see [11, Theorems 2.4 and 2.6]. Hence, since by (3.C) $\widehat{r}^{-1}(\partial D_i)$ is covered by two open sets biholomorphic to $\widehat{p}^{-1}(\mathbb{D}\cup\gamma)$, using (3) we obtain

(3.D) $\widehat{r}^{-1}(\partial D_i)$ is the disjoint union of the set containing all analytic disks and a compact totally disconnected set.

Moreover, all analytic disks in $\widehat{r}^{-1}(\partial D_i)$ lie in the union of closures of interpolating sequences for $H^\infty(\widetilde{A}_{ij})$, $j = 1, 2$. We strengthen this as follows (see §7 for the proof).

Theorem 3.1. *A point belongs to an analytic disk in $\widehat{r}^{-1}(\partial D_i)$ if and only if it is in the closure of an interpolating sequence for $H^\infty(\widetilde{R})$.*

Finally, in connection with the classical Hoffman theorem [19] for $\mathfrak{M}(H^\infty(\mathbb{D}))$, it is natural to ask *whether every maximal analytic disk in $\widehat{r}^{-1}(\partial D_i)$ is a Gleason part of $\mathfrak{M}(H^\infty(\widetilde{R}))$ and the complement in $\widehat{r}^{-1}(\partial D_i)$ of the set of maximal analytic disks consists of one-point Gleason parts of $\mathfrak{M}(H^\infty(\widetilde{R}))$.*

§4. Other properties of $\mathfrak{M}(H^\infty(\widetilde{R}))$

Let $\mathfrak{M}_a \subset \mathfrak{M}(H^\infty(\widetilde{R}))$ be the union of $E(R, \beta F)$ and the set of maximal analytic disks in $\widehat{r}^{-1}(\partial D_i)$, $1 \leqslant i \leqslant k$, and let $\mathfrak{M}_s = M(H^\infty(\widetilde{R})) \setminus \mathfrak{M}_a$. Then (3.D) implies that $\mathfrak{M}_s$ is totally disconnected. Also, by the construction of the proof of [11, Theorem 2.4], one deduces that $\mathfrak{M}_s$ has the same cardinality as the set of trivial Gleason parts of $\mathfrak{M}(H^\infty(\mathbb{D}))$, i.e., $2^\mathfrak{c}$. In turn, $\mathfrak{M}_a$ is an open dense subset of $\mathfrak{M}(H^\infty(\widetilde{R}))$ satisfying the following property.

(4.A) The image of a connected and locally connected space under a continuous map into $\mathfrak{M}_a$ lies either in a maximal analytic disk in $r^{-1}(\partial D_i)$, $i = 1, \dots, k$, or in a path-component of $E(R, \beta F)$ (see (2.G)).

This result follows straightforwardly from [3, Proposition 2.3 (c)] and [11, Theorem 2.5(d)] applied to the open subsets $E(R, \beta F)$ and $\widehat{r}^{-1}(A_{ij} \cup \partial_o A_{ij}) \setminus \mathfrak{M}_s$ of $\mathfrak{M}_a$.

Next, due to [4, Theorem 1.9] every fiber of $r\colon \widetilde{R} \to R$ is an interpolating sequence for $H^\infty(\widetilde{R})$. Since these fibers are dense in fibers of $\widehat{r}\colon E(R, \beta F) \to R$, see §2, each point of $E(R, \beta F)$ lies in the closure of an interpolating sequence for $H^\infty(\widetilde{R})$. Combining this with Theorem 3.1 we get the following fact.

(4.B) A point belongs to $\mathfrak{M}_a$ if and only if it is in the closure of an interpolating sequence for $H^\infty(\widetilde{R})$.

Next, we describe $\mathfrak{M}(H^\infty(\widetilde{R}))$ as the end compactification of $\mathfrak{M}_a$.

Let X be a semicompact Hausdorff space, i.e., every point of X has arbitrarily small neighborhoods with compact boundaries. The *Freudenthal compactification* $\gamma(X)$ of X is a unique (up to homeomorphism) Hausdorff compactification of X, i.e., a compact Hausdorff space containing X as an open dense subset, having the following properties:

(a) $\gamma(X)\setminus X$ is zero-dimensionally embedded in $\gamma(X)$, i.e., for each point $x\in\gamma(X)\setminus X$ and a neighborhood U of x there exists a neighborhood $V\subset U$ of x whose boundary lies in X;

(b) $\gamma(X)$ is maximal with respect to (a). That is, if $c(X)$ is a Hausdorff compactification of X such that $c(X)\setminus X$ is zero-dimensionally embedded in $c(X)$, then the identity map on X has a continuous extension from $\gamma(X)$ to $c(X)$.

Also, the Freudenthal compactification $\gamma(X)$ can be defined as follows.

Let $\bar{C}_{\rm fin}(X)$ be the closure in $C_b(X)$ of the algebra $C_{\rm fin}(X)$ of all functions f for which there is a compact subset $K\subset X$ such that $f(X\setminus K)\subset\mathbb{C}$ is finite. Then the maximal ideal space of $\bar{C}_{\rm fin}(X)$ is homeomorphic to $\gamma(X)$.

The following result is analogous to [7, Theorem 4.1].

Theorem 4.1. $\mathfrak{M}(H^\infty(\widetilde{R}))$ *coincides with* $\gamma(\mathfrak{M}_a)$.

The proof of this result is given in §7.

Now, we formulate a result on the dimension and cohomology of $\mathfrak{M}(H^\infty(\widetilde{R}))$ proved in [10, Theorem 1.3] and similar to results of Suárez [27] for $\mathfrak{M}(H^\infty(\mathbb{D}))$.

(4.C) The covering dimension $\dim\mathfrak{M}(H^\infty(\widetilde{R}))$ equals 2, and the Čech cohomology group $H^2(\mathfrak{M}(H^\infty(\widetilde{R})),\mathbb{Z})$ is 0.

Further, as proved in [13, Theorem 1.1] by means of the Treil theorem [30], the *Bass stable rank* of $H^\infty(\widetilde{R})$ is 1 (this means that for every $f,g\in H^\infty(\widetilde{R})$ satisfying $f_1f+g_1g=1$ for some $f_1,g_1\in H^\infty(\widetilde{R})$, there exists $G\in H^\infty(\widetilde{R})$ such that $\Phi=f+gG$ is invertible in $H^\infty(\widetilde{R})$, see [1]). From here by the cohomological interpretation of the Bass stable rank, see [16], we deduce the following property of $\mathfrak{M}(H^\infty(\widetilde{R}))$.

Let $Z(I)=\{\xi\in\mathfrak{M}(H^\infty(\widetilde{R})):\widehat{f}(\xi)=0\ \forall f\in I\}$ be the zero set of a proper ideal $I\subset H^\infty(\widetilde{R})$.

(4.D) The inclusion $Z(I)\hookrightarrow\mathfrak{M}(H^\infty(\widetilde{R}))$ induces an epimorphism of the cohomology groups $H^1(\mathfrak{M}(H^\infty(\widetilde{R})),\mathbb{Z})\to H^1(Z(I),\mathbb{Z})$.

Let $H^\infty_I=\mathbb{C}+I$ be the unital closed subalgebra of $H^\infty(\widetilde{R})$ generated by a proper closed ideal $I\subset H^\infty(\widetilde{R})$. In particular, if I is a maximal ideal, then $H^\infty_I=H^\infty(\widetilde{R})$. The corona theorem for H^∞_I can be derived from the corona theorem for $H^\infty(\widetilde{R})$ by the argument of the proof of Theorem 1.6 in [24]. Then, using (4.C) and (4.D) and arguing exactly as in the proof of [9, Proposition 2.1] one deduces that there is a continuous surjective map

$$Q_I\colon\mathfrak{M}(H^\infty(\widetilde{R}))\to\mathfrak{M}(H^\infty_I)$$

sending $Z(I)$ to a point and one-to-one outside of $I(Z)$. Moreover,

(4.E) $\dim \mathfrak{M}(H^\infty_I) = 2$ and $H^2(\mathfrak{M}(H^\infty_I), \mathbb{Z}) = 0$.

§5. Applications

5.1. In this section, we formulate a corollary to [14, Theorem 1.7], which relies heavily on property (4.E) of $\mathfrak{M}(H^\infty_I)$.

In what follows, for a complex Banach algebra $\mathcal{A}$ with unit 1 we denote by $\mathcal{A}^{-1}$ the group of invertible elements. It is a complex Banach Lie group. By $\mathcal{A}_0^{-1} \subset \mathcal{A}^{-1}$ we denote the connected component containing 1. Also, by $\mathcal{A}_l^{-1} = \{a \in \mathcal{A} : \exists\, b \in \mathcal{A} \text{ such that } ba = 1\}$ we denote the set of left-invertible elements of $\mathcal{A}$ and by $\operatorname{id}\mathcal{A} = \{a \in \mathcal{A} : a^2 = a\}$ the set of idempotents of $\mathcal{A}$.

The group $\mathcal{A}_0^{-1}$ acts on A_l^{-1} by left multiplications:

$$\mathcal{A}_0^{-1} \times \mathcal{A}_l^{-1} \ni (g, a) \mapsto ga \in \mathcal{A}_l^{-1}.$$

The stabilizer of $a \in A_l^{-1}$ under this action is a complex Lie subgroup $\mathcal{A}_0^{-1}(a) \subset \mathcal{A}_0^{-1}$ of elements g satisfying $ga = a$.

In turn, $\mathcal{A}_0^{-1}$ acts on $\operatorname{id}\mathcal{A}$ by similarity transformations:

$$\mathcal{A}_0^{-1} \times \operatorname{id}\mathcal{A} \ni (g, a) \mapsto gag^{-1} \in \operatorname{id}\mathcal{A},$$

and the stabilizer of $a \in \operatorname{id}\mathcal{A}$ under this action is a complex Lie subgroup $\mathcal{A}_0^{-1}(a) \subset \mathcal{A}_0^{-1}$ of elements g satisfying $gag^{-1} = a$.

Let $H^\infty_I \subset H^\infty(\widetilde{R})$ be the subalgebra generated by a proper ideal $I \subset H^\infty(\widetilde{R})$. For a complex Banach algebra A with unit 1, we denote by $C(\mathfrak{M}(H^\infty_I), A)$ the Banach algebra of A-valued continuous functions F on $\mathfrak{M}(H^\infty_I)$ equipped with the norm

$$\|F\| := \max_{\xi \in \mathfrak{M}(H^\infty_I)} \|F(\xi)\|_A.$$

Let $H^\infty_I \widehat{\otimes}_\varepsilon A$ be the injective tensor product of H^∞_I and A. It is naturally identified with the closure in $C(\mathfrak{M}(H^\infty_I), A)$ of the subalgebra of functions of the form

$$\sum_{i=1}^{n} f_i a_i, \quad f_i \in H^\infty_I, \quad a_i \in L(X), \quad n \in \mathbb{N}.$$

Now, using property (4.E), we obtain (see [14, Theorem 1.7] for the proof):

(5.A) let $F \in \operatorname{id}(H^\infty_I \widehat{\otimes}_\varepsilon A)$ be such that for some $\xi \in \mathfrak{M}(H^\infty_I)$ the stabilizer $A_0^{-1}(F(\xi)) \subset A_0^{-1}$ of $F(\xi) \in \operatorname{id} A$ is connected, then there exists $G \in (H^\infty_I \widehat{\otimes}_\varepsilon A)_0^{-1}$ such that $GFG^{-1} = F(\xi)$;

(5.B) let $F \in (H^\infty_I \widehat{\otimes}_\varepsilon A)_l^{-1}$ be such that for some $\xi \in \mathfrak{M}(H^\infty_I)$ the stabilizer $A_0^{-1}(F(\xi)) \subset A_0^{-1}$ of $F(\xi) \in A_l^{-1}$ is connected, then there exist $G \in (H^\infty_I \widehat{\otimes}_\varepsilon A)_0^{-1}$ such that $GF = F(\xi)$.

5.2. Example. Let $L(X)$ be the Banach algebra of bounded linear operators on a complex Banach space X equipped with the operator norm. By $I_X \in L(X)$ we denote the identity operator and by $GL(X) \subset L(X)$ the set of invertible bounded linear operators on X. Clearly, $L(X)^{-1} = GL(X)$. By $GL_0(X) \subset GL(X)$ we denote the connected component of I_X. Each $E \in \operatorname{id} L(X)$ determines a direct sum decomposition $X = X_E^0 \oplus X_E^1$, where $X_E^0 = \ker E$ and $X_E^1 = \ker (I_X - E)$. It is easily seen that the stabilizer $GL_0(X)(E) \subset GL_0(X)$ consists of operators $B \in GL_0(X)$ such that $B(X_E^k) \subset X_E^k$, $k = 0, 1$. In particular, restrictions of operators in $GL_0(X)(E)$ to X_E^k determine a monomorphism of complex Banach Lie groups $S_E \colon GL_0(X)(E) \to GL(X_E^0) \oplus GL(X_E^1)$. Moreover, S_E is an isomorphism if $GL(X_E^k)$, $k = 0, 1$, are connected.

Now, (5.A) leads to the following statement.

(A) Let $F \in \operatorname{id}(H_I^\infty \widehat{\otimes}_\varepsilon L(X))$ be such that for some $\xi \in \mathfrak{M}(H_I^\infty)$ and $E = F(\xi) \in \operatorname{id} L(X)$, the groups $GL(X_E^0)$ and $GL(X_E^1)$ are connected. Then there exists $G \in (H_I^\infty \widehat{\otimes}_\varepsilon L(X))_0^{-1}$ such that $GFG^{-1} = E$.

In particular, this is valid if X is one of the spaces: a finite-dimensional space, a Hilbert space, c_0 or ℓ^p, $1 \leqslant p \leqslant \infty$. Indeed, $GL(X)$ is connected if $\dim_{\mathbb{C}} X < \infty$ and contractible for other spaces of the list, see, e.g., [23] and references therein. Moreover, each subspace of a Hilbert space is Hilbert, and if X is either c_0 or ℓ^p, $1 \leqslant p \leqslant \infty$, then its infinite-dimensional complemented subspaces are isomorphic to X as well, see [26, 22]. This provides the required condition in (A). Note also that there are complex Banach spaces X with disconnected groups $GL(X)$, see, e.g., [17].

Further, each $E \in L(X)_l^{-1}$ determines a complemented subspace $X_1 = \operatorname{ran} E \subset X$ isomorphic to X. Then the stabilizer $GL_0(X)(E) \subset GL_0(X)$ of E consists of operators $B \in GL_0(X)$ such that $B|_{X_1} = I_{X_1}$. If $X_2 \subset X$ is a complement to X_1, then each $B \in GL_0(X)(E)$ has a form

$$B = \begin{pmatrix} I_{X_1} & C \\ 0 & D \end{pmatrix}, \quad \text{where} \quad D \in GL(X_2) \text{ and } C \in L(X_2, X_1); \qquad (4)$$

here $L(X_2, X_1)$ is the Banach space of bounded linear operators $X_2 \to X_1$. Thus $GL_0(X)(E)$ is homotopy equivalent to the subgroup of $GL(X_2)$ (isomorphic to $GL(X/X_1)$) of operators D such that $\operatorname{diag}(I_{X_1}, D) \in GL_0(X)$. In particular, this subgroup coincides with $GL(X_2)$ if the latter is connected.

Now, (5.B) leads to the following statement.

(B) Let $F \in (H_I^\infty \widehat{\otimes}_\varepsilon L(X))_l^{-1}$ be such that for some $\xi \in \mathfrak{M}(H_I^\infty)$ and $E = F(\xi) \in L(X)_l^{-1}$ the corresponding group $GL(X/X_1)$ is connected. Then there exists $G \in (H_I^\infty \widehat{\otimes}_\varepsilon L(X))_0^{-1}$ such that $F = GE$.

We identify X with X_1 by E and regard F as an element of

$H_I^\infty \widehat{\otimes}_\varepsilon L(X_1, X_1 \oplus X_2)$ and E as an operator in $L(X_1, X_1 \oplus X_2)$. Then

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} I_{X_1} \\ 0 \end{pmatrix},$$

where

$$F_i \in H_I^\infty \widehat{\otimes}_\varepsilon L(X_1, X_i) \quad \text{and} \quad G_{ii} \in H_I^\infty \widehat{\otimes}_\varepsilon L(X_i), \quad i = 1, 2,$$

$$G_{ij} \in H_I^\infty \widehat{\otimes}_\varepsilon L(X_j, X_i), \quad i, j \in \{1, 2\}, \quad i \neq j.$$

Now, the equation $F = GE$ implies that

$$F_1 = G_{11}, \quad F_2 = G_{21},$$

i.e., F extends to an invertible element

$$G \in (H_I^\infty \widehat{\otimes}_\varepsilon L(X_1 \oplus X_2))_0^{-1}.$$

Thus, statement (B) and the Bochner–Phillips–Allan–Markus–Sementsul theory, see, e.g., [25, Theorem 9.2.7], yield:

(B′) let $U \subset \mathfrak{M}(H_I^\infty)$ be a dense subset and let $\xi \in U$. Suppose that

$$F \in H_I^\infty \widehat{\otimes}_\varepsilon L(Y_1, Y_2),$$

where Y_i, $i = 1, 2$, are complex Banach spaces, is such that for each $\lambda \in U$ there is a left inverse G_λ of $F(\lambda) \in L(Y_1, Y_2)$ satisfying

$$\sup_{\lambda \in U} \|G_\lambda\|_{L(Y_2, Y_1)} < \infty. \tag{5}$$

Let $Y = \ker G_\xi$ be such that $GL(Y)$ is connected. Then, there exist elements $H \in H_I^\infty \widehat{\otimes}_\varepsilon L(Y_1 \oplus Y, Y_2), G \in H_I^\infty \widehat{\otimes}_\varepsilon L(Y_2, Y_1 \oplus Y)$ such that for all $\nu \in \mathfrak{M}(H_I^\infty)$ we have

$$H(\nu)G(\nu) = I_{Y_2}, \quad G(\nu)H(\nu) = I_{Y_1 \oplus Y} \quad \text{and} \quad H(\nu)|_{Y_1} = F(\nu).$$

Statement (B′) is obtained from (B) if we set $X_1 = Y_1$, $X_2 = Y$, and $X = Y_2$.

Note that $GL(Y)$ is connected in the following cases (see, e.g., [6, Corollary] for the references): (1) $\dim_{\mathbb{C}} Y < \infty$; (2) Y_2 is isomorphic to a Hilbert space or c_0 or one of the spaces ℓ^p, $1 \leqslant p \leqslant \infty$; (3) Y_2 is isomorphic to one of the spaces $L^p[0, 1]$, $1 < p < \infty$, or $C[0, 1]$ and Y_1 is not isomorphic to Y_2.

§6. Interpolating sequences for $H^\infty(\widetilde{R})$

We recall the following definition.

Let X be a complex analytic space such that the algebra of bounded holomorphic functions $H^\infty(X)$ separates points of X. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is

said to be interpolating for $H^\infty(X)$ if for every bounded sequence $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ there is a function $f \in H^\infty(X)$ with

$$f(x_n) = c_n \quad \text{for all} \quad n \in \mathbb{N}. \tag{6}$$

By the Banach open mapping theorem such an $f \in H^\infty(X)$ can be chosen to satisfy

$$\|f\|_{H^\infty(X)} := \sup_X |f| \leqslant L \sup_{n\in\mathbb{N}} |c_n|$$

for some L depending only on $\{x_n\}_{n\in\mathbb{N}}$. The optimal L here is called the constant of interpolation of $\{x_n\}_{n\in\mathbb{N}}$.

Let $r_u\colon \mathbb{D} \to \widetilde{R}$ be the universal covering map. Due to [4, Theorem 1.9], for every $z \in \widetilde{R}$ the sequence $r_u^{-1}(z) \in \mathbb{D}$ is interpolating for $H^\infty(\mathbb{D})$. Thus, we can define a Blaschke product $B_z \in H^\infty(\mathbb{D})$, $z \in \widetilde{R}$, with simple zeros at all points of $r_u^{-1}(z)$. If B_z' is another Blaschke product with the same property, then $B_z' = aB_z$ for some $a \in \mathbb{C}$, $|a| = 1$. In particular, the logarithmically subharmonic function $|B_z|$ is invariant with respect to the action on $\mathbb{D}$ of the deck transformation group $\pi_1(\widetilde{R})$. Hence, there is a nonnegative logarithmically subharmonic function P_z on $\widetilde{R}$ with the only zero at z such that $r_u^* P_z = |B_z|$. It is also clear that $P_z(y) = P_y(z)$ for any $y, z \in \widetilde{R}$, and $\sup_{\widetilde{R}} P_z = 1$. Now, Proposition 1.13 of [4] asserts that

(6.A) a sequence $\{z_i\}_{i\in\mathbb{N}} \subset \widetilde{R}$ is interpolating for $H^\infty(\widetilde{R})$ if and only if

$$\inf_j \left\{ \prod_{k\,:\,k\neq j} P_{z_k}(z_j) \right\} =: \delta > 0.$$

The number δ, the *characteristic* of the interpolating sequence $\{z_i\}_{i=1}^\infty$, plays the same role as its counterpart for $H^\infty(\mathbb{D})$, see, e.g., [18], [4, Subsection 1.3].

Next, as follows from [4, Theorem 1.9],

(6.B) a sequence in $\widetilde{R}$ is interpolating for $H^\infty(\widetilde{R})$ if and only if its preimage under r_u is an interpolating sequence for $H^\infty(\mathbb{D})$.

Now, we present a new result on interpolating sequences for $H^\infty(\widetilde{R})$ that will be used in the proof of Theorem 3.1.

Given $\mathcal{F} = \{f_1, \dots, f_n\} \subset H^\infty(\widetilde{R})$, we set

$$\Pi_\nu(\mathcal{F}) := \Big\{ z \in \widetilde{R} \,:\, \max_{1\leqslant i\leqslant n} |f_i(z)| < \nu \Big\}, \quad \nu > 0. \tag{7}$$

Theorem 6.1. *A sequence $\{z_i\}_{i\in\mathbb{N}} \subset \Pi_\nu(\mathcal{F})$ is interpolating for $H^\infty(\Pi_{\nu'}(\mathcal{F}))$ for some $\nu' > \nu$ if and only if it is interpolating for $H^\infty(\widetilde{R})$.*

§7. Proofs

7.1. Proof of Theorem 6.1.

Proof. If $\zeta \subset \Pi_\nu(\mathcal{F})$ is an interpolating sequence for $H^\infty(\widetilde{R})$, then clearly it is interpolating for $H^\infty(\Pi_{\nu'}(\mathcal{F}))$ for all $\nu' > \nu$. Thus, we must prove the converse statement only.

To this end, due to (6.B) it suffices to prove that if

$$\zeta = \{z_i\}_{i\in\mathbb{N}} \subset \Pi_\nu(\mathcal{F})$$

is an interpolating sequence for $H^\infty(\Pi_{\nu'}(\mathcal{F}))$ for some $\nu' > \nu$, then

$$\omega = \{w_i = r_u^{-1}(z_i)\}_{i\in\mathbb{N}}$$

is an interpolating sequence for $H^\infty(\mathbb{D})$.

By definition, ω is a subset of $\Pi_\nu(\widetilde{\mathcal{F}})$, $\widetilde{\mathcal{F}} = \{r^*f_1, \dots, r^*f_n\} \subset H^\infty(\mathbb{D})$. Since ζ is an interpolating sequence for $H^\infty(\Pi_{\nu'}(\mathcal{F}))$, arguing exactly as in the proof of Theorem 1.9 of [4] we conclude that ω is an interpolating sequence for $H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))$.

Hereafter, for a function $h \in \ell^\infty(\omega)$ its norm is denoted by $\|h\|_\infty$.

Since ω is an interpolating sequence for $H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))$, we get

(7.A) for every $h \in \ell^\infty(\omega)$ there is a function $H \in H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))$ such that $\|H\|_{H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))} \leqslant L\|h\|_\infty$ and $H|_\omega = h$; here L is the constant of interpolation of ω.

We also use the following simple corollary to [12,Theorem 1.10] (with $\mathcal{M}=\mathbb{C}$):

(7.B) given $H \in H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))$, there is $\widetilde{H} \in H^\infty(\mathbb{D})$ of norm at most $C\|H\|_{H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))}$, where C is independent of H, such that

$$\left\|(\widetilde{H} - H)|_{\Pi_\nu(\widetilde{\mathcal{F}})}\right\|_{H^\infty(\Pi_\nu(\widetilde{\mathcal{F}}))} \leqslant \frac{1}{2L}\|H\|_{H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))}.$$

Next, given $g \in \ell^\infty(\omega)$ we define sequences

$$\{g_k\}_{k\in\mathbb{N}} \subset \ell^\infty(\omega), \quad \{G_k\}_{k\in\mathbb{N}} \subset H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}})), \quad \text{and} \quad \{\widetilde{G}_k\}_{k\in\mathbb{N}} \subset H^\infty(\mathbb{D})$$

by the following algorithm:

$g_1 = g$, and for every $k \in \mathbb{N}$: (i) G_k is the function H in (7.A) for $h = g_k$; (ii) $\widetilde{G}_k$ is the function $\widetilde{H}$ in (7.B) for $H = G_k$; (iii) $g_{k+1} = g_k - \widetilde{G}_k|_\omega$.

Due to norm estimates in (7.A) and (7.B), for $k \in \mathbb{N}$ we obtain

$$\|G_k\|_{H^\infty(\Pi_{\nu'}(\widetilde{\mathcal{F}}))} \leqslant L\|g_k\|_\infty, \quad \|\widetilde{G}_k\|_{H^\infty(\mathbb{D})} \leqslant CL\|g_k\|_\infty, \quad \|g_{k+1}\|_\infty \leqslant \frac{1}{2}\|g_k\|_\infty.$$

These imply that the series $\sum_{k=1}^\infty \widetilde{G}_k$ converges uniformly on $\mathbb{D}$ to a function $\widetilde{G} \in H^\infty(\mathbb{D})$ such that $\widetilde{G}|_\omega = g$.

Thus, we have proved that ω is an interpolating sequence for $H^\infty(\mathbb{D})$ and so ζ is an interpolating sequence for $H^\infty(\widetilde{R})$ due to (6.B).

The proof of the theorem is complete. □

7.2. Proof of Theorem 3.1.

Proof. Let $\xi \in \mathfrak{M}_a \cap \widehat{r}^{-1}(\partial_o A_{ij})$, see (3.A). Then there is an interpolating sequence $\{z_k\}_{k\in\mathbb{N}}$ for $H^\infty(\widetilde{A}_{ij})$ whose closure in $\mathfrak{M}(H^\infty(\widetilde{R}))$ contains ξ, see [11, Theorem 2.4]. By definition, $\widehat{r}(\xi) \in \partial_o A_{ij}\,(\subset \partial D_i)$. Let f be a bounded holomorphic function defined in an open neighborhood $O \subsetneq S$ of $\bar{R}$ and having a zero at $\widehat{r}(\xi)$. Then there is some $\varepsilon > 0$ such that

$$\Pi_\varepsilon(\{f\}) := \{z \in R : |f(z)| < \varepsilon\} \subset A_{ij}.$$

Passing to a subsequence of $\{z_k\}_{k\in\mathbb{N}}$ whose closure contains ξ, if necessary, without loss of generality we may assume that

$$\{z_k\}_{k\in\mathbb{N}} \subset \Pi_{\frac{\varepsilon}{2}}\big(\{r^*f\}\big).$$

Moreover, it is an interpolating sequence for $H^\infty(\Pi_\varepsilon(\{r^*f\}))$. Therefore, due to Theorem 6.1, $\{z_k\}_{k\in\mathbb{N}}$ is an interpolating sequence for $H^\infty(\widetilde{R})$. Since $\partial D = \partial_o A_{i1} \cup \partial_o A_{i2}$, the last statement implies that every point of an analytic disk in $\widehat{r}^{-1}(\partial D_i)$ belongs to the closure of an interpolating sequence for $H^\infty(\widetilde{R})$.

Conversely, assume that $\xi \in \widehat{r}^{-1}(\partial D)$ belongs to the closure of an interpolating sequence $\{z_k\}_{k\in\mathbb{N}}$ for $H^\infty(\widetilde{R})$. Then there is a subsequence $\{z_{k_n}\}_{n\in\mathbb{N}}$ of $\{z_k\}_{k\in\mathbb{N}}$ lying in some $\widetilde{A}_{ij}$, $j = 1, 2$, whose closure contains ξ as well. Since clearly $\{z_{k_n}\}_{n\in\mathbb{N}}$ is an interpolating sequence for $H^\infty(\widetilde{A}_{ij})$, by [11, Theorem 2.4] ξ belongs to an analytic disk.

The proof of the theorem is complete. □

7.3. Proof of Theorem 4.1.

Proof. The proof is based on the following claim.

Lemma 7.1. *Let $U \subset \mathfrak{M}(H^\infty(\widetilde{R}))$ be an open set and let $f \in H^\infty(U \cap \widetilde{R})$. Then there exists a unique function $\widehat{f} \in C(U)$ that extends f.*

Proof. Since $\mathfrak{M}(H^\infty(\widetilde{R}))$ is covered by the open set $E(R, \beta F)$ and finitely many open sets biholomorphic to an open subset of $\mathfrak{M}(H^\infty(\mathbb{D}\times F))$, it suffices to prove the result for open subsets U of these two sets. Now, if $U \subset E(R, \beta F)$, the required result follows by [3, Lemma 3.1] and the fact that the base of topology of $E(R, \beta F)$ consists of open sets biholomorphic to $\mathbb{D} \times H$, where $H \subset \beta F$ is clopen. In turn, if $U \subset \mathfrak{M}(H^\infty(\mathbb{D}\times F))$, the required result is given in [11, Lemma 8.1]. □

Using that lemma and the fact that $\mathfrak{M}_s$ is totally disconnected and arguing word-for-word as in the proof of Theorem 2.7 of [11], we complete the proof of the theorem. □

References

[1] Bass H., K-*theory and stable algebra*, Inst. Hautes Etudes Sci. Publ. Math. **22** (1964), 5–60.

[2] Behrens M., *The corona conjecture for a class of infinitely connected domains*, Bull. Amer. Math. Soc. **76** (1970), no. 2, 387–391.

[3] Brudnyi A., *Topology of the maximal ideal space of* H^∞, J. Funct. Anal. **189** (2002), no. 1, 21–52.

[4] Brudnyi A., *Projections in the space* H^∞ *and the corona theorem for subdomains of coverings of finite bordered Riemann surfaces*, Ark. Mat. **42** (2004), 31–59.

[5] Brudnyi A., *Grauert- and Lax-Halmos-type theorems and extension of matrices with entries in* H^∞, J. Funct. Anal. **206** (2004), 87–108.

[6] Brudnyi A., *On the completion problem for algebra* H^∞, J. Funct. Anal. **266** (2014), no. 7, 4293–4313.

[7] Brudnyi A., *Topology of the maximal ideal space of* H^∞ *revisited*, Adv. Math. **299** (2016), 931–939.

[8] Brudnyi A., *Banach-valued holomorphic functions on the maximal ideal space of* H^∞, Invent. Math. **193** (2013), no. 1, 187–227.

[9] Brudnyi A., *Oka principle on the maximal ideal space of* H^∞, Алгебра и анализ **31** (2019), №5, 24–89.

[10] Brudnyi A., *Projective freeness of algebras of bounded holomorphic functions on infinitely connected domains*, Алгебра и анализ **33** (2021), №4, 49–65.

[11] Brudnyi A., *Structure of the maximal ideal space of* H^∞ *on the countable disjoint union of open disks*, Алгебра и анализ **32** (2020), №6, 58–71.

[12] Brudnyi A., *Dense stable rank and Runge type approximation theorems for* H^∞ *maps*, J. Aust. Math. Soc., `https://doi.org/10.1017/S1446788721000045`.

[13] Brudnyi A., *On stable rank of* H^∞ *on coverings of bordered Riemann surfaces*, Rev. Mat. Iberoam., `DOI: 10.4171/RMI/1293`.

[14] Brudnyi A., *On homotopy invariants of tensor products of Banach algebras*, Integral Equations Operator Theory **92** (2020), no. 2, Paper No. 19, 14 pp.

[15] Carleson L., *Interpolations by bounded analytic functions and the corona theorem*, Ann. Math. (2) **76** (1962), 547–559.

[16] Corach G., Suárez F. D., *Extension problems and stable rank in commutative Banach algebras*, Topology Appl. **21** (1985), 1–8.

[17] Douady A., *Un espace de Banach dont le groupe linéaire n'est pas connexe*, Indag. Math. **27** (1965), 787–789.

[18] Garnett J. B., *Bounded analytic functions*, Pure Appl. Math., vol. 96, Acad. Press, New York, 1981.

[19] Hoffman K., *Bounded analytic functions and Gleason parts*, Ann. Math. (2) **86** (1967), 74–111.

[20] Hoffman K., *Analytic functions and logmodular Banach algebras*, Acta Math. **108** (1962), 271–317.

[21] Husemoller D., *Fibre bundles*, Grad. Texts in Math., vol. 20, Springer-Verlag, New York, 1994.
[22] Lindenstrauss J., *On complemented subspaces of m*, Israel J. Math. **5** (1967), 153–156.
[23] Митягин Б. С., *Гомотопическая структура линейной группы банахова пространства*, Успехи мат. наук **25** (1970), №5, 63–106.
[24] Mortini R., Sasane A., Wick B. D., *The Corona theorem and stable rank for the algebra* $\mathbb{C} + BH^\infty$, Houston J. Math. **36** (2010), no. 1, 289–302.
[25] Nikolski N. K., *Operators, functions, and systems: an easy reading*. Vol. 1. *Hardy, Hankel, and Toeplitz*, Math. Surveys Monogr., vol. 92, Amer. Math. Soc., Providence, RI, 2002.
[26] Pełczyński A., *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.
[27] Suárez D., *Čech cohomology and covering dimension for the* H^∞ *maximal ideal space*, J. Funct. Anal. **123** (1994), no. 2, 233–263.
[28] Suárez D., *Trivial Gleason parts and the topological stable rank of* H^∞, Amer. J. Math. **118** (1996), no. 4, 879–904.
[29] Stout E. L., *Bounded holomorphic functions on finite Riemann surfaces*, Trans. Amer. Math. Soc. **120** (1965), 255–285.
[30] Treil S., *The stable rank of* H^∞ *is equal to one*, J. Funct. Anal. **109** (1992), 130–154.

Department of Mathematics
and Statistics
University of Calgary
Calgary, Alberta, Canada
T2N 1N4
E-mail: abrudnyi@ucalgary.ca

Поступило 19 августа 2021 г.