E. Brugallé, G. Mikhalkin, J.-J. Risler, K. Shaw, Nonexistence of torically maximal hypersurfaces, *Алгебра и анализ*, 2018, том 30, выпуск 1, 20–31

Алгебра и анализ
Том 30 (2018), №1

# NONEXISTENCE OF TORICALLY MAXIMAL HYPERSURFACES

© E. BRUGALLÉ, G. MIKHALKIN, J.-J. RISLER, K. SHAW

Torically maximal curves (known also as simple Harnack curves) are real algebraic curves in the projective plane such that their logarithmic Gauss map is totally real. In this paper it is shown that the hyperplanes in projective spaces are the only torically maximal hypersurfaces of higher dimensions.

## §1. Introduction

Torically maximal curves, also known as simple Harnack curves, were introduced and studied in [Mik00]. Since then, they have appeared in several areas of mathematics. Finding their reasonable higher-dimensional counterparts is an open and challenging problem (cf., e.g., [AIM06]). In this paper we explore a direct generalization of toric maximality for projective hypersurfaces proposed in [Mik01, Section 3.4]. We show that when $n \geqslant 3$, the hyperplanes in projective spaces are the only torically maximal hypersurfaces in this sense.

Let $X$ be an algebraic hypersurface of $(\mathbb{C}^*)^n$ defined by the equation

$$P(z_1, \dots, z_n) = 0.$$

We denote by $\Delta(X)$ the Newton polytope of the polynomial $P(z_1, \dots, z_n)$, and by $\overline{X}$ the topological closure of $X$ in the toric variety $\mathrm{Tor}(X)$ defined by $\Delta(X)$. Note that $X \subset (\mathbb{C}^*)^n$ determines $\Delta(X)$ only up to a translation in $\mathbb{Z}^n$, however this does not play a role in what follows.

---

*Key words:* Simple Harnack curves, real algebraic toric hypersurfaces.

Part of this work was done during the research stay of E. B. and J.-J. R. at the Centre Interfacultaire Bernoulli (Lausanne) in the framework of the program "Tropical geometry in its complex and symplectic aspects". The authors are grateful to CIB for the support and excellent working conditions. Research of G. M. was supported in part by the grant TROPGEO of the European Research Council, by the grants 141329 and 159240 of the Swiss National Science Foundation, and by the NCCR SwissMAP of the Swiss National Science Foundation. Research of K. S. was supported by an Alexander von Humboldt Postdoctoral Fellowship.

**Definition 1.1.** We say that a hypersurface $X \subset (\mathbb{C}^*)^n$ is *torically nonsingular* if the polytope $\Delta(X)$ is $n$-dimensional and the intersection of $\overline{X}$ with each torus orbit of $\mathrm{Tor}(X)$ is nonsingular in this orbit. If $X$ is torically nonsingular, then $\overline{X}$ is transverse to all torus orbits of $\mathrm{Tor}(X)$.

We say that $X \subset (\mathbb{C}^*)^n$ is *torically projective* if $\mathrm{Tor}(X) = \mathbb{C}P^n$.

Following [Kap91], we define the *logarithmic Gauss map* of a nonsingular hypersurface $X \subset (\mathbb{C}^*)^n$ as

$$\begin{array}{cccc} \gamma_X : & X & \longrightarrow & \mathbb{C}P^{n-1} \\ & (z_1,\dots,z_n) & \longmapsto & [z_1\frac{\partial P}{\partial z_1}(z_1,\dots,z_n) : \dots : z_n\frac{\partial P}{\partial z_n}(z_1,\dots,z_n)] \end{array} .$$

The map $\gamma_X$ is simply the usual Gauss map after the reparameterisation of $X$ with the help of a local branch of the holomorphic logarithm restricted to $X$; clearly the map $\gamma_X$ does not depend on the chosen branch of the logarithm. In [Mik00, Sec. 3.2] it was proved that when $X$ is torically nonsingular, the map $\gamma_X$ extends to an algebraic map

$$\overline{\gamma}_X : \overline{X} \to \mathbb{C}P^{n-1}$$

such that

$$\deg(\overline{\gamma}_X) = \mathrm{Vol}_n(\Delta(X)), \tag{1}$$

where $\mathrm{Vol}_n$ denotes the lattice volume of an $n$-dimensional polytope (i.e., $n!$ times the Euclidean volume).

Since $\gamma_X$ is a map between two manifolds of the same dimension, the fiber $\gamma_X^{-1}(y)$ is finite for almost all $y$ in $\mathbb{C}P^{n-1}$. Our first result is that $\gamma_X$ is actually finite in the case of torically projective hypersurfaces, i.e., $\gamma_X^{-1}(y)$ is finite for *any* $y \in \mathbb{C}P^{n-1}$.

**Theorem 1.2.** *If $X \subset (\mathbb{C}^*)^n$ is a torically nonsingular projective hypersurface, then the logarithmic Gauss map $\overline{\gamma}_X : \overline{X} \to \mathbb{C}P^{n-1}$ is finite.*

Then we investigate the existence of *torically maximal hypersurfaces*. Given a real algebraic subvariety $X$ of a complex toric variety, we denote by $\mathbb{R}X$ the real part of $X$. We say that a real algebraic map $f : X \to Y$ between two real algebraic varieties is *almost totally real* if $f^{-1}(x) \subset \mathbb{R}X$ for any $x \in \mathbb{R}Y \setminus S$, where $S$ is some subspace of $f(X) \cap \mathbb{R}Y$ of positive codimension. If $S$ is empty, then the map $f$ is said to be *totally real*.

**Definition 1.3.** A torically nonsingular real algebraic hypersurface $X$ of $(\mathbb{C}^*)^n$ is said to be *almost torically maximal* if the map $\gamma_X$ is almost totally real.

A torically nonsingular real algebraic hypersurface $X$ of $(\mathbb{C}^*)^n$ is said to be *torically maximal* if $\overline{X}$ is nonsingular and the map $\overline{\gamma}_X$ is totally real.

**Remark 1.4.** Note that the above definition of (almost) torically maximal is axiomatizing [Mik01, Proposition 26], rather than making use of [Mik01, Definition 10]. In particular, we do not require that $X$ or $\overline{X}$ be maximal in the sense the Smith–Thom inequality (see for example [BR90] for the Smith–Thom inequality).

Any almost torically maximal hypersurface is torically maximal if $n \leqslant 2$. For $n = 1$, the variety $X$ is torically maximal if and only if all roots of $P(z_1)$ in $\mathbb{C}^*$ are simple and real. When $n = 2$, the real curve $X$ is torically maximal if and only if $\mathbb{R}X$ is a simple Harnack curve, see [PR11, Lemma 2.2 and Theorem 3.5]. In [Mik00] it was proved that the topological type of the pair $((\mathbb{R}^*)^2, \mathbb{R}X)$ is uniquely determined by $\Delta(X)$ when $X$ is a simple Harnack curve (see also [Bru15] for an alternative proof).

Since the logarithmic Gauss map of a hyperplane in $\mathbb{C}P^n$ has degree 1, any real hyperplane is almost torically maximal, and hence torically maximal by Theorem 1.2. The next theorem asserts that this is the only possible example of almost torically maximal projective hypersurfaces as soon as $n \geqslant 3$.

**Theorem 1.5.** *Let $n \geqslant 3$, and let $X \subset (\mathbb{C}^*)^n$ be an almost torically maximal projective hypersurface. Then $\overline{X}$ is a hyperplane.*

In the case of torically maximal hypersurfaces, the preceding theorem can be extended to any Newton polytope.

**Theorem 1.6.** *Let $n \geqslant 3$, and let $X \subset (\mathbb{C}^*)^n$ be a torically maximal hypersurface. Then $Tor(X) = \mathbb{C}P^n$ and $\overline{X}$ is a hyperplane.*

**Remark 1.7.** Note that Theorem 1.5 can be deduced as a consequence of Theorem 1.2 and Theorem 1.6. Nevertheless, its direct proof is quite simple, so we prove it independently of Theorem 1.6.

We make some comments about Theorems 1.5 and 1.6 and further generalizations of simple Harnack curves. First, we do not know whether there exist almost torically maximal hypersurfaces $X \subset (\mathbb{C}^*)^n$ that are not torically maximal. However, in § 4 we provide an example of a singular hypersurface for which the logarithmic Gauss map is almost totally real but not totally real. Therefore the smoothness assumption on $\overline{X}$ (which is a part of the definition of toric maximality) is essential in Theorem 1.6.

Next, Theorems 1.5 and 1.6 may be a hint that the direct generalization of toric maximality proposed in [Mik01, Section 3.4] in dimension at least 3 can be weakened. For example, relaxing the smoothness assumption on $X \subset (\mathbb{C}^*)^n$ in Definition 1.3 may produce meaningful objects (see [Lan15] for the case of generalised simple Harnack curves). Additionally, it is worthwhile to consider real subvarieties of higher codimension. There is a natural generalization of the logarithmic Gauss map where the target is now a Grassmannian, and also a generalization of (almost) torically maximal real algebraic varieties of any codimension. Products of torically maximal hypersurfaces give examples of torically maximal subvarieties of codimension greater than 1. So far we do not know of other examples.

**Acknowledgment.** We are grateful to Benoît Bertrand, Christian Haase, Ilia Itenberg, Michael Joswig, Mario Kummer, and Lucia López de Medrano for helpful discussions. We also wish to thank an anonymous referee whose comments helped us to improve our paper.

## §2. Properties of the logarithmic Gauss map

If $X \subset (\mathbb{C}^*)^n$ is a torically nonsingular hypersurface, then $\overline{X} \cap Y$ is by definition also a torically nonsingular hypersurface for any torus orbit $Y$ of $\mathrm{Tor}(X)$. In particular the logarithmic Gauss map $\gamma_{\overline{X}\cap Y}$ is well defined. The following lemma is straightforward.

**Lemma 2.1.** *Let $X \subset (\mathbb{C}^*)^n$ be a torically nonsingular hypersurface. Then for any torus orbit $Y$ of $Tor(X)$, the logarithmic Gauss map of $\overline{X} \cap Y$ coincides with the restriction of $\overline{\gamma}_X$ to $\overline{X} \cap Y$. Furthermore, if the face of $\Delta(X)$ corresponding to $Y$ is parallel to a linear space $L \subset \mathbb{R}^n$, then the image of the restriction of $\overline{\gamma}_X$ to $\overline{X} \cap Y$ lies in the projectivization of $L \otimes \mathbb{C}$ in $\mathbb{C}P^{n-1}$.*

**Proof of Theorem 1.2.** Lemma 2.1 implies that for any $x \in \mathbb{C}P^{n-1}$, the fiber $\overline{\gamma}_X^{-1}(x)$ is disjoint from at least one toric divisor of $\mathbb{C}P^n$, which is a hyperplane. Since any positive-dimensional subvariety of $\mathbb{C}P^n$ intersects any hyperplane, each fiber $\overline{\gamma}_X^{-1}(x)$ must be a finite collection of points. □

**Remark 2.2.** When $X$ is torically nonsingular, but not necessarily projective, the above argument can be used to show that any curve contained in the fiber $\overline{\gamma}_X^{-1}(x)$ must be contained in the closure of a subtorus translate of $(\mathbb{C}^*)^n$.

Theorem 1.2 immediately implies the following.

**Corollary 2.3.** *If $X \subset (\mathbb{C}^*)^n$ is an almost torically maximal projective hypersurface, then $X$ is torically maximal.*

The following theorem about totally real morphisms is used to restrict the topology of $\mathbb{R}\overline{X}$. Note that in [KS15] a totally real morphism was called real fibered.

**Theorem 2.4.** [KS15, Theorem 2.19] *Let $X$ and $Y$ be nonsingular real algebraic varieties of the same dimension, and let $\phi : X \to Y$ be a totally real morphism. Then $d_x\phi : T_x\mathbb{R}X \to T_{\phi(x)}\mathbb{R}Y$ is an isomorphism for all $x \in \mathbb{R}X$.*

We outline the proof of the above theorem for completeness, referring the reader to [KS15] for the details. Since it is a local statement, we may assume that both $X$ and $Y$ are real open neighborhoods of 0 in $\mathbb{C}^n$. First, observe that the statement is true when $X$ and $Y$ are 1-dimensional: if a real map $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is ramified at 0, then $\phi$ is locally given by $z \mapsto z^d$ for $d \geqslant 2$, which is clearly not totally real.

For $n > 1$, if $d_x\phi$ is not injective for some $x \in \mathbb{R}X$, then we choose a real line $L \subset \mathbb{C}^n$ such that $\phi(x) \in L$ and $T_{\phi(x)}\mathbb{R}L \cap d_x\phi(T_x\mathbb{R}X) = \{0\}$. Consider the real algebraic curve $C = \phi^{-1}(L) \subset X$ and its normalization $\pi : \widetilde{C} \to C$. The composition $\phi \circ \pi : \widetilde{C} \to L$ is also a totally real map. Since the theorem is true for maps between curves, this map is unramified over the real locus. However, for any point $\widetilde{x} \in \mathbb{R}\widetilde{C}$ such that $\pi(\widetilde{x}) = x \in \mathbb{R}C \subset \mathbb{R}X$, the differential satisfies $d_{\widetilde{x}}(\phi \circ \pi) = d_x\phi \circ d_{\widetilde{x}}\pi$. Therefore, the image of $d_{\widetilde{x}}(\phi \circ \pi)$ is zero by the assumption that $T_{\phi(x)}\mathbb{R}L \cap d_x\phi(T_x\mathbb{R}X) = \{0\}$. This gives a contradiction and the theorem follows.

From Theorem 2.4 it follows that the logarithmic Gauss map induces a covering map $\mathbb{R}\overline{X} \to \mathbb{R}P^{n-1}$ if $X \subset (\mathbb{C}^*)^n$ is a torically maximal hypersurface. For $n > 1$, there are only two connected coverings of $\mathbb{R}P^n$, namely $\mathbb{R}P^n \to \mathbb{R}P^n$ of degree 1 and $S^n \to \mathbb{R}P^n$ of degree 2. Hence, the degree of the covering map $\mathbb{R}\overline{X} \to \mathbb{R}P^{n-1}$ is determined by the topology of $\mathbb{R}\overline{X}$ when $n \geqslant 3$, and formula (1) implies the following.

**Corollary 2.5.** *Let $X \subset (\mathbb{C}^*)^n$ be a torically maximal hypersurface with $n \geqslant 3$. Then $\mathbb{R}\overline{X}$ is a disjoint union of $k$ connected components homeomorphic to $S^{n-1}$ and $l$ connected components homeomorphic to $\mathbb{R}P^{n-1}$. Furthermore, the integers $k$ and $l$ satisfy*

$$\deg(\gamma) = Vol_n(\Delta(X)) = 2k + l.$$

## §3. Torically maximal hypersurfaces

Let $X \subset (\mathbb{C}^*)^3$ be a torically maximal surface. By Lemma 2.1, for each 2-dimensional torus orbit $Y$ of $\mathrm{Tor}(X)$, the curve $Z = \overline{X} \cap Y$ is a simple Harnack curve. By [Mik00], the intersections of $\overline{Z}$ with the toric boundary divisors of $\mathrm{Tor}(Z)$ are real and contained in a single component of $\mathbb{R}\overline{Z}$. We call this connected component the *outer circle* of the curve $\overline{Z}$ and denote it by $O(Z)$.

**Lemma 3.1.** *Let $X \subset (\mathbb{C}^*)^3$ be a torically maximal surface. Then there exists a connected component of $\mathbb{R}\overline{X}$ containing all outer circles of $\overline{X}$.*

This connected component of $\mathbb{R}\overline{X}$ will be called the *outer component* of $X$.

**Proof.** Each outer circle is an embedded circle contained in some connected component of $\mathbb{R}\overline{X}$. Facets $F$ and $F'$ of $\Delta(X)$ intersect in an edge $E$ if and only if the corresponding outer circles $O(Z)$ and $O(Z')$ intersect transversally at exactly $\mathrm{Length}(E)$ points, where $\mathrm{Length}(E)$ is the lattice length of $E$, i.e.,

$$\mathrm{Length}(E) := |E \cap \mathbb{Z}^3| - 1.$$

In particular, $O(Z)$ and $O(Z')$ are contained in the same connected component of $\mathbb{R}\overline{X}$. Since the facets of $\Delta(X)$ are connected via edges, there exists a single connected component $\mathbb{R}\overline{X}$ containing the outer circles of all boundary curves of $\overline{X}$. □

**Proposition 3.2.** *Let $X \subset (\mathbb{C}^*)^3$ be a torically maximal surface. Then the outer component of $X$ is homeomorphic to $\mathbb{R}P^2$, and $\Delta(X)$ is a tetrahedron with all edges of lattice length* 1.

**Proof.** Let $\mathcal{C}$ denote the outer component of $X$. By Corollary 2.5, it is homeomorphic either to $S^2$ or to $\mathbb{R}P^2$. Suppose $\mathcal{C}$ is homeomorphic to $S^2$. Since any two closed curves in $S^2$ intersecting transversally do so at an even number of points, we deduce that each edge of $\Delta(X)$ has an even lattice length. Since a facet $F$ of $\Delta(X)$ has at least 3 edges, the lattice perimeter of every facet $F$ satisfies

$$\sum_{E \in \mathcal{E}(F)} \mathrm{Length}(E) \geqslant 6,$$

where $\mathcal{E}(F)$ denotes the set of edges of $F$. On the other hand, by [Mik00] we have

$$\deg(\overline{\gamma}_X|_{O(Z)}) = \sum_{E \in \mathcal{E}(\Delta(Z))} \mathrm{Length}(E) - 2 \tag{2}$$

for any outer circle $O(Z)$ of $\overline{X}$. Since the restriction of the logarithmic Gauss map $\overline{\gamma}_X$ to $\mathcal{C}$ has degree 2, identity (2) gives

$$\sum_{E \in \mathcal{E}(\Delta(Z))} \mathrm{Length}(E) - 2 \leqslant 2.$$

Therefore,

$$\sum_{E \in \mathcal{E}(F)} \mathrm{Length}(E) \leqslant 4$$

for any facet $F$ of $\Delta(X)$, which yields a contradiction to the lower bound of the lattice perimeter of $F$ given above.

So, $\mathcal{C}$ is homeomorphic to $\mathbb{R}P^2$ and the restriction of the logarithmic Gauss map $\overline{\gamma}_X|_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}P^2$ is one-to-one. Formula (2) gives

$$\sum_{E \in \mathcal{E}(\Delta(Z))} \mathrm{Length}(E) - 2 = 1,$$

which implies that each facet $F$ of $\Delta(X)$ is a lattice triangle, and that $\text{Length}(E) = 1$ for all edges $E$ of $\Delta(X)$. In particular, each outer circle $O(Z)$ intersects some other outer circle $O(Z')$ transversally at a single point. Hence, each outer circle realizes a nonzero class in $H_1(\mathcal{C}; \mathbb{Z}/2\mathbb{Z})$. But then any two outer circles intersect, that is to say each pair of faces of $\Delta(X)$ must share an edge. This implies that $\Delta(X)$ is a lattice tetrahedron, and the proposition is proved. □

**Proof of Theorem 1.5.** By Corollary 2.3, the hypersurface $X$ is torically maximal. Therefore, the case where $n = 3$ follows immediately from Proposition 3.2. If $X \subset (\mathbb{C}^*)^n$ is torically maximal and projective for $n > 3$, then, by intersecting $\overline{X}$ with a 3-dimensional torus orbit of $\text{Tor}(X)$, we would obtain a torically maximal projective surface, which is a plane by the above. Therefore, $\overline{X}$ must be a hyperplane. □

Recall that, for a given lattice polytope $F$ of dimension $k$ in $\mathbb{R}^n$, its *lattice volume* is defined as

$$\text{Vol}_k(F) = \frac{\text{Vol}_k^E(F)}{\text{Vol}_k^E(\Pi_F)},$$

where $\text{Vol}_k^E$ denotes any Euclidean volume in the affine span $V_F$ of $F$, and $\Pi_F$ is any lattice simplex whose vertices form an affine basis of $V_F \cap \mathbb{Z}^n$. We say that $F$ is unimodular if $\text{Vol}_k(F) = 1$.

An $n$-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$ is said to be *smooth in dimension* 1 if for every 1-dimensional face $E$ of $\Delta$, there exist $n-1$ outward primitive integer normal vectors to the facets adjacent to $E$ that can be completed up to a basis of $\mathbb{Z}^n$. If $\Delta$ is smooth in dimension 1, then the corresponding toric variety has singularities only at 0-dimensional torus orbits. If $X \subset (\mathbb{C})^n$ is a torically maximal hypersurface, then its Newton polytope $\Delta(X)$ is smooth in dimension 1 because $\overline{X}$ is nonsingular.

**Lemma 3.3.** *Let $n \geqslant 3$, and let $\Delta \subset \mathbb{R}^n$ be an $n$-dimensional lattice simplex smooth in dimension* 1 *such that all its facets are unimodular. Then $\Delta$ is unimodular.*

**Proof.** We denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$ and choose a facet $F$ of $\Delta$. Since $F$ is unimodular, it can be assumed, up to an integer affine transformation of $\mathbb{R}^n$, that the vertices of $F$ are $0, e_1, \dots, e_{n-1}$. There is one additional vertex $a$ of $\Delta$ with

$$a = (a_1, \dots, a_{n-1}, v) \in \mathbb{Z}^n.$$

Note that

$$\text{Vol}_n(\Delta) = |\det(e_1, \dots, e_{n-1}, a)| = v.$$

By assumption, the facet of $\Delta$ that is the convex hull of all vertices of $\Delta$ except $e_i$ is also unimodular, so that there is a vector

$$c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$$

satisfying

$$\det(c, e_1, \ldots, \widehat{e_i}, \ldots, e_{n-1}, a) = \pm(vc_i - a_i c_n) = \pm 1.$$

Therefore, the primitive integer normal vectors to this facet are $\pm(a_i e_n - v e_i)$.

The condition that $\Delta$ is smooth in dimension 1 implies that at each edge $E$ of $\Delta$, the primitive integer outward normal vectors of the facets of $\Delta$ adjacent to $E$ form a subset of a basis of $\mathbb{Z}^n$. Applying this condition at the edge $[0, e_1]$ of $\Delta$, we deduce that there must exist

$$c' = (c'_1, \ldots, c'_n) \in \mathbb{Z}^n,$$

such that

$$\det(c', a_2 e_n - v e_2, \ldots, a_{n-1} e_n - v e_{n-1}, e_n) = \pm c'_1 \cdot v^{n-2} = \pm 1.$$

Therefore, $\mathrm{Vol}_n(\Delta) = v = 1$ and $\Delta$ is unimodular, as stated. □

**Remark 3.4.** In dimension 3, there are tetrahedra with unimodular faces that are not unimodular. For example, the convex hull of

$$(0,0,0), (1,0,0), (0,1,0), (1,p,q),$$

for every pair of $p, q$ with $\gcd(p,q) = 1$, has unimodular facets, but the volume of the tetrahedron is $q$. So this polytope is not unimodular for $q > 1$. Notice that it fails to be smooth in dimension 1 along every edge.

**Proof of Theorem 1.6.** The theorem is proved by induction on $n$, starting with $n = 3$ as the base case. Recall that $\Delta(X)$ is smooth in dimension 1 if $X \subset (\mathbb{C})^n$ is a torically maximal hypersurface.

Let $X \subset (\mathbb{C}^*)^3$ be a torically maximal surface. By Corollary 2.5, the real part $\mathbb{R}\overline{X}$ is a disjoint union of $k$ connected components homeomorphic to $S^{n-1}$ and $l$ connected components homeomorphic to $\mathbb{R}P^{n-1}$, and that

$$\mathrm{Vol}_n(\Delta(X)) = 2k + l.$$

By Proposition 3.2, the outer component of $X$ is homeomorphic to $\mathbb{R}P^2$, and $\Delta(X)$ is a tetrahedron with all edge lengths equal to 1. In particular, we have

$$\sum_{E \in \mathcal{E}(\Delta)} \mathrm{Length}(E) = 6.$$

Let $\beta_*(M; \mathbb{Z}/2\mathbb{Z})$ denote the sum of all $\mathbb{Z}/2\mathbb{Z}$ Betti numbers of a manifold $M$. The Smith–Thom inequality states that

$$\beta_*(\mathbb{R}\overline{X}; \mathbb{Z}/2\mathbb{Z}) \leqslant \beta_*(\overline{X}; \mathbb{Z}/2\mathbb{Z}) \tag{3}$$

(see, e.g., [BR90]). The total sums of the Betti numbers for $\mathbb{R}P^2$ and $S^2$ are

$$\beta_*(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = 3 \qquad \text{and} \qquad \beta_*(S^2;\mathbb{Z}/2\mathbb{Z}) = 2,$$

so that

$$\beta_*(\mathbb{R}\overline{X};\mathbb{Z}/2\mathbb{Z}) = 3l + 2k = 2l + \mathrm{Vol}_3(\Delta). \tag{4}$$

Moreover, Khovanskiĭ's formula [Kho78] for the Euler characteristic of the complex hypersurface $\overline{X}$ gives

$$\beta_*(\overline{X};\mathbb{Z}/2\mathbb{Z}) = \mathrm{Vol}_3(\Delta(X)) - \sum_{F\in\mathcal{F}(\Delta(X))} \mathrm{Area}(F) + \sum_{E\in\mathcal{E}(\Delta(X))} \mathrm{Length}(E), \tag{5}$$

where $\mathcal{F}(\Delta(X))$ denotes the set of facets of $\Delta(X)$. Combining (4), (5), and (3), we get

$$2l + \mathrm{Vol}_3(\Delta) \leqslant \mathrm{Vol}_3(\Delta) - \sum_{F\in\mathcal{F}(\Delta)} \mathrm{Area}(F) + \sum_{E\in\mathcal{E}(\Delta)} \mathrm{Length}(E),$$

which further implies that

$$\sum_{F\in\mathcal{F}(\Delta)} \mathrm{Area}(F) \leqslant 4.$$

Therefore, each facet of $\Delta$ is unimodular. Since $\Delta$ is also nonsingular in dimension 1, Lemma 3.3 shows that $\Delta$ is itself unimodular. Hence, $\mathrm{Tor}(X) = \mathbb{C}P^3$ and $\overline{X}$ is a hyperplane.

Now we proceed by induction for $n > 3$. Suppose $X \subset (\mathbb{C}^*)^n$ is a torically maximal hypersurface. By induction, each facet $F$ of $\Delta(X)$ is unimodular, the corresponding toric divisor $T_F$ is $\mathbb{C}P^n$, and $\overline{X} \cap T_F$ is a hyperplane. In particular, the intersection $\mathbb{R}\overline{X} \cap T_F$ is connected for all facets $F$.

Therefore, like in Lemma 3.1, there is a single connected component $\mathcal{C}$ of $\mathbb{R}\overline{X}$ that contains all intersections $\mathbb{R}\overline{X} \cap T_F$ when $F$ runs over all faces of $\Delta(X)$. Let $F$ be a facet of $\Delta(X)$, and let $A$ be a 2-dimensional face of $\Delta(X)$ intersecting $F$ along an edge $E$. Hence, $\mathbb{R}\overline{X} \cap T_F$ and $\mathbb{R}\overline{X} \cap T_A$ intersect transversally, and their intersection is $\mathbb{R}\overline{X} \cap T_E$, which is a single point by the unimodularity of $F$. Hence, the class realized by $\mathbb{R}\overline{X} \cap T_F$ in $H_{n-2}(\mathcal{C};\mathbb{Z}/2\mathbb{Z})$ is nontrivial. In particular,

$$H_{n-2}(\mathcal{C};\mathbb{Z}/2\mathbb{Z}) \neq 0,$$

and $\mathcal{C}$ is homeomorphic to $\mathbb{R}P^{n-1}$. Furthermore, for any two facets $F$ and $F'$ of $\Delta(X)$, the intersection of $\mathbb{R}\overline{X} \cap T_F$ and $\mathbb{R}\overline{X} \cap T_{F'}$ realizes the nonzero homology class in

$$H_{n-3}(\mathcal{C};\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

In particular, the intersection is not empty and has dimension $n-3$. On the other hand, the intersection of $\mathbb{R}\overline{X} \cap T_F$ and $\mathbb{R}\overline{X} \cap T_{F'}$ is of codimension 2 if and only if $F$ and $F'$ intersect in a face of $\Delta(X)$ of codimension 2. This implies that every pair of facets of $\Delta(X)$ must meet in face of codimension 2. Therefore, the polytope $\Delta(X)$ has at most $n+1$ facets, all of which are $(n-1)$-dimensional unimodular lattice simplexes. Since $\Delta(X)$ is also smooth in dimension 1 by assumption, applying Lemma 3.3 completes the proof. □

## §4. A singular torically maximal surface

We end the paper with an example showing that the assumption that any singularities of $\mathrm{Tor}(X)$ are contained in the 0-dimensional torus orbits is essential in Theorem 1.6. Let $\Delta \subset \mathbb{R}^3$ be the simplex with vertices

$$(0,0,0),(1,0,0),(0,1,0),(0,0,2),$$

and let $X \subset (\mathbb{C}^*)^3$ be a nonsingular real algebraic surface with $\Delta(X) = \Delta$. Up to a real toric change of coordinates, the surface $X$ has equation

$$az_3^2 + z_3 + z_2 + z_1 + 1 = 0$$

with $a \in \mathbb{R}^\times$.

The variety $\mathrm{Tor}(X)$ is singular along the orbit $Y$ corresponding to the edge

$$e = [(1,0,0),(0,1,0)].$$

Namely, the surface $\overline{X}$ has an ordinary double point at $p = \overline{X} \cap Y$. The blow-up of $\mathrm{Tor}(X)$ along $Y$ is a nonsingular toric variety $Z$, and the proper transform $\widetilde{X}$ of $\overline{X}$ is nonsingular. Note that $\widetilde{X}$ is simply the blow-up of $\overline{X}$ at the point $p$. We denote by $C$ the corresponding $(-2)$-curve in $\widetilde{X}$. The logarithmic Gauss map

$$\gamma_X : X \to \mathbb{C}P^2$$

extends to a map

$$\widetilde{\gamma}_X : \widetilde{X} \to \mathbb{C}P^2$$

that contracts the exceptional curve $C$ to a point. In particular, the map

$$\widetilde{\gamma}_X : \widetilde{X} \to \mathbb{C}P^2$$

is the composition of the blow-down map with the map $\overline{\gamma}_X$.

**Proposition 4.1.** *The map $\overline{\gamma}_X$ is totally real for $a \in (0, \frac{1}{4})$.*

Note that even if $a \in (0, \frac{1}{4})$, the map $\widetilde{\gamma}_X$ is almost totally real, but not totally real because it contracts the curve $C$ to a point.

**Proof.** We have

$$\gamma_X(z_1, z_2, z_3) = [z_1 : z_2 : 2az_3^2 + z_3].$$

For given $(\gamma_1 : \gamma_2 : \gamma_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, determining the real points in the fiber $\gamma_X^{-1}([\gamma_1 : \gamma_2 : \gamma_3])$ reduces to solving the system

$$(S) \quad \begin{cases} az_3^2 + z_3 + z_2 + z_1 + 1 & = 0 \\ z_1 & = s\gamma_1 \\ z_2 & = s\gamma_2 \\ 2az_3^2 + z_3 & = s\gamma_3 \end{cases}$$

in the variables $z_1, z_2, z_3 \in \mathbb{R}$ and $s \in \mathbb{R}^*$.

Since the triangle with vertices $(0,0,0), (1,0,0), (0,1,0)$ is unimodular, from Lemma 2.1 it follows that $\overline{\gamma}_X^{-1}([\gamma_1 : \gamma_2 : 0])$ contains at least 1 real point. Since it is a degree 2 map, the whole fiber must be contained in $\mathbb{R}\overline{X}$. Similarly, we have

$$\overline{\gamma}_X^{-1}([\gamma_1 : \gamma_2 : \gamma_3]) \subset \mathbb{R}\overline{X}$$

if $2\gamma_1 + 2\gamma_2 + \gamma_3 = 0$.

Assume now that $\gamma_3 = 1$ and $2\gamma_1 + 2\gamma_2 + 1 \neq 0$. Then system $(S)$ reduces to the system

$$\begin{cases} -az_3^2 + s(\gamma_1 + \gamma_2 + 1) + 1 & = 0, \\ 2az_3^2 + z_3 & = s. \end{cases}$$

Substituting $s = 2az_3^2 + z_3$ in the first equation, we obtain

$$a(2\gamma_1 + 2\gamma_2 + 1)z_3^2 + (\gamma_1 + \gamma_2 + 1)z_3 + 1 = 0.$$

This is a degree 2 equation in the variable $z_3$ whose discriminant is

$$(\gamma_1 + \gamma_2 + 1)^2 - 4a(2\gamma_1 + 2\gamma_2 + 1)$$
$$= (\gamma_1 + \gamma_2 + 1)^2 - 8a(\gamma_1 + \gamma_2 + 1) + 4a.$$

The discriminant of the polynomial $P(x) = x^2 - 8ax + 4a$ is

$$16a(4a - 1),$$

so that it is negative if $a \in (0, \frac{1}{4})$. In this case $P(\gamma_1 + \gamma_2 + 1)$ is positive, and $\overline{\gamma}_X^{-1}([\gamma_1 : \gamma_2 : 1])$ is composed of two points in $\mathbb{R}\overline{X}$. □

## References


[AIM06] *Extreme forms of real algebraic varieties*, ARCC. workshop, Amer. Inst. Math., Palo Alto, CA, April 6–9, 2006.

[BR90] Benedetti R., Risler J. J., *Real algebraic and semi-algebraic sets*, Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1990.

[Bru15] Brugallé E., *Pseudoholomorphic simple Harnack curves*, Enseign. Math. **61** (2015), no. 3–4, 483–498.

[Kap91] Kapranov M. M., *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), no. 2, 277–285.

[Kho78] Хованский А. Г., *Многогранники Ньютона и род полных пересечений*, Функц. анал. и его прил. **12** (1978), №1, 51–61.
[KS15] Kummer M., Shamovich E., *Real fibered morphisms and ulrich sheaves*, arXiv:1507.06760, 2015.
[Lan15] Lang L., *A generalization of simple Harnack curves*, arXiv:1504.07256, 2015.
[Mik00] Mikhalkin G., *Real algebraic curves, the moment map and amoebas*, Ann. of Math. (2) **151** (2000), no. 1, 309–326.
[Mik01] Mikhalkin G., *Amoebas of algebraic varieties*, arXiv:math/0108225, 2001.
[PR11] Passare M., Risler J-J., *On the curvature of the real amoeba*, Proc. of the Gökova Geometry-Topology Conference 2010, Int. Press, Somerville, MA, 2011, pp. 129–134.

Поступило 30 января 2017

École Polytechnique,
Centre Mathématiques
Laurent Schwartz,
91 128 Palaiseau Cedex,
France
*E-mail:* erwan.brugalle@math.cnrs.fr

Section de mathématiques,
Université de Genève,
Villa Battelle,
1227 Carouge, Suisse
*E-mail:* grigory.mikhalkin@unige.ch

Université Pierre et Marie Curie,
4 Place Jussieu,
75 005 Paris, France

Max Planck Institute
for Mathematics in the Sciences,
Inselstrasse 22,
04103 Leipzig, Germany
*E-mail:* kshaw@mis.mpg.de