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ASYMPTOTIC VALUES OF FUNCTIONS, ANALYTIC IN PLANAR DOMAINS[1]

Abstract. In [1] W. Gross constructed the example of an entire function of infinite order whose set of asymptotic values is equal to the extended complex plain. We obtain an analog of Gross' result for functions, analytic in planar domains of arbitrary connectivity with isolated boundary fragment.

Key words: *analytic function, asymptotic value, isolated boundary fragment.*

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In 1918 W. Gross constructed the example of an entire function Φ of infinite order whose set of asymptotic values is equal to $\overline{\mathbb{C}}$ (see [1]). This means that for every $a \in \overline{\mathbb{C}}$ there exists an open arc Γ_a with endpoint at infinity such that

$$\lim_{\Gamma_a \ni z \to \infty} \Phi(z) = a.$$

In this note, using Gross' example, we construct a function, analytic in preassigned planar domain, having $\overline{\mathbb{C}}$ as the set of asymptotic values at a given boundary point.

Let D be an arbitrary domain, $D \subset \mathbb{C}$, $z_0 \in \partial D$ is an accessible point (i. e. there exists an open arc $\Gamma \subset D$ with endpoint z_0), f is analytic in D function.

Definition 1. [2], [3, p. 8] *We say that $a \in \overline{\mathbb{C}}$ is an asymptotic value of f at the point z_0 if there exists an open arc $\Gamma_a \subset D$ with endpoint z_0 such that*

$$\lim_{\Gamma_a \ni z \to z_0} f(z) = a.$$

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We shall say that Γ_a is an *asymptotic curve* corresponding to the asymptotic value a. The set of all asymptotic values of a function f at a point z_0 we shall denote by $\mathrm{As}(f, z_0)$.

This definition restricts the choice of point z_0. It is possible to define asymptotic values of function not at all boundary points. In the sequel, we assume that z_0 belongs to an isolated boundary fragment.

Definition 2. [4] *A domain $D \subset \mathbb{C}$ has an isolated boundary fragment if one of the following conditions holds:*

(I) There exists a continuum $K \subset \partial D$, different from a point, and an open set U such that $K \subset U$ and $(\partial D \setminus K) \cap U = \emptyset$.

(II) There exists a Jordan arc $\Gamma \subset \partial D$ with distinct ends ξ, η and an open disk B such that $\xi, \eta \in \partial B$, $\Gamma \setminus \{\xi, \eta\} \subset B$ and $(\partial D \setminus \Gamma) \cap B = \emptyset$.

(III) There exist a point $a \in \partial D$ and an open disc $B(a)$ centered at a such that $(B(a) \setminus \{a\}) \cap \partial D = \emptyset$, i. e. a is an isolated point of the set ∂D.

Using Gross' example we construct a functions $f = \Phi \circ \varphi$, analytic in a domain D with isolated boundary fragment, with condition

$$\mathrm{As}(f, z_0) = \overline{\mathbb{C}}, \quad z_0 \in \partial D.$$

Here Φ is an entire function, φ is either injective analytic function from D into $\mathbb{C}$ or surjective at most 3-valent analytic function from D onto $\mathbb{C}$.

Theorem 1. *Let D be an arbitrary domain with isolated boundary fragment and a point z_0 belong to this fragment. If it is the fragment of type (I) we assume in addition that z_0 is an accessible point and it corresponds to the prime end, which is equal to z_0. Then there exist an injective analytic function $\varphi : D \to \mathbb{C}$ and an entire function Φ such that*

$$\mathrm{As}(\Phi \circ \varphi, z_0) = \overline{\mathbb{C}} \quad \text{and} \quad \overline{\lim_{z \to z_0}} \frac{\ln \ln |\Phi(\varphi(z))|}{\ln |\varphi(z)|} = \infty.$$

Proof. Let Φ be Gross' function. Asymptotic curves Γ_a, corresponding to asymptotic values $a \in \overline{\mathbb{C}}$ of function Φ, are rays, lying in the set

$$\{z \in \mathbb{C} : |\arg z| \le \pi/4 \text{ or } |\pi - \arg z| \le \pi/4\}$$

(see [1]).

Case 1. If z_0 is an isolated boundary fragment of type (III) we take

$$\varphi(z) = \frac{1}{z - z_0}.$$

By γ_a denote a connected subset of $\varphi^{-1}(\Gamma_a \cap \varphi(D))$ with endpoint at z_0. Then

$$\lim_{\gamma_a \ni z \to z_0} \Phi(\varphi(z)) = \lim_{\Gamma_a \ni w \to \infty} \Phi(w) = a \quad \forall a \in \overline{\mathbb{C}} \tag{1}$$

and

$$\overline{\lim_{\gamma_a \ni z \to z_0}} \frac{\ln \ln |\Phi(\varphi(z))|}{\ln |\varphi(z)|} = \overline{\lim_{w \to \infty}} \frac{\ln \ln |\Phi(w)|}{\ln |w|} = \infty. \tag{2}$$

Case 2. Let z_0 belong to an isolated boundary fragment Γ of type (II). Γ can be considered as an open arc. By the Riemann mapping theorem there exists a biholomorphic mapping ψ from D into the upper half-plane P such that

$$\{z \in \mathbb{C} : |z| < \rho, \mathrm{Im} z > 0\} \subset D_1 = \psi(D) \text{ for some } \ \rho > 0,$$

the point $0 \in \partial D_1$ corresponds to the z_0 and some interval $(\alpha, \beta) \ni 0$ corresponds to Γ (see [4] for details). It follows from the Caratheodory theorem that ψ can be extended to homeomorphism from $(D \cup \Gamma)$ onto $(\psi(D) \cup (\alpha, \beta))$.

The univalent in D_1 function

$$w_\pm = \frac{\pm i}{z^2}$$

(the sign will be chosen later) maps D_1 onto a domain D_2, containing the rays

$$\Gamma'_a = \Gamma_a \cap \left\{ w \in \mathbb{C} : |w| \geq \frac{1}{\rho^2} \right\} \quad \forall a \in \overline{\mathbb{C}}.$$

Since Φ is the function of infinite order then there exists a sequence $w_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \frac{\ln \ln |\Phi(w_n)|}{\ln |w_n|} = \infty.$$

We can choose a subsequence w_n (let us save the notation) that doesn't belong to one of the following sets $\{w \in \mathbb{C} : \mathrm{Re}\, w = 0, \mathrm{Im}\, z \geq 0\}$ or $\{w \in \mathbb{C} : \mathrm{Re}\, w = 0, \mathrm{Im}\, z \leq 0\}$. Therefore, we can choose the sign for the function $w_\pm$ such that $w_n \in D_2$.

Let φ be the constructed above univalent function from D onto D_2, $\gamma_a = \varphi^{-1}(\Gamma'_a)$, $z_n = \varphi^{-1}(w_n)$. Then $\mathrm{As}(\Phi \circ \varphi, z_0) = \overline{\mathbb{C}}$ and

$$\lim_{n \to \infty} \frac{\ln \ln |\Phi(\varphi(z_n))|}{\ln |\varphi(z_n)|} = \infty.$$

Case 3. Let z_0 belong to isolated boundary fragment K of type (I). Consider the simply connected domain $D_0 \supset D$, $\partial D_0 = K$. By the Riemann mapping theorem there exists a univalent conformal mapping ψ of the domain D_0 onto the upper half-plane P. In addition, the origin corresponds to the accessible boundary point $z_0 \in \partial D_0$. It follows from the definition of isolated boundary fragment of type (I) that if $D_1 = \psi(D)$, then the set $\partial D_1 \setminus \mathbb{R}$ is a subset of some compact set $C \subset P$. Hence, we can find $\rho > 0$ such that

$$D_1 \supset \{z \in \mathbb{C} : |z| < \rho, \operatorname{Im} z > 0\}.$$

Since the point z_0 is a prime end of the domain D (this means that z_0 is a prime end of D_0), then ψ^{-1} can be continuously extended from D_1 to $D_1 \cup \{0\}$, $\psi^{-1}(0) = z_0$ (see [5, ch. II, §3, p. 41]). Consequently, ψ is a homeomorphism from $D \cup \{z_0\}$ to $D_1 \cup \{0\}$. Now, as in Case 2, we put $\varphi = w_\pm \circ \psi$, choosing an appropriate sign. Then for all $a \in \overline{\mathbb{C}}$ conditions (1) and (2) are fulfilled for φ and $\gamma_a = \varphi^{-1}(\Gamma'_a)$. $\square$

It was shown in [4] that the domain D from Theorem 1 can be mapped onto $\mathbb{C}$ locally biholomorphically and at most 3-valently. The mapping function $F : D \to \mathbb{C}$ is the composition of the univalent function φ from Theorem 1 and an 3-valent function g. Here 3-valence of a function F means that for every $w \in \mathbb{C}$ the equation $F(z) = w$ has at most three solutions and it has exactly three solutions for some $w \in \mathbb{C}$. For the polynomial $Q(z) = z^3 - 3z$ the Riemann surface $Q(\mathbb{C})$ contains all rays $\Gamma'_a = \Gamma_a \cap \{w \in \mathbb{C} : |w| > \rho\}$ for sufficiently large $\rho > 2$, here Γ_a are rays from Gross' example. Thus, it follows from the construction of mapping g that the rays Γ'_a belong to the Riemann surface $F(D)$. Like in the proof of Case 3, Theorem 1 we can show that F is continuous at the prime end z_0 and

$$\lim_{D \ni z \to z_0} F(z) = \infty.$$

Therefore, for every $a \in \overline{\mathbb{C}}$ there exists an open arc $\gamma_a \subset D$, $F(\gamma_a) = \Gamma'_a$ such that

$$\lim_{\gamma_a \ni z \to z_0} \Phi(F(z)) = \lim_{\Gamma_a \ni w \to \infty} \Phi(w) = a$$

and

$$\overline{\lim_{z \to z_0}} \frac{\ln \ln |\Phi(F(z))|}{\ln |F(z)|} = \infty.$$

Consequently, the univalent function from Theorem 1 can be replaced by the at most 3-valent function F (it was proved in [6] that there is no a 2-valent locally biholomorphic mapping of D onto $\mathbb{C}$).

In such way we have obtained

Theorem 2. *Let D and z_0 be as in Theorem 1, $D \neq \mathbb{C} \setminus \{z_0\}$. Then there exist a locally biholomorphic at most 3-valent surjective function $F : D \to \mathbb{C}$ and an entire function Φ of infinite order such that*

$$\mathrm{As}(\Phi \circ F, z_0) = \overline{\mathbb{C}} \quad \textit{and} \quad \overline{\lim_{z \to z_0}} \frac{\ln \ln |\Phi(F(z))|}{\ln |F(z)|} = \infty.$$

References

[1] Gross W. *Eine ganze Funktion für die jede Komplexe Zahl Konvergenzwert ist* // Math. Ann. 1918. V. 79. P. 201–208.

[2] *Encyclopedia of Mathematics*. [S. l.]: Kluwer Academic Publishers, 1987. Vol. 1 (A—B).

[3] Collingwood E. F., Lohwater A. J. *The theory of cluster sets*. Cambridge: Collingwood Lohwater Cambridge University Press, 1966.

[4] Liczberski P., Starkov V. V. *On locally biholomorhic mappings from multi-connected onto simply connected domains* // Ann. Polon. Math. 2005. V. 85. No. 2. P. 135–143.

[5] Goluzin G. M. *Geometric Theory of Functions of a Complex Variable*. Providence, R. I.: American Mathematical Scociety, 1969.

[6] Starkov V. V. *Locally biholomorphic mappings of multiconnected domains* // Sib. Math. J. 2007. V. 48. No. 4. P. 733–739.

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