Алгебра и анализ
Том 34 (2022), №3

To N. K. Nikolski on the occasion of his 80th birthday

SUMMATION METHOD IN OPTIMAL CONTROL PROBLEM WITH DELAY

© P. BARKHAYEV, YU. BELOV, YU. LYUBARSKIĬ

A summation procedure is described for the construction of the optimal solution in the null controllability problem for a differential equation with distributed delay.

§1. Introduction

We consider the control problem for differential equation with distributed delay

$$\dot{x}(t) = x(t-1) + \int_{-1}^{0} x(t+\tau)\varphi(\tau)\,d\tau + u(t), \quad t > 0 \tag{1.1}$$

with the initial conditions

$$x(t) = x_0(t), \quad t \in (-1,0); \quad x(0) = \xi, \quad \xi \in \mathbb{C}, \quad x_0 \in L^2(-1,0). \tag{1.2}$$

The distribution of delay is defined by the function $\varphi \in L^2(-1,0)$ and the control function u is to be chosen to steer the initial condition $(\xi, x_0(t))$ to the desired state.

Let $\mathcal{M} = \mathbb{C} \times L^2(-1,0)$ be the Hilbert space of all initial data

$$\mathcal{M} = \left\{ \vec{x} = \begin{pmatrix} \xi \\ x(t) \end{pmatrix}, \quad \xi \in \mathbb{C}, \quad x_0 \in L^2(-1,0) \right\}$$

Ключевые слова: delay differential equation, null controllability, exponential system, summation method.

The work of P. Barkhayev and Yu. Lyubarskiĭ was supported by Norwegian Research Council project №275113, "COMAN". The work of Yu. Belov was supported by a grant of the Goverment of the Russian Federation for the state support of scientific research, carried under the supervision of leading scientists, agreement 075-15-2021-602.

with the natural inner product:

$$\langle \vec{x}, \vec{y} \rangle_{\mathcal{M}} = \xi\bar{\eta} + \int_{-1}^{0} x(t)\overline{y(t)}\,dt \quad \text{for} \quad \vec{x} = \begin{pmatrix} \xi \\ x(t) \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} \eta \\ y(t) \end{pmatrix}.$$

Equation (1.1) always has a solution $x(t) = x(t; \vec{x}, u)$ and we deal with the *null controlability* problem: given $T > 1$ and $\vec{x} \in \mathcal{M}$, find a control function u that steers the initial data to zero, i.e., $x(t; \vec{x}, u) = 0$, $t \in (T-1, T)$.

It is known (see, e.g., [1]) that such a control function always exists, and can be given by a bounded operator. Actually there exist many such functions. We consider the problem of finding the optimal control. In other words we look for the problem

$$\|u\|_{L^2(0,T)} \to \min, \quad u \in U(\vec{x}_0; T), \quad \vec{x}_0 = \begin{pmatrix} \xi \\ x_0(t) \end{pmatrix}, \tag{1.3}$$

here $U(\vec{x}_0; T)$ stays for all control functions that steer the initial data (1.2) to zero at the moment T.

Problems of this kind have been investigated in various settings by many authors. We refer the reader to [5, 7, 8, 10] to mention a few. We also refer the reader to [5, 7] for a comprehensive survey and description of the current stay of the art.

We combine the semigroup techniques from [6] with the projection techniques, see, e.g., [2] and [9]. These techniques lead one to the study of the eigenfunction expansions for the corresponding nonselfadjoint operator. Since the corresponding systems of eigenfunctions do not form a Riesz basis, the expansions diverge, generally speaking. In [2] the authors proved the pointwise convergence on the inner intervals of the segment $[T-1, T]$, yet the problem of construction of the optimal control on the whole interval remained open. In this article we suggest the reconstruction procedure which converges in the whole $L^2(0, T)$.

We adjust the techniques in [4] for this case, this techniques cannot be applied directly because the generating function does not satisfy the Muckenhoupt condition. Also it is worth to mention that when constructing the optimal control for a single eigenvector we arrived at an already familiar problem, which appeared in the study of the hereditary completeness of sets of exponentials: given a set of exponents $\Lambda \in \mathbb{C}$ such that the corresponding exponentials $\{\exp(i\lambda t)\}$ are incomplete in $L^2(0, T)$ we need to construct a function that is in the span of $\{\exp(i\lambda t)\}$ and is orthogonal to all but one of these exponentials. The existence of such function is, of course straightforward, simply take any function in $L^2(0, T)$ that is orthogonal to all but one functions in $\{\exp(i\lambda t)\}$ and then project this function to the span of this system.

However for the explicit construction of the problem one needs the explicit expression for this projection. Finding this expression is a complicated task, generally speaking. In the present article we present its construction in the simplest model case where $\varphi = 0$. It should be also mentioned that in a very general setting (and from a different perspective) similar functions appeared in [3] where the authors considered a system $\{\exp(i\mu t)\}_{\mu\in M}$ that is complete and minimal in the whole space $L^2(0,T)$. They split the set of exponential into two disjoint sets $M = M_1 \cup M_2$ and studied the functions that are orthogonal to the exponentials $\{\exp(i\mu t)\}_{\mu\in M_1}$ and to the elements of the biorthogonal system $\{\psi_\mu\}_{\mu\in M_2}$.

§2. The homogeneous problem

The corresponding homogeneous problem has the form

$$\dot{x}(t) = x(t-1) + \int_{-1}^{0} x(t+\tau)\varphi(\tau)\,d\tau, \quad t > 0,$$

with the initial conditions

$$\binom{x(0)}{x(\cdot)} = \vec{x}_0 \in \mathcal{M}.$$

This problem admits the unique solution $x(t) = x(t;\vec{x}_0)$ and also generates the natural semigroup

$$S_t = e^{At}\colon \mathcal{M} \to \mathcal{M}, \quad S_t\vec{x}_0 = \binom{x(t)}{x(t+\cdot)}.$$

The infinitesimal generator of the semigroup is defined by the relation

$$A\colon \vec{x}_0 \mapsto \begin{pmatrix} x_0(-1) + \int\limits_{-1}^{0} x_0(\tau)\varphi(\tau)\,d\tau \\ \dot{x}_0(\tau) \end{pmatrix}. \tag{2.1}$$

Its domain is given by

$$\mathcal{D}_A = \Big\{\binom{\xi}{x(t)}, \quad x \in W_1^2(-1,0), \quad \xi = x(0)\Big\}.$$

Let

$$D(z) = -iz + e^{-iz} + \int_{-1}^{0} e^{i\tau z}\varphi(\tau)\,d\tau$$

be the characteristic function of the homogeneous problem. Its indicator diagram is $[0,i]$. We denote the set of all zeros of D by $\Lambda = \{\lambda_n\}$. It can easily be seen that

$$\lambda_n = -\pi/2 + 2\pi n + i\log|n| + o(1), \quad n \in \mathbb{Z}.$$

Here and in what follows we refer the reader to [11] for facts about entire functions. For simplicity, we assume that $\Lambda \subset \mathbb{C}_+$ and that D has no multiple zeros, this may be not true for a finite number of zeros only.

We also have

$$|D(z)| \asymp \begin{cases} |z| & \text{for} \quad z = x + iy \in \mathbb{C}_+, \quad y < \log x, \quad \operatorname{dist}(z, \Lambda) > \epsilon, \\ e^y & \text{for} \quad z = x + iy \in \mathbb{C}_+, \quad y > \log x, \quad \operatorname{dist}(z, \Lambda) > \epsilon. \end{cases} \tag{2.2}$$

The propositions below are straightforward (again we refer to [11] for the proof of completeness and minimality).

Proposition 2.1. *The spectrum of A is simple and coincides with $i\Lambda$. The corresponding eigenvectors are of the form $\vec{e}_\lambda = \binom{1}{\exp(i\lambda\tau)}, \lambda \in \Lambda$.*

Let $\widetilde{\mathcal{E}}_\Lambda = \{\vec{e}_\lambda\}_{\lambda\in\Lambda}$ be the set of all eigenfunctions.

Proposition 2.2. *The system $\widetilde{\mathcal{E}}_\Lambda$ is complete and minimal in $\mathcal{M}$.*

By $\mathcal{X}_\Lambda = \{\vec{x}_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{M}$, $\vec{x}_\lambda = \binom{\xi_\lambda}{x_\lambda(\tau)}$ we denote the system biorthogonal to $\widetilde{\mathcal{E}}_\Lambda$.

We will use the relations

$$e^{A\tau}\vec{e}_\lambda = e^{i\lambda\tau}\vec{e}_\lambda, \quad e^{A^*\tau}\vec{x}_\lambda = e^{-i\bar{\lambda}\tau}\vec{x}_\lambda, \quad \lambda \in \Lambda.$$

Proposition 2.3. *We have the following relations*:

$$\overline{\xi_\lambda} = \frac{-i}{D'(\lambda)}, \tag{2.3}$$

$$\int_{-1}^{0} e^{izt}\overline{x_\lambda(t)}\,dt = \frac{1}{D'(\lambda)} \frac{e^{-iz} + \int_{-1}^{0} e^{iz\tau}\varphi(\tau)\,d\tau + i\lambda}{z - \lambda}. \tag{2.4}$$

Since $D(\lambda) = 0$, one can rewrite (2.4) as

$$\int_{-1}^{0} e^{izt}\overline{x_\lambda(t)}\,dt = \frac{1}{D'(\lambda)}\left[\frac{e^{-izt} - e^{-i\lambda t}}{z - \lambda} + \int_{-1}^{0} \frac{e^{-izt} - e^{-i\lambda t}}{z - \lambda}\varphi(t)\,dt\right]. \tag{2.5}$$

In what follows we use both (2.4) and (2.5).

Even after being normalised, the system $\widetilde{\mathcal{E}}_\Lambda$ does not form a Riesz basis in $\mathcal{M}$. However, for each $\vec{x} \in \mathcal{M}$ we can consider it formal Fourier series

$$\vec{x} \sim \sum_{\lambda\in\Lambda} \langle \vec{x}, \vec{x}_\lambda\rangle \vec{e}_\lambda. \tag{2.6}$$

§3. Structure of the optimal solution

Denote the operator $\mathcal{B}\colon \mathbb{C} \to \mathcal{M}$ by $\mathcal{B}a = \binom{a}{0}$, $a \in \mathbb{C}$. Problem (1.1), (1.2) can now be written as

$$\begin{cases} \dot{\vec{x}}(t) = A\vec{x}(t) + \mathcal{B}u(t), \quad t > 0, \\ \vec{x}(0) = \vec{x}_0. \end{cases} \tag{3.1}$$

Its solution has the form

$$\vec{x}(t) = e^{At}\vec{x}_0 + \int_0^t e^{A(t-\tau)}\mathcal{B}u(\tau)\,d\tau.$$

Fix now $T > 1$. The admissibility condition $u \in U(T; \vec{x}_0)$ reads as

$$e^{AT}\vec{x}_0 = -\int_0^T e^{A(T-\tau)}\mathcal{B}u(\tau)\,d\tau.$$

The set of exponentials $\mathcal{E}_\Lambda = \{e^{i\lambda t}\}$ is incomplete in $L^2(0,T)$. Let $\mathcal{E}_\Lambda(T)$ be the closure of its span in $L^2(0,T)$, and let $\mathcal{E}_\Lambda(T)^\perp = L^2(0,T) \ominus \mathcal{E}_\Lambda(T)$.

The following theorem was proved in [2], see also [9, Chapter 7.4]. We include the proof in order to make the presentation self-contained.

Theorem 3.1. *Let $u_\lambda \in L^2(0,T)$ solve problem (1.3) for $\vec{x}_0 = \vec{e}_\lambda$, and let $v_\lambda(t) = -u_\lambda(T-t)$. Then*

$$v_\lambda \in \mathcal{E}_\Lambda(T),$$

and also

$$\int_0^T e^{i\mu t}\overline{v_\lambda(t)}\,dt = \delta_{\lambda,\mu}, \quad \mu \in \Lambda.$$

Lemma 3.2. *Let $v \in L^2(0,T)$ be such that*

$$0 = \int_0^T e^{At}\mathcal{B}v(t)\,dt. \tag{3.2}$$

Then $v \in \mathcal{E}_\Lambda(T)^\perp$.

Proof. Given $\lambda \in \Lambda$, relation (3.2) yields

$$0 = \int_0^T \langle e^{At}\mathcal{B}v(t), \vec{x}_\lambda\rangle\,dt = \int_0^T \langle \mathcal{B}v(t), e^{A^*t}\vec{x}_\lambda\rangle\,dt = \int_0^T \langle \mathcal{B}v(t), e^{-i\bar{\lambda}t}\vec{x}_\lambda\rangle\,dt.$$

It remains to observe that

$$\mathcal{B}v(t) = \left(\begin{smallmatrix} v(t) \\ 0 \end{smallmatrix}\right), \quad \vec{x}_\lambda = \left(\begin{smallmatrix} \xi_\lambda \\ x_\lambda(t) \end{smallmatrix}\right),$$

therefore $\langle \mathcal{B}v(t), e^{-i\bar{\lambda}t}\vec{x}_\lambda \rangle = \overline{\xi_\lambda} v(t) e^{-i\bar{\lambda}t}$ and $\xi_\lambda \neq 0$. □

Lemma 3.3. *Let $\lambda \in \Lambda$ and $v \in L^2(0,T)$ be such that*

$$e^{AT}\vec{e}_\lambda = \int_0^T e^{At}\mathcal{B}v(t)\,dt.$$

Then

$$\int_0^T v(t)e^{-i\bar{\mu}t}\,dt = \bar{\xi}_\lambda^{-1} e^{i\lambda t}\delta_{\lambda,\mu}, \quad \mu \in \Lambda.$$

Proof. The proof is similar to that of the previous lemma. We write

$$e^{i\lambda t}\delta_{\lambda,\mu} = \left\langle e^{AT}\vec{e}_\lambda, \vec{x}_\mu \right\rangle = \int_0^T \left\langle e^{At}\mathcal{B}v(t), \vec{x}_\mu \right\rangle dt$$

and then apply similar arguments. □

Corollary 3.4. *The optimal control linearly depends on the initial data: if u_1 and u_2 are solutions of problem (1.3) for $\vec{x} = \vec{x}_1, \vec{x}_2$ respectively, then $u_1 + u_2$ solves this problem for $\vec{x} = \vec{x}_1 + \vec{x}_2$.*

Now in order to solve the general problem we need to reconstruct an arbitrary initial data $\vec{x} \in \mathcal{M}$ from its expansion (2.6).

§4. Summation method

4.1. Formulation of the result. In this section we describe a linear summation method for the series (2.6), i.e., we describe matrices $(w_n(\lambda))_{n\in\mathbb{N},\lambda\in\Lambda}$ with the following properties:

$$w_n(\lambda) \to 1 \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad \lambda \in \Lambda; \tag{4.1}$$

$$\#\{n : w_n(\lambda) \neq 0\} < \infty, \quad \text{for each} \quad n \in \mathbb{N}; \tag{4.2}$$

$$S_n\vec{x} \to \vec{x} \quad \text{as} \quad n \to \infty, \quad \text{here} \quad S_n\vec{x} = \sum_{\lambda\in\Lambda} w_n(\lambda)\langle \vec{x}, \vec{x}_\lambda \rangle \vec{e}_\lambda. \tag{4.3}$$

We start with any sequences $\{l_n\}$, $\{R_n\}$ satisfying the conditions

$$l_n, \quad R_n \to \infty \quad \text{as} \quad n \to \infty, \tag{4.4}$$

$$l_n^2/n \to 0, \quad n/R_n \to 0 \quad \text{as} \quad n \to \infty, \tag{4.5}$$

$$e^{-\pi l_n/2} R_n \to 0 \quad \text{as} \quad n \to \infty. \tag{4.6}$$

It suffices, for example, to take $l_n = |n|^{1/4}$, $R_n = n^4$.

Consider the function

$$W_n(z) = e^{-l_n\pi - i l_n \log \frac{z-n}{z+n}}, \quad z \in \mathbb{C}_+.$$

Theorem 4.1. *For $\lambda \in \Lambda$, let*

$$w_n(\lambda) = \begin{cases} W_n(\lambda), & |\lambda| < R_n, \\ 0, & \text{otherwise}, \end{cases} \tag{4.7}$$

and let the operator S_n be defined by

$$S_n \vec{x} = \sum_{\lambda \in \Lambda} w_n(\lambda) \langle \vec{x}, \vec{x}_\lambda \rangle \vec{e}_\lambda. \tag{4.8}$$

Then $S_n \vec{x} \to \vec{x}$ as $n \to \infty$.

4.2. Preliminary estimates.

Lemma 4.2.

$$W_n(z) \rightrightarrows 1 \text{ as } n \to \infty, \text{ on each compact set } K \subset \mathbb{C}_+. \tag{4.9}$$

Lemma 4.3.

$$|W_n(z)| < e^{-\pi l_n/2}, \quad z \in \mathbb{C}_+, |z| = R_n. \tag{4.10}$$

These lemmas are straightforward.

On the real line we have

$$W_n(t) = \begin{cases} e^{i l_n \log \left|\frac{t-n}{t+n}\right|}, & |t| < n, \\ e^{i l_n \log \left|\frac{t-n}{t+n}\right| - l_n \pi}, & |t| > n. \end{cases}$$

Lemma 4.4.

$$\sup_n \left| \int\limits_{1<|t|<n} W_n(t) \frac{dt}{t} \right| < \infty.$$

Lemma 4.5.

$$\sup_n \left| \int\limits_{|t|>n} W_n(t) \frac{dt}{t} \right| < \infty.$$

Proof of Lemma 4.4. We have

$$I_n := \int\limits_{1<|t|<n} W_n(t)\frac{dt}{t} = \left(\int\limits_{1<|t|<n/l_n} + \int\limits_{n/l_n<|t|<n} \right) = I_{n,1} + I_{n,2}.$$

The estimate of $I_{n,1}$ is straightforward. For $|t| < n/l_n$, we have

$$l_n \log\left|\frac{t-n}{t+n}\right| \prec \frac{l_n}{n}|t| \ll 1,$$

Respectively,

$$I_{n,1} = \int\limits_{1<|t|<n/l_n} \left[1 + O\left(\frac{l_n}{n}|t|\right)\right]\frac{dt}{t} = O(1).$$

When estimating $I_{2,n}$, we consider each interval $\pm[n/l_n, n]$ separately. Consider the integral

$$I'_{n,2} = \int\limits_{n/l_n<t<n} e^{il_n \log\left|\frac{t-n}{t+n}\right|}\frac{dt}{t}.$$

By changing variables

$$u = \frac{n-t}{n+t}; \quad \tau = 1-u; \quad v = \log(1-\tau),$$

and extracting the main term on each step, we finally arrive at the integral $\int_{1/l_n}^{\infty} e^{il_n v}\frac{dv}{v}$, which is bounded uniformly in n. □

Proof of Lemma 4.5. Proof goes in the same vein. We write

$$\int\limits_{|t|>n} W_n(t)\frac{dt}{t} = \left(\int\limits_{|t|>nl_n} + \int\limits_{n<|t|<nl_n} \right) W_n(t)\frac{dt}{t}$$

and use similar arguments. □

4.3. Proof of Theorem 4.1.

a. Let an operator S_n be defined by (4.3). From (4.9), it follows that $S_n\vec{x} \to \vec{x}$ as $n \to \infty$ on the dense set of finite linear combinations of $\vec{e}_\lambda$. So it suffices to prove the uniform boundedness of all operators S_n. Consider instead the adjoint operator

$$S^*\vec{y} = \sum_\lambda \overline{w_n(\lambda)}\langle \vec{y}, \vec{e}_\lambda\rangle \vec{x}_\lambda.$$

Actually it is more convenient to deal with the conjugate objects:

$$\overline{S^*\vec{y}} = \sum_\lambda w_n(\lambda)\langle \vec{e}_\lambda, \vec{y}\rangle \overline{\vec{x}_\lambda} =: \vec{s}_n.$$

Put $\vec{s}_n = \binom{\sigma_n}{s_n(t)}$ and $\vec{y} = \binom{\eta}{y(t)}$. Since $\vec{x}_\lambda = \binom{\xi_\lambda}{x_\lambda(t)}$, we have

$$\sigma_n = \sum_\lambda w_n(\lambda)\bar{\xi}_\lambda + \sum_\lambda w_n(\lambda)\bar{\xi}_\lambda \int_{-1}^{0} e^{i\lambda t}\overline{y(t)}\,dt = \sigma_n^{(1)} + \sigma_n(2), \tag{4.11}$$

$$s_n(t) = \sum_\lambda w_n(\lambda)\overline{x_\lambda(t)} + \sum_\lambda w_n(\lambda)\int_{-1}^{0} e^{i\lambda\tau}\overline{y(\tau)}\,d\tau \quad \overline{x_\lambda(t)} = s_n^{(1)}(t) + s_n^{(2)}(t). \tag{4.12}$$

Each of the four summands on the right-hand side of (4.11), (4.12) should be estimates separately.

Consider the contour $\Gamma_n = L_n \cup C_n$. Here $L_n R_n, R_n]$, $C_n = \{R_n e^{i\theta}; \theta \in [0,\pi]\}$. The arcs C_n satisfy the Carleson condition uniformly in n, hence, for any function h in the Hardy space $H^2(\mathbb{C}_+)$, we have

$$\Big(\int_{C_n} |h(\zeta)|^2 |d\zeta|\Big)^{1/2} \leqslant \text{Const}\,\|h\|.$$

We also assume that $\operatorname{dist}(\Lambda, \Gamma_n) > 0$ uniformly in n.

b. Estimate of $\sigma_n^{(1)}$. We use (2.3). Calculation of the residues gives

$$2i\pi\sigma_n^{(1)} = \int_{\Gamma_n} \frac{W_n(\zeta)}{D(\zeta)}d\zeta = \Big(\int_{C_n} + \int_{L_n}\Big)\frac{W_n(\zeta)}{D(\zeta)}d\zeta = I_n + J_n. \tag{4.13}$$

That I_n is bounded (and even tends to zero as $n \to 0$) is very clear, this follows from (4.6), (4.10), and (2.2). In order to see the boundedness of J_n, it suffices to mention that, according to Lemma 4.5, the integral

$$\int_{L_n} \frac{W_n(t)}{it+1}\,dt$$

is bounded uniformly in n. Therefore, it suffices to establish the boundedness of

$$\int_{L_n} W_n(t)\left\{\frac{1}{D(t)} - \frac{1}{it+1}\right\}dt.$$

The latter is evident because the expression in braces is uniformly $O(t^{-2})$ as $t \to \infty$.

c. Estimate of $\sigma_n^{(2)}$ Denote

$$Y(z) = \int_{-1}^{0} e^{iz\tau}\overline{y(\tau)}\,d\tau. \tag{4.14}$$

As before, we have

$$2i\pi\sigma_n^{(2)} = \int_{\Gamma_n} W_n(\zeta)\frac{Y(\zeta)}{D(\zeta)}d\zeta = \Big(\int_{C_n} + \int_{L_n}\Big)W_n(\zeta)\frac{Y(\zeta)}{D(\zeta)}d\zeta = I_n + J_n.$$

The estimate of J_n is now straightforward because $Y \in L^2(\mathbb{R})$ uniformly in y in the unit ball of $L^2(-1,0)$ and $|D(t)| \asymp |t|$ as $t \to \pm\infty$.

In order to estimate I_n, we observe that the function $\Psi(\zeta) = e^{iz}Y(\zeta)$ belongs to the Hardy space $H^2(\mathbb{C}_+)$ and has the same $L^2(\mathbb{R})$ norm as Y. Also, from (2.2) it follows that

$$\Big|\frac{Y(\zeta)}{D(\zeta)}\Big| \prec |\Psi(\zeta)|, \quad \zeta \in C_n.$$

Now we use that all C_n's are uniformly Carleson curves in $\mathbb{C}_+$ and also (4.10). We obtain

$$[J_n| \prec e^{-l_n\pi/2}\int_{C_n}|\Psi(\zeta)||d\zeta| \leqslant e^{-l_n\pi/2}|C_n|^{1/2}\Big(\int_{C_n}|\Psi(\zeta)|^2|d\zeta|\Big)^{1/2} \prec R_n^{1/2}e^{-l_n\pi/2}\|\Psi\|.$$

It remains to refer to (4.6).

d. Estimate of $\|s_n^{(1)}\|$. We apply the Fourier transform and use the representation (2.5):

$$(s_n^{(1)})\widehat{\ }(x) = \sum_{\lambda} w_n(\lambda)\frac{1}{D'(\lambda)}\Big(\frac{e^{-izt} - e^{-i\lambda t}}{z-\lambda} + \int_{-1}^{0}\frac{e^{-izt} - e^{-i\lambda t}}{z-\lambda}\varphi(t)\,dt\Big).$$

Therefore it suffices to obtain estimates of $\|\Phi_{n,t}\|_{L^2(\mathbb{R})}$ uniform in t where

$$\Phi_{n,t}(x) = \sum_{\Lambda} w_n(\lambda)\frac{1}{D'(\lambda)}\frac{e^{-ixt} - e^{-i\lambda t}}{x-\lambda}.$$

Since $\Lambda \subset \mathbb{C}_+$, this function belongs to the Hardy space $H^2(\mathbb{C}_-)$ and

$$\|\Phi_{n,t}\|_{L^2(\mathbb{R})} = \sup\Big\{\Big|\int_{-\infty}^{\infty}\Phi_{n,t}(x)h(x)dx\Big| \;:\; h \in H^2(\mathbb{C}_+), \quad \|h\| = 1\Big\}.$$

We obtain two terms to be estimated:

$$\sum_{\Lambda} \frac{w_n(\lambda)}{D'(\lambda)} e^{i\lambda t} \int_{-\infty}^{\infty} \frac{h(x)}{x-\lambda} dx = 2i\pi \sum_{\Lambda} \frac{w_n(\lambda)}{D'(\lambda)} e^{i\lambda t} h(\lambda) =: a_{1,n},$$

and

$$\sum_{\Lambda} \frac{w_n(\lambda)}{D'(\lambda)} \int_{-\infty}^{\infty} \frac{e^{ixt} h(x)}{x-\lambda} dx =: a_{2,n}. \tag{4.15}$$

Once again we have

$$a_{1,n} = \left(\int_{L_n} + \int_{C_n} \right) W_n(\zeta) \frac{e^{i\zeta t}}{D(\zeta)} h(\zeta) d\zeta.$$

The estimate of the first integral follows bluntly from the Schwarz inequality. In order to estimate the second one we observe that $\left|e^{it\zeta}/D(\zeta)\right| \prec 1$, $\zeta \in C_n$ uniformly in t and n and then again use the fact that all C_n's are uniformly Carlesonian together with (4.6):

$$\left| \int_{C_n} W_n(\zeta) \frac{e^{i\zeta t}}{D(\zeta)} h(\zeta) d\zeta \right| \prec \int_{C_n} |W_n(\zeta)||h(\zeta)||d\zeta| \prec e^{-\pi l_n/2} R_n^{1/2} \|h\| \prec \|h\|.$$

In order to estimate $a_{n,2}$ we represent h as $h(x) = \int_0^\infty e^{i\xi x} \omega(\xi)\, d\xi$ for some $\omega \in L^2(0,\infty)$.

For $t \in [-1, 0]$ we have

$$h(x) = \int_t^0 e^{i\xi x} \omega(\xi - t)\, d\xi + \int_0^\infty e^{i\xi x} \omega(\xi - t)\, d\xi = h_t^-(x) + h_t^+(x). \tag{4.16}$$

Then $h_t^\pm \in H^2(\mathbb{C}_\pm)$, $\|h_t^+\| \leqslant \|h\|$, and

$$a_{2,n} = \sum_{\lambda} w_n(\lambda) \frac{h_t^+(\lambda)}{D'(\lambda)}.$$

The rest of the estimate is similar to that of $a_{n,1}$.

e. Estimate of $\|s_n^{(2)}\|$. Again, we apply the Fourier transform but now use the representation (2.4):

$$\widehat{(s_n^{(2)})}(x) = \sum_{\lambda} w_n(\lambda) \frac{Y(\lambda)}{D'(\lambda)} \frac{e^{-ix} + \int_{-1}^{0} e^{ix\tau} \varphi(\tau)\, d\tau + i\lambda}{x-\lambda},$$

here $Y(\lambda)$ was defined in (4.14).

The duality reasons yield

$$\|\widehat{s_n^{(2)}}\| = \sup\left\{\left|\int\limits_{-\infty}^{\infty} \widehat{s_n^{(2)}}(x)h(x)dx\right|, \quad h \in H^2(\mathbb{C}_+), \quad \|h\| \leqslant 1\right\}.$$

Let the functions h_t be defined as in (4.16). Consider the function

$$f(x) = h_{-1} + \int\limits_{-1}^{0} h_t(x)\varphi(t)\,dt.$$

We have $f \in H^2(\mathbb{C}^+)$, $\|f\| \prec 1$, and

$$\int\limits_{-\infty}^{\infty} \widehat{s_n^{(2)}}(x)h(x)dx = \sum_{\lambda} w_n(\lambda)\frac{1+i\lambda}{D'(\lambda)}Y(\lambda)f(\lambda).$$

This expression can be estimated much as $\|s_{n,1}\|$. □

We have proved the following statement.

Theorem 4.6. *Given the control problem* (1.1), (1.2), *let* $u_0(t)$ *be the optimal solution defined by* (1.3). *Let, further,* Λ *and* $\widetilde{\mathcal{E}}_\Lambda$ *be the corresponding eigenvalues and eigenvectors and also let*

$$\vec{x}_0 \sim \sum_{\lambda\in\Lambda}\langle\vec{x}_0, \vec{x}_\lambda\rangle\vec{e}_\lambda$$

be the formal Fourier expansion of the initial data $\vec{x}_0$.

Then

$$u_0(t) = \lim_{n\to\infty}\sum_{\lambda} w_n(\lambda)\langle\vec{x}_0, \vec{x}_\lambda\rangle u_\lambda(t),$$

here the $w_n(\lambda)$ *are defined in* (4.7) *and the functions* $u_\lambda \in L^2(0,T)$ *are defined by the relations*

$$u_\lambda(t) = -\overline{v_\lambda(T-t)}; \quad v_\lambda \in \mathcal{E}(\Lambda), \quad \int\limits_0^T e^{i\mu t}\overline{v_\lambda(t)}\,dt = \delta_{\lambda,\mu}, \quad \mu \in \Lambda.$$

The limit exists in the $L^2(0,T)$ *sense.*

§5. A model case

a. In this section we consider the simplest model case

$$\dot{x}(t) = x(t-1) + u(t), \quad t > 0; \tag{5.1}$$

$$x(t) = x_0(t), \quad t \in (-1, 0), \quad x(0) = \xi_0. \tag{5.2}$$

For this case we give an explicit construction of the corresponding functions v_λ for $T \in (1,2)$. We denote $\delta = T - 1$. The characteristic function has the form

$$D(z) = -iz + e^{-iz}. \tag{5.3}$$

b. Description of $\mathcal{E}(\Lambda)^\perp$. Let $g \in L^2(0,T)$, $g \in \mathcal{E}(\Lambda)^\perp$. Then the entire function

$$G(z) = \int_0^T e^{izt}\overline{g(t)}\,dt$$

vanishes on Λ and the standard arguments on comparing its growth with that of $D(z)$ yield

$$G(z) = e^{iz}D(z)\Omega(z), \tag{5.4}$$

where Ω runs through the set of all functions of the form

$$\Omega(z) = \int_0^\delta e^{izt}\overline{\omega(t)}\,dt, \quad x\Omega(x) \in L^2(\mathbb{R}). \tag{5.5}$$

The last folmulas yield

$$\omega \in W_1^2(0,\delta), \quad \omega(0) = \omega(\delta) = 0, \tag{5.6}$$

and, by (5.3)–(5.4), we obtain

$$g(t) = \begin{cases} \omega'(t-1), & t \in (1, 1+\delta), \\ \omega(t), & t \in (0,\delta), \\ 0, & \text{otherwise.} \end{cases}$$

This is the description of $\mathcal{E}(\Lambda)^\perp$.

c. Description of the biorthogonal system. Fix $\lambda \in \Lambda$ and let a function $v_\lambda \in L^2(0,T)$ satisfy

$$\int_0^T e^{i\mu t}\overline{v_\lambda(t)}\,dt = \delta_{\lambda,\mu}, \quad \mu \in \Lambda.$$

Denote

$$V_\lambda(z) = \int_0^T e^{izt}\overline{v_\lambda(t)}\,dt.$$

As before we have

$$V_\lambda(z) = \frac{1}{e^{i\lambda}D'(\lambda)}\frac{e^{iz}D(z)}{z-\lambda}P_\lambda(z),$$

where the function

$$P_\lambda(z) = \int_0^\delta e^{izt}\overline{p_\lambda(t)}\,dt, \quad p_\lambda \in L^2(0,\delta)$$

satisfies

$$P_\lambda(\lambda) = \int_0^\delta e^{i\lambda t}\overline{p_\lambda(t)}\,dt = 1. \tag{5.7}$$

We remind the reader that $D(\lambda) = 0$. Therefore

$$\frac{e^{ix}D(x)}{x-\lambda} = -ie^{ix} - i\frac{e^{i(x-\lambda)}-1}{x-\lambda} = -i\left[e^{ix} + \int_0^1 e^{ixt}e^{-i\lambda t}\,dt\right]$$

and, denoting $a_\lambda = 1/(e^{i\lambda}D'(\lambda)$, we obtain

$$v_\lambda(t) = -ia_\lambda\left[\overline{p_\lambda(t-1)} + \int_{t-1}^t e^{-i\lambda(t-u)}\overline{p_\lambda(u)}du\right].$$

When p_λ runs through $L^2(0,\delta)$, the function v_λ runs through all functions in $L^2(0,T)$ orthogonal to $\mathcal{E}_\Lambda \setminus \{e^{i\lambda t}\}$. The condition $v_\lambda \in \mathcal{E}_\lambda^\perp$ leads one to a condition on p_λ.

d. We assume that the functions ω, p_λ are defined on the whole $\mathbb{R}$ and vanish outside of $(0,\delta)$. The condition $v_\lambda \in \mathcal{E}_\Lambda$ takes the form

$$0 = \int_0^T [\omega(t) + \omega'(t-1)]\left[\overline{p_\lambda(t-1)} + e^{-i\lambda t}\int_{t-1}^t e^{-i\lambda\tau}\overline{p_\lambda(\tau)}\,d\tau\right]dt, \tag{5.8}$$

for all $\omega \in W_1^2(0,\delta)$, $\omega(0) = \omega(\delta) = 0$.

When opening the parenthesis, we obtain four summands A_i, $i = 1,2,3,4$. Consider each of them separately.

$$A_1 = \int_0^T \omega(t)\overline{p_\lambda(t-1)}\,dt = 0,$$

because the supports of $\omega(t)$ and $p_\lambda(t-1)$ are disjoint.

Denote $\varepsilon_\lambda(t) = e^{-i\lambda t}\chi_{(0,\infty)}(t)$. Then

$$A_2 = \int_0^T \omega(t)\int_{t-1}^t \overline{p_\lambda(\tau)}e^{-i\lambda(t-\tau)}\,d\tau = \int_0^\delta \omega(t)(\overline{p_\lambda} * \varepsilon_\lambda)(t)\,dt;$$

$$A_3 = \int_0^T \omega'(t-1)\overline{p_\lambda(t-1)}\,dt = -\int_0^T \omega(t-1)\overline{p'_\lambda(t-1)}\,dt.$$

Here we assume that p'_λ exists. With this assumption we will arrive to equation with respect to p'_λ which has smooth solution. This suffices, because we already know that there is only one p_λ satisfying all conditions.

When considering A_4 we use (5.7). We have

$$\begin{aligned}
A_4 &= \int_0^T \omega'(t-1)e^{-i\lambda t}\int_{t-1}^{t} \overline{p_\lambda(\tau)}e^{i\lambda\tau}\,d\tau\,dt \\
&= \int_1^T \omega'(t-1)e^{-i\lambda t}\int_{t-1}^{\delta} \overline{p_\lambda(\tau)}e^{i\lambda\tau}\,d\tau\,dt \\
&= e^{-i\lambda}\int_0^\delta \omega'(t)e^{-i\lambda t}\left(1 - \int_0^t \overline{p_\lambda(\tau)}e^{i\lambda\tau}\,d\tau\right)dt \\
&= e^{-i\lambda}\left[i\lambda\int_0^\delta \omega(t)e^{-i\lambda t}\,dt - \int_0^\delta \omega(t)(\overline{p_\lambda} * \varepsilon_\lambda)'(t)\,dt\right].
\end{aligned}$$

We collect these relations together and remark that ω can be any function in $W_1^2(0,\delta)$ satisfying (5.6). We then arrive at the equation

$$-\overline{p_\lambda}'(t) - e^{-i\lambda}(\overline{p_\lambda} * \varepsilon_\lambda)'(t) + (\overline{p_\lambda} * \varepsilon_\lambda)(t) - e^{-i\lambda}\varepsilon'_\lambda(t) = 0, \quad 0 < t < \delta. \quad (5.9)$$

The left-hand side of this equation is well defined for all $t > 0$. Denote it by $\alpha(t)$. Let $\mathcal{P}_\lambda(s)$ and $e^{-\delta s}\mathcal{A}(s)$ be the Laplace transform of $\overline{p_\lambda}$ and α respectively. Taking once again into account that $D(\lambda) = 0$, we transform (5.9) into an equation for $\mathcal{P}_\lambda(s)$:

$$\mathcal{P}_\lambda(s)(1-s^2) = \left(\lambda^2 - i\lambda\overline{p_\lambda(0)}\right) + s\overline{p_\lambda(0)} + e^{-\delta s}\mathcal{A}(s).$$

The last summand on the right-hand side does not influence the values of $\overline{p_\lambda}$ on $(0,\delta)$. Therefore

$$\overline{p_\lambda}(t) = \left[\lambda^2 - i\lambda\overline{p_\lambda}(0)\right]\sinh t + \overline{p_\lambda}(0)\cosh t, \quad t \in (0,\delta).$$

The value $\overline{p_\lambda}(0)$ should now be chosen to meet (5.7). We omit the corresponding calculation.

References

[1] Banks H. T., Jacobs M. Q., Langenhop C. E., *Characterization of the controlled states in $W(1)2$ of linear hereditary systems*, SIAM J. Control **13** (1975), 611–649.

[2] Banks H. T., Manitius A., *Projection series for retarded functional differential equations with applications to optimal control problems* J. Differential Equations **18** (1975), no. 2, 296–332.

[3] Baranov A., Belov Yu., Borichev A., *Hereditary completeness for systems of exponentials and reproducing kernels*, Adv. Math. **235** (2013), 525–554.

[4] Belov Yu., Lyubarskiĭ Yu., *On summation of nonharmonic Fourier series*, Constr. Approx. **43** (2016), no. 2, 291–309.

[5] Boccia A., Vinter R. B., *The maximum principle for optimal control problems with time delays*, SIAM J. Control Optim. **55** (2017), no. 5, 2905–2935.

[6] Burns J. A., Herdman T. L., Stech H. W., *Linear functional-differential equations as semigroups on product spaces*, SIAM J. Math. Anal. **14** (1983), no. 1, 98–116.

[7] Delfour M. C., *The linear-quadratic optimal control problem with delays in state and control variables: a state space approach*, SIAM J. Control Optim. **24** (1986), no. 5, 835–883.

[8] Halanay A., *Optimal controls for systems with time lag*, SIAM J. Control **6** (1968), 215–234.

[9] Hale J. K., Verduyn Lunel S. M. *Introduction to functional-differential equations*, Appl. Math. Sci., vol. 99, Springer-Verlag, New York, 1993.

[10] Lee C. H., Yung S. P., *Sufficient conditions for optimal control problems with time delay*, J. Optim. Theory Appl. **88** (1996), no. 1, 157–176.

[11] Левин Б. Я., *Целые функции. Курс лекций.*, МГУ, М., 1971.

Поступило 15 декабря 2021 г.

Department of Mathematical Sciences
Norwegian University
of Science and Technology,
7491, Trondheim, Norway;
Institute for Low Temperature
Physics and Engineering,
47 Nauki ave, 61103
Kharkiv, Ukraine
E-mail: pbarhaev@gmail.com

St. Petersburg State University,
St. Petersburg, Russia
E-mail: j_b_juri_belov@mail.ru

Department of Mathematical Sciences,
Norwegian University
of Science and Technology,
NO-7491 Trondheim, Norway
E-mail: yuralyu@gmail.com