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Second Order Krotov Method for Discrete-Continuous Systems

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The paper is dedicated to the blessed memory of our teachers, professors V. F. Krotov and V. I. Gurman

Abstract. In the late 1960s and early 1970s, a new class of problems appeared in the theory of optimal control. It was determined that the structure of a number of systems or processes is not homogeneous and can change over time. Therefore, new mathematical models of heterogeneous structure have been developed.

Research methods for this type of system vary widely, reflecting various scientific schools and thought. One of the proposed options was to develop an approach that retains the traditional assumptions of optimal control theory. Its basis is Krotov's sufficient optimality conditions for discrete systems, formulated in terms of arbitrary sets and mappings.

One of the classes of heterogeneous systems is considered in this paper: discrete-continuous systems (DCSs). DCSs are used for case where all the homogeneous subsystems of the lower level are not only connected by a common functional but also have their own goals.

In this paper a generalization of Krotov's sufficient optimality conditions is applied. The foundational theory is the Krotov method of global improvement, which was originally proposed for discrete processes. The advantage of the proposed method is that its conjugate system of vector-matrix equations is linear; hence, its solution always exists, which allows us to find the desired solution in the optimal control problem for DCSs.

Keywords: discrete-continuous systems, sufficient optimality conditions, control improvement method.

1. Introduction

Scientific developments and discoveries lead to both new technologies and modifications of old ones. Traditionally, mathematical models have been used to solve optimization problems. Often, these models do not fully reflect the investigated processes and require completion. For example, the problem of extending an old investment policy to a new period may require additional research on a separate mathematical model, not just a change in some of the parameters. In such situations, the mathematical model used becomes two-level, and, therefore, a new class of optimization problems appears.

Such systems with a heterogeneous structure are widespread in practice and have different names. These include discrete-continuous, logical-dynamic, impulse, hybrid, and a number of other systems [1; 3; 7; 11; 14]. Further examples are given in [2; 4]. Such systems continue to attract the attention of researchers in various scientific areas, which has been reflected in the subject matter of scientific conferences in recent years.

The approach proposed in [3], based on the interpretation of an abstract model of a multistep controlled process [8] as a discrete-continuous system (DCS), made it possible to construct a two-level model by decomposing an inhomogeneous system into homogeneous subsystems. And then, based on a generalization of the known optimality conditions, it was possible to construct optimization algorithms similar to those developed for homogeneous systems. Here, by homogeneous systems, we mean systems with an unchanged structure that are studied in the classical theory of optimal control. All homogeneous subsystems in such a model are connected by a common goal, the role of which is played by the functional. This does not exclude the fact that each homogeneous subsystem can have its own goal. For such a case, when intermediate criteria for homogeneous lower-level models are available, a generalization of the previously obtained sufficient optimality conditions is given in [13]. They are presented in this paper for a better understanding, and on their basis, a second-order method for control improvement is constructed. It can be considered as a development of the Krotov method of global improvement [10], proposed initially for ordinary discrete processes. The theorem on the improvability of the initial approximation is formulated and proved here.

The advantage of the proposed method is that its conjugate system of vector-matrix equations is linear; therefore, its solution always exists. It does not contain the Riccati matrix equation, as in [12; 15]. The proposed method may not have a solution and will require the development of an additional procedure to eliminate the problem. To demonstrate the operability of the method, an illustrative example is considered.

2. Discrete-Continuous System Model

Let us consider abstract controlled system [8], all of its objects of arbitrary nature (possibly different):

$$x(k+1) = f(k, x(k), u(k)), \quad k \in \mathbf{K} = \{k_I, k_I + 1, ..., k_F\}. \tag{2.1}$$

where k is the number of the step (stage), x and u are respectively variables of state and control, f is the operator, $\mathbf{U}(k, x)$ is the set given for each k and x, k_I, k_F are the initial and final steps, respectively.

On some subset $\mathbf{K}' \subset \mathbf{K}$, $k_F \notin \mathbf{K}'$, a continuous low-level system operates in the role of a control component

$$\dot{x}^c = \frac{dx^c}{dt} = f^c\left(z, t, x^c, u^c\right), \quad t \in \mathbf{T}\left(z\right) = [t_I(z), t_F(z)], \tag{2.2}$$

$$x^c(k,t) \in \mathbf{X}^c(z,t) \subset \mathbb{R}^{n(k)},\ u^c(k,t) \in \mathbf{U}^c\left(z,t,x^c\right) \subset \mathbb{R}^{p(k)},\ z = \left(k,\ x,\ u^d\right).$$

for the system (2.2) an intermediate goal is defined on the interval $[t_I(z), t_F(z)]$ in the form of a functional:

$$I^k = \int\limits_{\mathbf{T}(z(k))} f^k(t, x^c(k,t), u^c(k,t))dt \to \inf.$$

For each $k \in \mathbf{K}'$ the right-hand side operator (2.1) is the following $f\left(k, x(k), u(k)\right) = \theta\left(z, \gamma^c\right)$, where

$$\gamma^c = (t_I,\ x_I^c,\ t_F,\ x_F^c) \in \mathbf{\Gamma}^c(z),$$

$$\mathbf{\Gamma}^c(z) = \{\gamma^c\colon t_I = \tau(z),\ x_I^c = \xi(z),\ (t_F, x_F^c) \in \mathbf{\Gamma}_F^c(z)\}.$$

Here, $z = \left(k,\ x,\ u^d\right)$ is a set of upper-level variables (parameters at the lower level), u^d is a control variable of arbitrary nature, $t_I = \tau(z),\ x_I^c = \xi(z)$ are given functions of z.

The solution of this two-level system is the set $m = (x(k),\ u(k))$ (called a *discrete-continuous process*), where for $k \in \mathbf{K}'$:

$$u(k) = \left(u^d(k),\ m^c(k)\right), m^c(k) \in \mathbf{D}^c\left(z(k)\right).$$

For the element m $m^c(k)$ is a continuous process $(x^c(k,t),\ u^c(k,t))$, $t \in \mathbf{T}(z(k))$, and $\mathbf{D}^c(z)$ is the set of admissible processes m^c, complying with the differential system (2.2) with additional restrictions for piecewise continuous $u^c(k,t)$ and piecewise smooth $x^c(k,t)$ (at each discrete step k). It is assumed that the functions f^k have all properties required for the existence of the functionals I^k. Let us denote the set of elements m satisfying all the above conditions by $\mathbf{D}$ and call it a set of admissible discrete-continuous processes.

For the model (2.1), (2.2) we consider the problem of finding the minimum on $\mathbf{D}$ of the functional $I = F(x(k_F))$ for fixed initial and final steps $k_I = 0, \quad k_F = K, \quad x(k_I)$ and additional constraints

$$x(k) \in \mathbf{X}(k), \quad x^c \in \mathbf{X}^c(z,t), \tag{2.3}$$

where $\mathbf{X}(k),\ \mathbf{X}^c(z,t)$ are given sets.

Note that the construction of a discrete top-level model that connects homogeneous continuous systems operating at different time intervals is a kind of heuristic method and reflects the researcher's views on the problem under consideration. The model may not be the only one possible. The researcher has decided what information about the end of a stage should be transmitted to the upper level and what control actions the upper level passes to the lower level. There are no publications about the choice of a single top-level model.

The term DCS (or discrete-continuous process) was proposed in [3], when research on such systems was just beginning. This name is also used by other authors, for example, in the works of B. M. Miller and E. Ya. Rubinovich. The more common term is hybrid systems, especially abroad.

DCSs with intermediate criteria are characteristic of astronautics, chemical production, and economics. So, when traveling from one planet to another at different stages of movement, different systems of equations and different types of engines are used. For each stage, the task is to minimize fuel consumption. But in general, a soft fit is required. Other examples can be found in the works of A.S. Bortakovsky [2] and V.I. Gurman [4].

3. Optimality and Improvement Sufficient Conditions

The sufficient optimality conditions for this model were obtained in [13] and are as follows.

Theorem 1. *[13]. Let there be a sequence of discrete-continuous processes $\{m_s\} \subset \mathbf{D}$ and functionals $\varphi,\ \varphi^c$ such that:*

1) $\mu^c(z,t)$ is piecewise continuous for each z;

2) $R(k, x_s(k), u_s(k)) \to \mu(k),\ k \in \mathbf{K}$;

3) $\int_{\mathbf{T}(z_s)} (R^c(z_s, t, x_s^c(t), u_s^c(t^c)) - \mu^c(z_s,t))\,dt \to 0,\ k \in \mathbf{K}',\ t \in \mathbf{T}(z_s)$;

4) $G^c(z_s, \gamma_s^c) - l^c(z_s) \to 0,\ k \in \mathbf{K}'$;

5) $G(x_s(t_F)) \to l$.

Then the sequence $\{m_s\}$ is a minimizing sequence for I on $\mathbf{D}$.

The basic constructions of the theorem 1 1, representing a generalization of the constructions of sufficient Krotovs optimality conditions for homogeneous continuous and discrete systems [9], take the form:

$$G(x) = F(x) + \varphi(k_F, x) - \varphi(k_I, x(k_I)),$$

$$R\left(k,x,u\right)=\varphi\left(k+1,f\left(k,x,u\right)\right)-\varphi\left(k,x\right),$$
$$G^{c}\left(z,\gamma^{c}\right)=-\varphi\left(k+1,\theta\left(z,\gamma^{c}\right)\right)+\varphi\left(k,x\right)+$$
$$+\varphi^{c}\left(z,t_F,x_F^{c}\right)-\varphi^{c}\left(z,t_I,x_I^{c}\right),$$
$$R^{c}\left(z,t,x^{c},u^{c}\right)=\varphi_{x^{c}}^{c\mathrm{T}}f^{c}\left(z,t,x^{c},u^{c}\right)-f^{k}\left(z,t,x^{c},u^{c}\right)+\varphi_t^{c}\left(z,t,x^{c}\right).$$
$$\mu^{c}\left(z,t\right)=\sup\left\{R^{c}\left(z,t,x^{c},u^{c}\right):x^{c}\in\mathbf{X}^{c}(z,t),u^{c}\in\mathbf{U}^{c}\left(z,t,x^{c}\right)\right\},$$
$$l^{c}\left(z\right)=\inf\left\{G^{c}\left(z,\gamma^{c}\right):\ \gamma^{c}\in\mathbf{\Gamma}(z),\ x^{c}\in\mathbf{X}^{c}(z,t_F)\right\},$$
$$\mu\left(k\right)=\begin{cases}\sup\{R\left(k,x,u\right):x\in\mathbf{X}(k),u\in\mathbf{U}\left(k,x\right)\}, & t\in\mathbf{K}\backslash\mathbf{K}',\\ -\inf\{l^{c}\left(z\right):\ x\in\mathbf{X}\left(k\right),\ u^{d}\in\mathbf{U}^{d}\left(k,x\right)\}, & k\in\mathbf{K}',\end{cases}$$
$$l=\inf\{G\left(x\right):\ x\in\mathbf{\Gamma}\cap\mathbf{X}\left(K\right)\},$$
$$L=G\left(x\left(k_F\right)\right)-\sum_{\mathbf{K}\backslash\mathbf{K}'\backslash k_F}R(k,x(k),u(k))+$$
$$+\sum_{\mathbf{K}'}\Big(G^{c}(z(k),\gamma^{c}(z(k)))-\int\limits_{\mathbf{T}(z(k))}R^{c}(z(k),t,x^{c}(k,t),u^{c}(k,t))dt\Big),$$

where φ,φ^{c} are the Krotov functions for the upper and lower levels respectively , $\varphi_{x^{c}}^{c}$ is the gradient of φ^{c}, T is the transposition sign.

We note that $L=I$ on $\mathbf{D}$. This reflects the principle of the extension [9] and is one of the foundations for constructing the method.

Theorem 2. *[13]. For any element $m\in\mathbf{D}$ and any φ,φ^{c} the estimate is*

$$I(m)-\inf_{\mathbf{D}}I\leq\Delta=I(m)-l.$$

Let there be two processes $m^{\mathrm{I}}\in\mathbf{D}$ and $m^{\mathrm{II}}\in\mathbf{E}$ and functionals φ and φ^{c}, such that $L\left(m^{\mathrm{II}}\right)<L\left(m^{\mathrm{I}}\right)=I\left(m^{\mathrm{I}}\right)$, and $m^{\mathrm{II}}\in\mathbf{D}$.

Then $I(m^{\mathrm{II}})<I(m^{\mathrm{I}})$.

4. Krotov Method

Suppose that $k_I,x_I,K,t_I\left(k\right),t_F\left(k\right)$ are fixed, $\mathbf{X}\left(k\right)=R^{m}\left(k\right)$, $\mathbf{X}^{c}\left(k,t\right)=R^{n}\left(k\right)$, $\mathbf{\Gamma}(z)=R^{2m}$, $\mathbf{\Gamma}^{c}(z)=R^{2n}\left(k\right)$, $x_I^{c}\left(k\right)=\xi\left(k,x\left(k\right)\right)$, there are no constraints for state variables of both levels and upper-level control variables, lowerlevel subsystems do not depend on u^{d}, and the used constructions of sufficient optimality conditions are such that all the following operations are valid.

We will also assume that solutions of homogeneous systems (2.2) exist for each $k\in\mathbf{K}'$. The case of non-existence requires a change in the model and is not considered.

When constructing methods the problem of improving the element is used, which consists, essentially, in constructing some operator $\omega : \mathbf{D} \to \mathbf{D}$, such that $I(\omega(m)) \leq I(m)$ [5]. The problem of improving is following: we have an element $m^{\mathrm{I}} \in \mathbf{D}$ and we need to find an element $m^{\mathrm{II}} \in \mathbf{D}$ such that $I\left(m^{\mathrm{I}}\right) \geq I\left(m^{\mathrm{II}}\right)$.

We will lead search for an element m^{II} and corresponding functions $\varphi^{\mathrm{I}}\left(k, x\left(k\right)\right), \varphi^{c\mathrm{I}}\left(z, t, x^{c}\right)$ from the fulfillment of the conditions:

$$R\left(k, x(k), u^{I}(k)\right) \to \min_{x}, \tag{4.1}$$

$$G\left(x\right) \to \max, \tag{4.2}$$

$$R^{c}\left(z, t, x^{c}\left(k, t\right), u^{c\mathrm{I}}\left(k, t\right)\right) \to \min_{x^{c}}, \tag{4.3}$$

$$R^{c}\left(z, t, x^{c}\left(k, t\right), u^{c\mathrm{I}}\left(k, t\right)\right) \to \min_{x}, \tag{4.4}$$

$$G^{c}\left(z, x^{c}_{F}, x^{c}_{I}\right) \to \max_{x}. \tag{4.5}$$

Let

$$\tilde{u}\left(k, x\right) = \arg \max_{u \in \mathbf{U}(k,x)} R\left(k, x\left(k\right), u\left(k\right)\right), \tag{4.6}$$

$$\tilde{u}^{c}\left(z, t, x^{c}\right) = \arg \max_{u^{c} \in \mathbf{U}^{c}(z,t,x^{c})} R^{c}\left(z, t, x^{c}, u^{c}\right). \tag{4.7}$$

Then, from the given discrete-continuous system and the initial conditions for the obtained controls, the functions $x^{\mathrm{II}}(k)$, $x^{c\mathrm{II}}(k,t)$ and control programs are obtained:

$$u^{\mathrm{II}}\left(k\right) = \tilde{u}\left(k, x^{\mathrm{II}}(k)\right), u^{c\mathrm{II}}\left(k, t\right) = \tilde{u}^{c}\left(k, t, x^{\mathrm{II}}(k), x^{c\mathrm{II}}(k,t)\right),$$

i.e. an element m^{II}, such that $I\left(m^{\mathrm{II}}\right) \leq I\left(m^{\mathrm{I}}\right)$. Repeating iteratively these operations, we obtain an improving sequence $\{m_s\}$.

In this case the following theorem is valid.

Theorem 3. *If the element m^{I} is not a solution to the problem, then the inequality $L\left(m^{\mathrm{I}}\right) > L\left(m^{\mathrm{II}}\right)$ is valid.*

Proof. Let us show that $I\left(m^{\mathrm{II}}\right) - I\left(m^{\mathrm{I}}\right) = L\left(m^{\mathrm{II}}\right) - L\left(m^{\mathrm{I}}\right) < 0$, following the paper [6]. We obtain

$$L\left(m^{\mathrm{II}}, \varphi^{\mathrm{I}}, \varphi^{c\mathrm{I}}\right) - L\left(m^{\mathrm{I}}, \varphi^{\mathrm{I}}, \varphi^{c\mathrm{I}}\right) =$$

$$= G\left(x^{\mathrm{II}}\right) - G\left(x^{\mathrm{I}}\right) - \sum_{\mathbf{K} \setminus \mathbf{K}' \setminus k_F} (R(k, x^{\mathrm{II}}\left(k\right), u^{\mathrm{II}}\left(k\right), \varphi^{\mathrm{I}}) -$$

$$-R(k, x^{\mathrm{I}}(k), u^{\mathrm{I}}(k), \varphi^{\mathrm{I}})) + \sum_{\mathbf{K}'} \left(G^{c}\left(z^{\mathrm{II}}\left(k\right), \varphi^{\mathrm{I}}, \varphi^{c\mathrm{I}}\right)\right) -$$

$$-G^c(z^{\mathrm{I}}(k),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}})) - \int\limits_{\mathbf{T}(z)} (R^c(z^{\mathrm{II}}(k),t,x^{c\mathrm{II}}(t),u^{c\mathrm{II}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}) -$$

$$-R^c(z^{\mathrm{I}}(k),t,x^{c\mathrm{I}}(t),u^{c\mathrm{I}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}))dt = \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4,$$

where $\Delta_1 = G\left(x^{\mathrm{II}}\right) - G\left(x^{\mathrm{I}}\right) < 0$ by condition (4.2).

$$\Delta_2 = \sum_{\mathbf{K}\setminus\mathbf{K}'\setminus k_F} (R(k,x^{\mathrm{II}}(k),u^{\mathrm{II}}(k),\varphi^{\mathrm{I}}) - R(k,x^{\mathrm{I}}(k),u^{\mathrm{I}}(k),\varphi^{\mathrm{I}}))) =$$

$$= \sum_{\mathbf{K}\setminus\mathbf{K}'\setminus k_F} (R(k,x^{\mathrm{II}}(k),u^{\mathrm{II}}(k),\varphi^{\mathrm{I}}) - R(k,x^{\mathrm{II}}(k),u^{\mathrm{I}}(k),\varphi^{\mathrm{I}}))) +$$

$$+ \sum_{\mathbf{K}\setminus\mathbf{K}'\setminus k_F} (R(k,x^{\mathrm{II}}(k),u^{\mathrm{I}}(k),\varphi^{\mathrm{I}}) - R(k,x^{\mathrm{I}}(k),u^{\mathrm{I}}(k),\varphi^{\mathrm{I}}))) > 0$$

according to (4.1). Then we obtain

$$\Delta_3 = \sum_{\mathbf{K}'} \left(G^c\left(z^{\mathrm{II}}(k),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}\right) - G^c\left(z^{\mathrm{I}}(k),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}\right)\right) < 0,$$

and

$$\Delta_4 =$$

$$= \int\limits_{\mathbf{T}(z)} (R^c(z^{\mathrm{II}}(k),t,x^{c\mathrm{II}}(t),u^{c\mathrm{II}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}) -$$

$$- R^c(z^{\mathrm{I}}(k),t,x^{c\mathrm{I}}(t),u^{c\mathrm{I}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}))dt =$$

$$= \int\limits_{\mathbf{T}(z)} (R^c(z^{\mathrm{II}}(k),t,x^{c\mathrm{II}}(t),u^{c\mathrm{II}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}) -$$

$$- R^c(z^{\mathrm{II}}(k),t,x^{c\mathrm{II}}(t),u^{c\mathrm{I}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}))dt +$$

$$+ \int\limits_{\mathbf{T}(z)} (R^c(z^{\mathrm{II}}(k),t,x^{c\mathrm{II}}(t),u^{c\mathrm{I}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}) -$$

$$- R^c(z^{\mathrm{I}}(k),t,x^{c\mathrm{I}}(t),u^{c\mathrm{I}}(t),\varphi^{\mathrm{I}},\varphi^{c\mathrm{I}}))dt > 0$$

according to the conditions (4.3), (4.4) and (4.5).

Then

$$L\left(m^{\mathrm{II}}\right) - L\left(m^{\mathrm{I}}\right) = \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 < 0.$$

□

From the theorem 3 it follows that if the above conditions are satisfied, we can construct such an improving sequence m_s, that $I\left(m_{s+1}\right) \le I\left(m_s\right)$.

Let us consider the method of finding functions φ, φ^c. We are using principle of expansion [9] and the theorem 2. Conditions (4.1)—(4.5) mean, that the functional L, calculated for controls $u^I(k), u^{cI}(k,t)$, is investigated to the maximum. We consider the increment of the functional $L = I$, which we represent in the form:

$$\delta L \approx dG + \frac{1}{2}d^2G - \sum_{\mathbf{K}\setminus\mathbf{K}'\setminus k_F}\left(dR + \frac{1}{2}d^2R\right) +$$

$$+ \sum_{\mathbf{K}'\setminus k_F}\left(dG^c + \frac{1}{2}d^2G^c \ - \int\limits_{\mathbf{T}(z)}\left(dR^c + \frac{1}{2}d^2R^c\right)dt\right)$$

or

$$\Delta L \approx G_x^T\Delta x + \frac{1}{2}\Delta x^T G_{xx}\Delta x - \sum_{\mathbf{K}\setminus\mathbf{K}'\setminus k_F}\left(R_x^T\Delta x + \frac{1}{2}\Delta x^T R_{xx}\Delta x\right) +$$

$$+ \sum_{\mathbf{K}'\setminus k_F}(\ G^{cT}_{x^c{}_F}\Delta x^c_F +$$

$$+ \frac{1}{2}\Delta x^c_{F^T} G^c_{x^c_F x^c_F} + G^{cT}_x\Delta x + \frac{1}{2}\Delta x^T G^c_{xx}\Delta x + \Delta x^{c_F T} G^{c x^c_F x}\Delta x\ ) -$$

$$- \int\limits_{\mathbf{T}(\,z\,)}(\ R^{cT}_{x^c}\Delta x^c + R^{cT}_x\Delta x +$$

$$+ \frac{1}{2}\Delta x^{cT} R^c_{x^c x^c}\Delta x^c + \Delta x^T R^c_{xx^c}\Delta x^c + \frac{1}{2}\Delta x^T R^c_{xx}\Delta x\ )\,dt.$$

Here, the first and second derivatives of the functions R, G, R^c, G^c are calculated for $u = u^{\mathrm{I}}, u^c = u^{c\mathrm{I}}$, a $\Delta x = x - x^{\mathrm{I}}(k), \Delta x^c = x^c - x^{c\mathrm{I}}(k,t)$, $\Delta x^c_F = x^c_F - x^{c\mathrm{I}}_F$.

To fulfill the conditions (4.1)–(4.5) just assume:

$$G_x = 0,\ R_x = 0,\ R_{x^c} = 0,\ G^c_{x^c_F} = 0,\ G^c_x = 0,\ R^c_x = 0. \tag{4.8}$$

$$G_{xx} = -\Lambda^1,\ R_{xx} = \Lambda^2,\ R^c_{x^c x^c} = \Lambda^3,\ R^c_{xx^c} = 0,\ G^c_{x^c_F x^c_F} = -\Lambda^4, \tag{4.9}$$

$$G^c_{xx^c_F} = 0,\ G^c_{xx} = -\Lambda^5,\ R^c_{xx} = -\Lambda^6, \tag{4.10}$$

where $-\Lambda^1, \Lambda^2, \Lambda^3, -\Lambda^4, -\Lambda^5, -\Lambda^6$ are positive definite diagonal matrices. It is easy to see, that conditions (4.8)–(4.10) are sufficient conditions for the extrema of the functions G, G^c, R, R^c. We supplement the conditions (4.8) - (4.10) with the conditions of the first and second orders of level joining:

$$\frac{d}{dx}(\varphi(k+1, \theta(k, x, x^c_F, x^c_I)) - \varphi^c(k, x.t_F, x^c_F)) = 0, \tag{4.11}$$

$$\frac{d^2}{d^2x}(\varphi(k+1,\theta(k,x,x_F^c,x_I^c))-\varphi^c(k,x.t_F,x_F^c))=\Lambda^7, \tag{4.12}$$

Λ^7 is a positive definite diagonal matrix

We define the functions φ and φ^c in the following form:

$$\varphi\left(k,x\right)=\psi^T\left(k\right)x+\frac{1}{2}\Delta x^T\sigma\left(k\right)\Delta x,\ \varphi^c\left(z,t,x^c\right)=\lambda^T\left(k,t\right)x+\psi^{cT}(k,t)x^c+$$

$$+\frac{1}{2}\Delta x^{cT}\sigma^c\left(k,t\right)\Delta x^c+\frac{1}{2}\Delta x^T\sigma^d\left(k,t\right)\Delta x+\Delta x^T\Lambda\left(k,t\right)\Delta x^c,$$

where $\psi\left(k\right),\lambda\left(k,t\right),\psi^c\left(k,t\right)$ are vector functions of dimensions $m,n,n,$ and $\sigma^c\left(k,t\right),\ \sigma\left(k,t\right),\ \Lambda\left(k,t\right)$ are matrices of dimensions $n\times n,\ m\times m,$ $m\times n$, respectively. Then $\varphi_x=\psi\left(k\right),\ \varphi_{xx}=\sigma\left(k\right),\ \varphi_x^c=\lambda\left(k,t\right),\ \varphi_{x^c}^c=$ $\psi^c\left(k,t\right),\ \varphi_{xx}^c=\sigma^d\left(k,t\right),\ \varphi_{x^cx^c}^c=\sigma^c\left(k,t\right),\ \varphi_{x^cx}^c=\Lambda\left(k,t\right)$. In addition, we introduce the functions

$$H\left(k,x(k),\psi(k+1),u(k)\right)=\psi^T(k+1)f\left(k,x(k),u(k)\right),k\in\mathbf{K}\backslash\mathbf{K}'\backslash k_F,$$

$$H\left(k,x(k),\psi(k+1),x(k_I),x(k_F)\right)=\psi^T(k+1)\theta\left(k,x(k),x(k_I),x(k_F)\right),k\in\mathbf{K}',$$

$$\begin{aligned}&H^c\left(k,x(k),\psi^c(k,t),x^c(k,t),u^c(k,t)\right)=\\&\quad=\psi^{c\mathrm{T}}(k,t)f^c\left(k,x(k),x^c(k,t),u^c(k,t)\right)-f^k(t,x^c(k,t),u^c(k,t)).\end{aligned}$$

Taking into account the introduced notation from the conditions (4.8)–(4.10) and conditions of levels joining (4.11)–(4.12) we obtain:

$$\psi\left(k_F\right)=-F_x,\ \psi\left(k\right)=H_x,k\in\mathbf{K}\backslash\mathbf{K}'\backslash k_F,$$

$$\psi\left(k\right)=H_x+(H_{x^c}\xi_x)^T+\lambda(k,t_F)-\lambda(k,t_I)+\xi_x^T\psi^c(t_I),k\in\mathbf{K}'\backslash k_F,$$

$$\dot\lambda=-H_x^c,\ \lambda\left(k,t_F\right)=H_x+\xi_x^TH_{x_I^c},\dot\psi^c=-H_{x^c}^c,\ \psi^c(k,t_F)=H_{x_F^c},$$

$$\sigma^c\left(k,t_F\right)=\theta_{x^c}^T(t_F)\sigma\left(k+1\right)\theta_{x^c}(t_F)+H_{x^cx^c}+\Lambda^4,$$

$$\dot\sigma^c=-\sigma^cf_{x^c}^c-f_{x^c}^{c\mathrm{T}}\sigma^c-H_{x^cx^c}^c+\Lambda^3,\ \sigma^c(t_F)=\theta_{x_F^c}^T\sigma(k+1)\theta_{x_F^c},$$

$$\dot\sigma^d=-\frac{1}{2}\Lambda f_{x^cx}^c-\frac{1}{2}(\Lambda f_{x^cx}^c)^T-\frac{1}{2}\Lambda f_x^c-\frac{1}{2}(\Lambda f_x^c)^T-H_{xx}^c+\Lambda^5,$$

$$\sigma^d(k,t_F)=\theta_x^T\sigma\left(k+1\right)\theta_x+H_x+H_{xx_I^c}\xi_x+$$

$$+\theta_x^T\sigma(k+1)\theta_{x^c}\xi_x+\xi_x^TH_{x_I^cx}\xi_x+\xi_{xx}H_{x_I^c},$$

$$\dot\Lambda=-\frac{1}{2}\Lambda f_{x^c}^c-\frac{1}{2}(\Lambda f_{x^c}^c)^T-\frac{1}{2}f_x^{cT}\sigma^c-\frac{1}{2}\sigma^cf_x^c-H_{xx^c}^c,$$

$$\Lambda\left(k,t_F\right)=\theta_x^T\sigma\left(k+1\right)\theta_{x^c}(t_F)+H_{xx^c}$$

$$\sigma\left(k\right)=f_x^T\sigma\left(k+1\right)f_x+H_{xx}+\Lambda^2,k\in\mathbf{K}\backslash\mathbf{K}'\backslash k_F,$$

$$\sigma(k) = H_{xx} + H_{xx^c}(t_I)\xi_x + \theta_x^T \sigma(k+1)\theta_x + \xi_x^T \sigma^c(t_I)\xi_x +$$
$$+ H^c_{x^c_I}(t_I)\xi_{xx} + \xi_x^T \theta_{x^c}^T(t_I)\sigma(k+1)\theta_{x^c}(t_I)\xi_x + \xi_x^T H_{x^c x^c}(t_I)\xi_x +$$
$$+ H^c \xi_{xx}\theta(t_I) + \Lambda^5,\ k \in \mathbf{K}'\backslash k_F,\ \sigma(k_F) = -F_{xx} + \Lambda^1.$$

5. The Algorithm of the Method

1. We define arbitrary functions $\varphi^{\mathrm{I}}(k, x(k))$ and $\varphi^{c\mathrm{I}}(z, t, x^c)$.
2. We calculate controls $\tilde{u}(k, x), \tilde{u^c}(k, t, x^c)$ using (4.6), (4.7).
3. We define trajectories $x^{\mathrm{I}}, x^{c\mathrm{I}}$ and controls $u^{\mathrm{I}}(k), u^{c\mathrm{I}}(k, t)$ (element m^{I}) using equations of a discrete-continuous process (2.1), (2.2). We calculate the value of the functional I^{I}.
4. We resolve the DCS from right to left with respect to functions ψ, ψ^c, λ and matrices σ, σ^c, σ^d, Λ. For definiteness, we can put all the λ^i_{jj} are equal to constant $\delta^i \leq 0, i = 2, 3, 5, \delta^i \geq 0, i = 1, 4$. Then we define new functions $\varphi^{\mathrm{II}}(k, x(k))$ and $\varphi^{c\mathrm{II}}(z, t, x^c)$. We back to the step 2.

Remark 1. If the functional has not improved, then the values λ^i, playing the role of regulators of the proximity of neighboring approximations must be increased.

Remark 2. The system of vector-matrix equations with respect to vector functions ψ, ψ^c, λ and matrices $\sigma, \sigma^c, \sigma^d, \Lambda$ is linear and therefore always has a solution.

Example 1. The following 2-stage problem is considered.

1st stage $t \in [0, 1] : \dot{x}^{c1} = (x^{c2})^2,\ \ \dot{x}^{c2} = u^{c1},\ \ |u^{c1}| \leq 1,$

$$x^{c1}(0) = 0, x^{c2}(0) = 1, f^1 = x^{c1}.$$

2nd stage $t \in [1, 2] : \dot{x}^{c1} = u^{c2} - (x^{c1})^2,\ \ |u^{c2}| \leq 2,\ I = x^{c1}(2) \to \inf.$

Let us consider this system as discrete-continuous. We obtain $k = 0, 1, 2, 3$. Since the role of the connecting variable in the two stages under consideration is played by x^{c1}, it is easy to write the upper level process in terms of this variable. We establish the correlation between the variables. At the beginning of the process $k = 0, t = 0, x(0) = x^{c1}(0) = 0$. Further, $x(1) = x^{c1}(0, 1), x^{c1}(1, 1) = x(1)$. Then $I = x(2)$. The last instant 3rd stage plays the role of a transmitter of information about the end of the whole process. Then $I = x^1(3) = x^1(2)$.

So, the DCS model has the form:

$$k = 0,\ \dot{x}^{c1}(0, t) = (x^{c2}(0, t))^2,\ \ \dot{x}^{c2}(0, t) = u^{c1}(0, t),\ \ |u^{c1}(0, t)| \leq 1,$$

$$x^{c1}(0, 0) = 0, x^{c2}(0, 0) = 1, f^1(0, t) = x^{c1}(0, t), x(0) = x^{c1}(0, 0) = 0, t \in [0, 1].$$

$$k=1,\ \dot{x}^{c1}(1,t)=u^{c2}(1,t)-(x^{c1}(1,t))^2,\quad |u^{c2}(1,t)|\le 2,$$
$$x(1)=x^{c1}(0,1),\ t\in[1,2],$$
$$k=2,\ x(2)=x^{c1}(1,2),\quad k=3,\quad I=x^1(3)=x^1(2).$$

Obviously, the set $\mathbf{K}'=\{1,2\}$ and functions $\theta(1)=x^{c1}(0,1),\xi(1)=x(1)$, $\theta(2)=x^{c1}(1,2),\xi(2)=x(2)$. We obtain the necessary constructions:

$$R^c(0,t,x^1,x^2,x^{c1},x^{c2},u^{c1})=\varphi^c_{x^{c1}}(x^{c2})^2+\varphi^c_{x^{c2}}u^{c1}-x^{c1}+\varphi^c_t,$$

$$R^c(1,t,x^1,x^{c1},u^{c2})=\varphi^c_{x^{c1}}(u^{c2}-(x^{c1})^2)+\varphi^c_t.$$

It is easy to see, that

$$\tilde{u}^{c1}=\mathrm{sign}\varphi^c_{x^{c2}},\ \tilde{u}^{c2}=2\mathrm{sign}\varphi^c_{x^{c1}},$$

$$H^c(0,t,\psi^{c1},\psi^{c2},x^{c1},x^{c2},u^{c1})=\psi^{c1}(x^{c2})^2+\psi^{c2}u^{c1}-x^{c1},$$
$$H^c(1,t,\psi^{c1},x^{c1},u^{c2})=\psi^{c1}(u^{c2}-(x^{c1})^2),$$
$$H(1,x,\psi(2))=\psi(2)x^{c1}(0,1),H(2,x,\psi(3))=\psi(3)x^{c1}(1,2).$$

Since the lower level process is independent of x, then $\lambda=0,\ \sigma^d=0,\ \Lambda=0$. Let $\sigma_{12}=\sigma_{21}=\sigma_{22}=0$ and define the functions φ and φ^c in the following form:

$$\varphi=\psi(1)x+\frac{1}{2}\sigma(1)(x-(x)^{\mathrm{I}})^2,k=0,$$

$$\varphi=\psi(2)x+\frac{1}{2}\sigma(2)(x-(x)^{\mathrm{I}})^2,k=1,$$

$$\varphi^c(0,t)=\psi^{c1}x^{c1}+\psi^{c2}x^{c2}+\frac{1}{2}\sigma^c_{11}(x^{c1}-(x^{c1})^{\mathrm{I}})^2,k=0,$$

$$\varphi^c(1,t)=\psi^{c1}x^{c1}+\frac{1}{2}\sigma^c_{11}(x^{c1}-(x^{c1})^{\mathrm{I}})^2,k=1.$$

Then the equations of the method will take the following form.

At the first stage for $k=0$

$$\dot{\psi}^{c1}=1,\dot{\psi}^{c2}=-2\psi^{c1}x^{c2},\psi^{c1}(1)=\psi^1(2),\psi^{c2}(1)=\psi(1),\dot{\sigma}^c_{11}=0,$$

$$\sigma^c_{11}(1)=\sigma(2)+\delta_4(2),\psi(1)=\psi^{c1}(1,1),\sigma_{11}(1)=\delta_1(1).$$

At the second stage for $k=1$

$$\dot{\psi}^{c1}=2\psi^{c1}x^{c1},\psi^{c1}(2)=\psi(2),\dot{\sigma}^c=4\sigma^c x^{c1}+2\psi^{c1},$$

$$\sigma^c(2)=\sigma(3)+\delta_4(3),\psi(3)=-1,\sigma(3)=\delta_1(3).$$

We set the initial approximation $u^{c1}(0,t)=u^{c2}(1,t)=0$. Then $x^{c1}=t,k=0$ and $x^{c1}=\frac{1}{t},k=1,I^0=0.5$

The solution to the example was obtained in one iteration and the functionality changed from 0.5 to -3.44. Control variables and trajectories

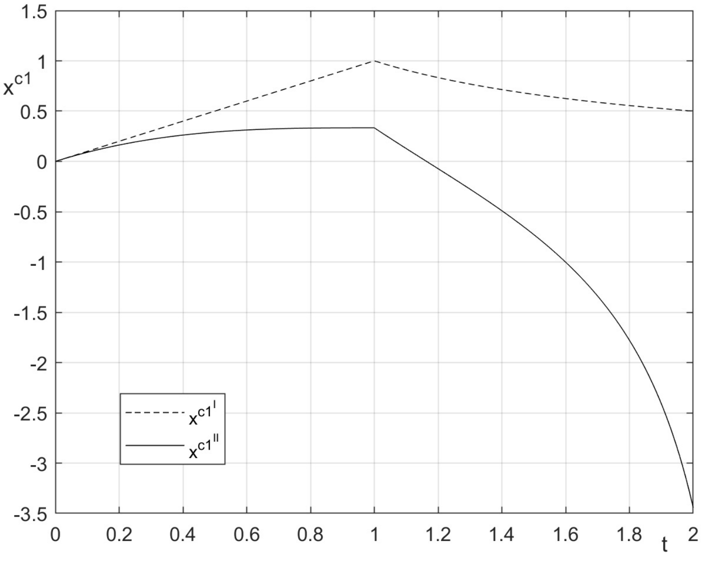

Figure 1. Trajectory change.

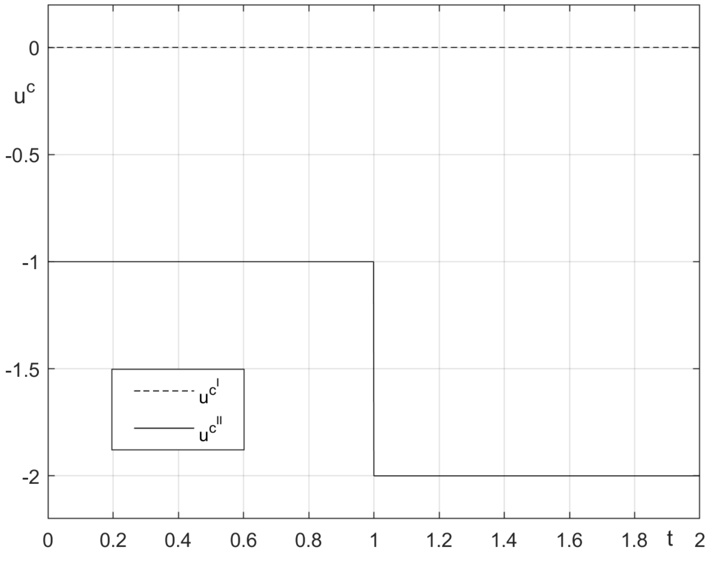

Figure 2. Control change.

are shown in Fig. 1 and Fig. 2. At the beginning of the calculations, the parameters δ_1, δ_4 were assumed to be equal to zero at both stages. Then they were changed over the interval [0;0.3], which did not affect the calculation results. So the functional at $\delta_1 = \delta_4 = 0.3$ at both stages changed from the value 0.5 to the previous value -3.44.

6. Conclusion

Thus, an analog of the well-known Krotov global improvement method is obtained. A second order method of improving for DCS with intermediate criteria is constructed, its algorithm is formulated and tested on an illustrative example. The calculation results confirm the efficiency of the method.

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Метод Кротова второго порядка для дискретно-непрерывных систем

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Аннотация. В конце 60-х и начале 70-х гг. прошлого века в теории оптимального управления появился новый класс задач. Оказалось, что структура описания ряда систем или рассматриваемых процессов не однородна и может изменяться с течением времени. Итог: появление новых математических моделей систем и процессов управления неоднородной структуры. Методы исследования таких систем очень разнообразны и отражают различные научные школы и направления.

Один из вариантов состоит в развитии подхода, позволяющего остаться в рамках традиционных предположений теории оптимального управления. Его основа – достаточные условия оптимальности В. Ф. Кротова для дискретных систем, сформулированные в терминах произвольных множеств и отображений.

В работе рассматривается одна из разновидностей неоднородных систем: дискретно-непрерывные системы (ДНС) для случая, когда все однородные подсистемы нижнего уровня не только связаны общим функционалом, но имеют и свои собственные цели.

Далее для построения метода применяется обобщение достаточных условий оптимальности В. Ф. Кротова. Идейной основой служит метод Кротова глобального улучшения, предложенный изначально для обычных дискретных процессов. Преимущество предлагаемого метода состоит в том, что его сопряженная система векторно-матричных уравнений линейная и, следовательно, ее решение всегда существует, что позволяет найти искомое решение в задаче оптимального управления для ДНС.

Ключевые слова: дискретно-непрерывные системы, достаточные условия оптимальности, метод улучшения управления.

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