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ON JOINT UNIVERSALITY OF DIRICHLET L-FUNCTIONS

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To the memory of Sergei Mikhailovich Voronin

Abstract.

In the paper, we present a probabilistic proof of the Voronin theorem on joint universality of Dirichlet L-functions, and prove the universality for some composite functions.

1 Introduction

For the first time, I met Sergei Mikhailovich Voronin in 1974 at the conference "Problems of analytic number theory and its applications"held in Vilnius. At that moment, I was a doctoral student, and his lecture on the functional independence of zeta-functions made a great impression for me. In 1975, the remarkable paper [13] on the universality of the Riemann zeta-function appeared, I begin to study it and obtained some related results. In 1983, during the conference "Theory of transcendental numbers and applications"at Moscow Lomonosov University, Sergei Mikhailovich invited me for a dinner, and from this moment we became good friends.

S. M. Voronin was extremely talented mathematician, and his early death is a very big loss.

The present paper is related to Voronin's work [14] on the joint universality and functional independence of Dirichlet L-functions.

Let χ be a Dirichlet character modulo q, and let $L(s,\chi)$, $s=\sigma+it$, denote the corresponding Dirichlet L-function. In [13], Voronin observed that the function $L(s,\chi)$ is universal in the following sense. Let $0<r<\frac{1}{4}$, and let $f(s)$ be a continuous

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non-vanishing function on the disc $|s| \leq r$ which is analytic for $|s| < r$. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s|\leq r} \left| L\left(s + \frac{3}{4} + i\tau, \chi\right) - f(s) \right| < \varepsilon.$$

The latter result has a joint generalization. First such a generalization in non-explicit form has been given in [14]. The full statement with proof is presented in [5].

THEOREM 1. *Let* $0 < r < \frac{1}{4}$, *let* $\chi_1, \dots, \chi_n$ *be pairwise non-equivalent Dirichlet characters, and let* $f_1(s), \dots, f_n(s)$ *be functions which are analytic for* $|s| < r$ *and continuous for* $|s| \leq r$ *and which have no zeros in the disc* $|s| < r$. *Then, for every* $\varepsilon > 0$, *there exists* $\tau > 0$ *such that for all* $j = 1, \dots, n$,

$$\max_{|s|\leq r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_j\right) - f_j(s) \right| < \varepsilon.$$

Gonek in [4] also obtained a joint universality theorem for Dirichlet L-functions. We remind a Bagchi's version [2] of Theorem 1.

THEOREM 2. *Let* $q \geq 1$, *and let* $\chi_1, \dots, \chi_n$ *be distinct Dirichlet characters modulo* q. *For* $j = 1, \dots, n$, *let* K_j *be a simply connected compact subset of the strip* $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, *and let* $f_j(s)$ *be a non-vanishing continuous function on* K_j *which is analytic in the interior of* K_j. *Then the set of all* $\tau \in \mathbb{R}$ *for which*

$$\sup_{1\leq j\leq n} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon$$

has a positive lower density for every $\varepsilon > 0$.

An another proof of Theorem 2 based on probabilistic arguments was given in [1].

A modern version of Theorem 1 is of the following form [11].

THEOREM 3. *Let* $\chi_1, \dots, \chi_n$ *be pairwise non-equivalent Dirichlet characters. For* $j = 1, \dots, n$, *let* K_j *be a compact subset of the strip* D *with connected complement, and let* $f_j(s)$ *be a non-vanishing continuous function on* K_j *which is analytic in the interior of* K_j. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Here $\text{meas}\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

As usual, denote by $H(G)$ the space of analytic functions on a region $G \subset \mathbb{C}$ equipped with the topology of uniform convergence on compacta. The space $H(G)$

is metrisable. It is well known that there exists a sequence of compact subsets $\{K_l : l \in \mathbb{N}\}$ of the region G such that

$$G = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, $l \in \mathbb{N}$, and, for every compact subset $K \subset G$, there exists K_l such that $K \subset K_l$. For $g_1, g_2 \in H(G)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then it is easily seen that ρ is a metric which induces the topology of uniform convergence on compacta in $H(G)$.

For $\underline{g}_1 = (g_{11}, \dots, g_{1n})$, $\underline{g}_2 = g_{21}, \dots, g_{2n}) \in H^n(G)$, $n \in \mathbb{N}$, we use the metric

$$\rho_n(\underline{g}_1, \underline{g}_2) = \max_{1 \le j \le n} \rho(g_{1j}, g_{2j}).$$

The aim of this paper is to describe the functions $F: \ H^n(D) \to H(D)$ such that the shifts $F(L(s + i\tau, \chi_1), \dots, L(s + i\tau, \chi_n))$ will be approximants of a given analytic function.

Let

$$S = \{g \in H(D): \ g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0\}.$$

Denote by U the class of continuous functions $F: \ H^n(D) \to H(D)$ such that, for every open set $G \in H(D)$,

$$(F^{-1}G) \cap S^n \neq \varnothing.$$

THEOREM 4. *Suppose that $\chi_1, \dots, \chi_n$ are pairwise non-equivalent Dirichlet characters, and that $F \in U$. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T]: \ \sup_{s \in K} |F(L(s + i\tau, \chi_1), \dots, L(s + i\tau, \chi_n)) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 4 is theoretical, it is difficult to check its hypotheses. We will give a simpler version of Theorem 4. Let $V > 0$ be an arbitrary number. Define a bounded strip

$$D_V = \{s \in \mathbb{C}: \ \frac{1}{2} < \sigma < 1, \ |t| < V\}.$$

Moreover, let

$$S_V = \{g \in H(D_V): \ g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0\}.$$

Denote by U_V the class of continuous functions $F: H(D_V) \to H(D_V)$ satisfying, for each polynomial $p = p(s)$, the condition

$$(F^{-1}\{p\}) \cap S_V^n \neq \varnothing.$$

THEOREM 5. *Let $\chi_1, \dots, \chi_n$ be pairwise non-equivalent Dirichlet characters, and K and $f(s)$ be the same as in Theorem* 4. Suppose that $V > 0$ is such that $K \subset D_V$, and that $F \in U_V$. Then the assertion of Theorem 4 is valid.

Theorems 4 and 5 describe the classes of universal operators on the space of analytic functions. It is easy to present an example. Let $n = 2$, and, for $f_1, f_2 \in H(D_V)$,

$$F(f_1, f_2) = c_1 f_1 + c_2 f_2, \quad c_1, c_2 \in \mathbb{C}, \quad c_1 c_2 \neq 0.$$

Then, clearly, the function F is continuous. Moreover, for each polynomial $p = p(s)$, there exist two polynomials $q_1(s)$ and $q_2(s)$ such that $(q_1, q_2) \in F^{-1}\{p\}$. Varying the constant terms of q_1 and q_2, we may obtain that the polynomials $q_1(s)$ and $q_2(s)$ do not vanish for $s \in D_V$. Thus, by Theorem 5, the linear combination $c_1 L(s, \chi_1) + c_2 L(s, \chi_2)$ is universal in the sense of Theorem 4. Similar assertion is also true for $c_1 L^2(s, \chi_1) + c_2 L^2(s, \chi_2)$.

Proofs of Theorems 4 and 5 are based on the probabilistic approach to universality of zeta and L-functions. Unfortunately, a probabilistic proof of Theorem 3 is not published, therefore, we will present some of its parts.

2 Limit theorems

A probabilistic way of the proof of universality for zeta and L-functions is based on limit theorems in the sense of weak convergence of probability measures with explicitly given limit measures in the space of analytic functions. Proofs of such theorems are standard, therefore, we state a joint limit theorem without proof.

As usual, denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : \ |s| = 1\}$ for each prime p. With the product topology and operation of pointwise multiplication, by the Tikhonov theorem, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^n(D)$-valued random element $\underline{L}(s, \omega, \underline{\chi})$, $\underline{\chi} = (\chi_1, \dots, \chi_n)$, by the formula

$$\underline{L}(s, \omega, \underline{\chi}) = (L(s, \chi_1), \dots, L(s, \chi_n)),$$

where

$$L(s,\chi_j)=\prod_p\left(1-\frac{\chi_j(p)\omega(p)}{p^s}\right)^{-1},\quad j=1,\dots,n.$$

Let $P_{\underline{L}}$ be the distribution of the element $\underline{L}(s,\omega,\underline{\chi})$, i. e.,

$$P_{\underline{L}}(A)=m_H\left(\omega\in\Omega:\underline{L}(s,\omega,\underline{\chi})\in A\right),\quad A\in\mathcal{B}(H^n(D)).$$

For $A\in\mathcal{B}(H^n(D))$, define

$$P_T(A)=\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_n))\in A\right\}.$$

LEMMA 1. *P_T converges weakly to $P_{\underline{L}}$ as $T\to\infty$.*

Lemma 1 implies a limit theorem in the space $H^n(D_V)$. To see this and for the sequel, we remind a simple property of weak convergence of probability measures. Let S_1 and S_2 be two metric spaces, and let $h: S_1\to S_2$ be $(\mathcal{B}(S_1),\mathcal{B}(S_2))$-measurable function. Then every probability measure P on $(S_1,\mathcal{B}(S_1))$ induces the unique probability measure Ph^{-1} on $(S_2,\mathcal{B}(S_2))$ defined by $Ph^{-1}(A)=P(h^{-1}A)$, $A\in\mathcal{B}(S_2)$.

LEMMA 2. *Suppose that P and P_n, $n\in\mathbb{N}$, be probability measures on $(S_1,\mathcal{B}(S_1))$, $h: S_1\to S_2$ is a continuous function, and P_n converges weakly to P as $n\to\infty$. Then P_nh^{-1} also converges weakly to Ph^{-1} as $n\to\infty$.*

The lemma is a particular case of Theorem 5.1 from [3].

For $V>0$, denote by $P_{T,V}$ and $P_{\underline{L},V}$ the restrictions to the space $(H(D_V),\mathcal{B}(H(D_V)))$ of the measures P_T and $P_{\underline{L}}$, respectively.

LEMMA 3. *For every $V>0$, $P_{T,V}$ converges weakly to $P_{\underline{L},V}$ as $T\to\infty$.*

PROOF. The lemma is a result of Lemmas 1 and 2 because the restriction of $H^n(D)$ to $H^n(D_V)$ is a continuous function. □

For $A\in\mathcal{B}(H(D))$, define

$$P_{T,F}(A)=\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\ F\left(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_n)\right)\in A\right\}.$$

LEMMA 4. *Suppose that $F: H^n(D)\to H(D)$ is a continuous function. Then the probability measure $P_{T,F}$ converges weakly to the distribution of the random element $F\left(\underline{L}(s,\omega,\underline{\chi})\right)$ as $T\to\infty$.*

PROOF. By the definitions of P_T and $P_{T,F}$, $P_{T,F}=P_TF^{-1}$. This, the continuity of F, and Lemmas 1 and 2 show that $P_{T,F}$ converges weakly to $P_{\underline{L}}F^{-1}$ as $T\to\infty$. However, for $A\in\mathcal{B}(H(D))$,

$$P_{\underline{L}}F^{-1}(A)\ =\ P_{\underline{L}}\left(F^{-1}A\right)=m_H\left(\omega\in\Omega:\ \underline{L}(s,\omega,\underline{\chi})\in F^{-1}A\right)$$

$$= \quad m_H\left(\omega \in \Omega : \; F\left(\underline{L}(s,\omega,\underline{\chi})\right) \in A\right).$$

Thus, $P_{T,F}$ converges weakly to the distribution of the random element $F\left(\underline{L}(s,\omega,\underline{\chi})\right)$ as $T \to \infty$. □

For $A \in \mathcal{B}(H(D_V))$, let

$$P_{T,F,V}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : F\left(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_n)\right) \in A\right\}.$$

Denote by $\underline{L}_V(s,\omega,\chi)$ the random element having the distribution $P_{\underline{L},V}$.

Lemma 5. *Suppose that $F: \; H^n(D_V) \to H(D_V)$ is a continuous function. Then $P_{T,F,V}$ converges weakly to the distribution of the random element $F\left(\underline{L}_V(s,\omega,\underline{\chi})\right)$ as $T \to \infty$.*

Proof of the lemma uses Lemmas 2 and 3, and completely coincides with that of Lemma 4.

3 Supports

For the proof of universality, we need to know the support of the limit measures in limit theorems in the space of analytic functions. Suppose that S is a separable metric space, and P is a probability measure on $(S,\mathcal{B}(S))$. We remind that the minimal closed set $S_P \subset S$, $P(S_P) = 1$, is called a support of the measure P. The support S_P consists of elements x such that, for every open neighbourhood G of x, $P(G) > 0$. The support of the distribution of a random element is called its support.

In this section, we will prove that the support of the $H^n(D)$-valued random element $\underline{L}(s,\omega,\underline{\chi})$ is the set S^n. For this, the non-equivalence of the characters $\chi_1, ..., \chi_n$ will be essentially used.

We start with statements of some known general results. Denote by S_ξ the support of a random element ξ.

Lemma 6. *Let $\{\underline{X}_m : \; m \in \mathbb{N}\}$ be a sequence of independent $H^n(D)$-valued random elements such that the series*

$$\sum_{m=1}^{\infty} \underline{X}_m$$

converges almost surely. Then the support of the sum of this series is equal to the closure of the set of all $\underline{g} \in H^n(D)$ that can be written as the sum of a convergent series,

$$\underline{g} = \sum_{m=1}^{\infty} \underline{g}_m, \quad \underline{g}_m \in S_{\underline{X}_m}.$$

The lemma is a particular case of Lemma 5 from [9].

LEMMA 7. *Suppose that the sequence* $\{\underline{g}_m : m \in \mathbb{N}\} = \{g_{1m}, \dots, g_{nm} : m \in \mathbb{N}\} \in H^n(D)$ *satisfies the following conditions:*

1° *If* $\mu_1, \dots, \mu_n$ *are complex-valued Borel measures on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *with compact supports contained in* D *and such that*

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} g_{jm}(s) \mathrm{d}\mu_j(s) \right| < \infty,$$

then

$$\int_{\mathbb{C}} s^l \mathrm{d}\mu_j(s) = 0$$

for all $j = 1, \dots, n$ *and* $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$*;*

2° *For every compact subset* $K \subset D$*,*

$$\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K} |g_{jm}(s)|^2 < \infty;$$

3° *The series*

$$\sum_{m=1}^{\infty} \underline{g}_m$$

is convergent in $H^n(D)$.

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m \underline{g}_m$$

with $|a_m| = 1$ *is dense in* $H^n(D)$.

The lemma is a particular case of Lemma 6 in [9].

LEMMA 8. *Let* μ *be a complex-valued Borel measure on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *with compact support contained in the half-plane* $\{s \in \mathbb{C} : \sigma > \sigma_0\}$*, and*

$$g(s) = \int_{\mathbb{C}} \mathrm{e}^{sz} \mathrm{d}\mu(z).$$

If $g(s) \not\equiv 0$*, then*

$$\limsup_{x \to \infty} \frac{\log |g(x)|}{x} > \sigma_0.$$

The lemma is Lemma 6.4.10 from [7].

We remind that an analytic function $g(s)$ in an angular region $|\arg s| \leq \theta_0$, $0 < \theta_0 \leq \pi$, is called a function of exponential type if

$$\limsup_{r\to\infty} \frac{\log |g(re^{i\theta})|}{r} < \infty$$

uniformly in θ, $|\theta| \leq \theta_0$.

Lemma **9.** *Suppose that $g(s)$ is a function of exponential type, and*

$$\limsup_{x\to\infty} \frac{\log |g(x)|}{x} > -1.$$

Then, for all coprime l and k,

$$\sum_{p\equiv l (\mathrm{mod}\ k)} |g(\log p)| = +\infty.$$

The lemma is Lemma 4.1 of [8].

We also recall the Hurwitz theorem.

Lemma **10.** *Suppose that $\{g_n(s) : n \in \mathbb{N}\}$ is a sequence of analytic functions in a region G bounded by a simple closed contour, and that*

$$\lim_{n\to\infty} g_n(s) = g(s)$$

uniformly on G, where $g(s) \not\equiv 0$. Then an interior point s_0 of G is a zero of $g(s)$ if and only if there exists a sequence $\{s_n\} \subset G$ such that $s_n \to s_0$ as $n \to \infty$ and $g_n(s_n) = 0$ for $n > n_0 = n_0(s_0)$.

The lemma is the Hurwitz theorem, for the proof, see [12].

Lemma **11.** *Suppose that $\varphi_1, \dots, \varphi_n$ are distinct homomorphisms from a group G to the multiplicative group of non-zero complex numbers. Then $\varphi_1, \dots, \varphi_n$ are linearly independent over $\mathbb{C}$.*

Proof of the lemma is given in [6].

Now we are ready to consider the support of the random element $\underline{L}$.

Lemma **12.** *Suppose that the characters $\chi_1, \dots, \chi_n$ are pairwise non-equivalent. Then the support of the random element $\underline{L}$ (or of the measure $P_{\underline{L}}$) is the set S^n.*

Proof. We have that

$$\underline{L}(s, \omega, \underline{\chi}) = \left(\prod_p \left(1 + g_{1p}(s, \omega)\right)^{-1}, \dots, \prod_p \left(1 + g_{np}(s, \omega)\right)^{-1} \right), \tag{1}$$

where

$$g_{jp}(s,\omega) = -\frac{\chi_j(p)\omega(p)}{p^s}, \quad j = 1, \dots, n.$$

For $|z| < 1$, define

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots .$$

Then the functions $\log(1 + g_{jp}(s,\omega))$ are well defined for $s \in D$, $j = 1, \dots, n$, and we can consider the support of the random element

$$\left(-\sum_p \log\left(1 + g_{1p}(s,\omega)\right), \dots, -\sum_p \log\left(1 + g_{np}(s,\omega)\right) \right). \tag{2}$$

The proof that

$$\prod_p \left(1 + g_{jp}(s,\omega)\right)^{-1}$$

is an $H(D)$-valued random element, $j = 1, \dots, r$, contains a statement that the product converges uniformly on compact subsets of the strip D for almost all $\omega \in \Omega$. Denoting

$$\underline{g}_p(s,\omega) = \left(g_{1p}(s,\omega), ..., g_{np}(s,\omega)\right),$$

hence we have that there exists a sequence $\underline{b} = \{b_p : |b_p| = 1\}$ such that the series

$$\sum_p \underline{g}(s,\underline{b}) \tag{3}$$

converges in $H^n(D)$. Moreover, for every compact subset $K \subset D$,

$$\sum_p \sum_{j=1}^n \sup_{s\in K} |g_{jp}(s,\underline{b})|^2 < \infty.$$

This, and the convergence of the series (3) show that the hypotheses 2° and 3° of Lemma 7 are satisfied by the sequence $\{\underline{g}_p(s,\underline{b})\}$. It remains to verify the hypothesis 1°.

Let p_0 be a fixed positive number. Suppose that $\mu_1, \dots, \mu_n$ are complex-valued Borel measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{p>p_0} \left| \sum_{j=1}^n \int_{\mathbb{C}} g_{jp}(s,\underline{b}) \mathrm{d}\mu_j(s) \right| < \infty. \tag{4}$$

Let d be the product of the moduli of the characters $\chi_1, \dots, \chi_n$. Using the definition of $g_{jp}(s,\underline{b})$, the periodicity of the characters $\chi_1, ..., \chi_n$ with period d, and (4), we find that

$$\sum_{\substack{p>p_0 \\ p\equiv l (\mathrm{mod}\ d)}} \left| \sum_{j=1}^n \chi_j(l) \int_{\mathbb{C}} \frac{1}{p^s} \mathrm{d}\mu_j(s) \right| < \infty \tag{5}$$

for all $l = 1, \dots, d$ with $(l, d) = 1$. For $A \in \mathcal{B}(\mathbb{C})$, define

$$\nu_l(A) = \sum_{j=1}^{n} \chi_j(l)\mu_j(A), \quad l = 1, ..., d, \quad (l, d) = 1.$$

Then $\nu_1, \dots, \nu_n$ are again complex-valued measures with compact supports contained in D, and (5) implies

$$\sum_{\substack{p>p_0 \\ p\equiv l (\mathrm{mod}\ d)}} |\rho_l(\log p)| < \infty, \quad l = 1, \dots, d, \quad (l, d) = 1, \tag{6}$$

where

$$\rho_l(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\nu_l(s), \quad l = 1, \dots, d, \quad (l, d) = 1.$$

Since the functions $\rho_l(z)$ are of exponential type, in view of Lemma 8, we have that either $\rho_l(z) \equiv 0$, or

$$\limsup_{x\to\infty} \frac{\log |\rho_l(x)|}{x} > -1, \quad l = 1, \dots, d, \quad (l, d) = 1.$$

If the last inequality holds for some $l = 1, \dots, d$, $(l, d) = 1$, then, by Lemma 9, for this l,

$$\sum_{\substack{p>p_0 \\ p\equiv l (\mathrm{mod}\ d)}} |\rho_l(\log p)| = \infty,$$

what contradicts (6). Thus, we have that $\rho(z) \equiv 0$ for all $l = 1, \dots, d$, $(l, d) = 1$, and, by the definitions of $\rho_l(z)$ and ν_l,

$$\sum_{j=1}^{n} \chi_j(l) \int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\mu_j(s) \equiv 0, \quad l = 1, ..., d, \quad (l, d) = 1. \tag{7}$$

In this place, we use the assumption that the characters $\chi_1, \dots, \chi_n$ are pairwise non-equivalent. This, Lemma 11 and (7) lead to the equality

$$\int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\mu_j(s) \equiv 0, \quad j = 1, \dots, n.$$

Differentiating this equality, we easily find that

$$\int_{\mathbb{C}} s^m \mathrm{d}\mu_j(s) = 0$$

for all $m \in \mathbb{N}_0$ and $j = 1, \dots, n$. This means that the hypothesis 1° of Lemma 7 also holds for the sequence $\{\underline{g}_p(s, \underline{b}) : \ p \geq p_0\}$. Therefore, the set of all convergent series

$$\sum_{p>p_0} \hat{a}(p)\underline{g}_p(s, \underline{b}) \tag{8}$$

with $|\hat{a}(p)| = 1$ is dense in $H^n(D)$.

Let $\underline{x}_0(s) = (x_{10}(s), ..., x_{n0}(s))$ be an arbitrary element of $H^n(D)$, $\varepsilon > 0$ and $K \subset D$ be arbitrary compact set. Then there exists p_0 such that

$$\max_{1\leq j\leq n} \sup_{s\in K} \left(\sum_{p>p_0} \sum_{k=2}^{\infty} \frac{|g_{jp}(s, \underline{a})|^k}{k} \right) < \frac{\varepsilon}{2} \tag{9}$$

with $\underline{a} = \{a(p) : \ |a(p)| = 1\}$. The denseness of the set of series (8) shows that there exists $\hat{\underline{a}} = \{\hat{a}(p) : |\hat{a}(p)| = 1\}$, such that

$$\max_{1\leq j\leq n} \sup_{s\in K} \left| x_{j0}(s) - \sum_{p\leq p_0} \log(1 + g_{jp}(s, 1)) - \sum_{p>p_0} \hat{a}(p) g_{jp}(s, \underline{a}) \right| < \frac{\varepsilon}{2}. \tag{10}$$

Denoting

$$a(p) = \begin{cases} \hat{a}(p)b_p & \text{if } \ p > p_0, \\ 1 & \text{if } \ p \leq p_0, \end{cases} \qquad \text{and} \quad \underline{a} = \{a(p)\},$$

we find from (9) and (10) that

$$\begin{aligned} &\max_{1\leq j\leq n} \sup_{s\in K} \left| x_{j0}(s) - \sum_{p} \log(1 + g_{jp}(s, \underline{a})) \right| \\ \leq \ &\max_{1\leq j\leq n} \sup_{s\in K} \left| x_{j0}(s) - \sum_{p\leq p_0} \log(1 + g_{jp}(s, 1)) - \sum_{p>p_0} \hat{a}(p) g_{jp}(s, \underline{b}) \right| \\ + \ &\max_{1\leq j\leq n} \sup_{s\in K} \left| \sum_{p>p_0} \sum_{k=2}^{\infty} \frac{g_{jp}^k(s, \underline{a})|}{k} + \sum_{p>p_0} \frac{\hat{a}(p)\chi_j(p)b_p}{p^s} - \sum_{p>p_0} \frac{\chi_j(p)a_p}{p^s} \right| < \varepsilon. \end{aligned}$$

This means that the set of all convergent series

$$-\sum_{p} (\log(1 + g_{1p}(s, \underline{a})), \dots, \log(1 + g_{np}(s, \underline{a}))) \tag{11}$$

with $|a(p)| = 1$ is dense in $H^n(D)$.

By the definition, $\{\omega(p)\}$ is a sequence of independent random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Hence,

$$\{\log(1 + g_{1p}(s, \underline{a})), \dots, \log(1 + g_{np}(s, \underline{a}))\}$$

is a sequence of independent $H^n(D)$-valued random elements on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Since the unit circle is the support of every random variable $\omega(p)$, the set

$$\{(g_0, \dots, g_n) \in H^n(D) : g_j(s) = \log(1 + g_{jp}(s, \underline{a})),\ j = 1, \dots, n\}$$

with $\underline{a} = \{a(p) :\ |a(p)| = 1\}$ is the support of the $H^n(D)$-valued random element

$$(\log(1 + g_{1p}(s, \omega)), \dots, \log(1 + g_{np}(s, \omega)))\,.$$

Thus, by Lemma 6, the support of the random element (2) is the closure of all convergent series (11). Since the latter set is dense in $H^n(D)$, the support of the random element (2) is the whole of $H^n(D)$.

Let $u :\ H^n(D) \to H^n(D)$ be given by the formula

$$u(g_1, \dots, g_n) = (\mathrm{e}^{g_1}, \dots, \mathrm{e}^{g_n}), \quad (g_1, \dots, g_n) \in H^n(D).$$

Then u is a continuous function sending the vector

$$\left(-\sum_p \log(1 + g_{1p}(s, \omega)), \dots, -\sum_p \log(1 + g_{np}(s, \omega))\right)$$

to the vector

$$\left(\prod_p (1 + g_{1p}(s, \omega))^{-1}, \dots, \prod_p (1 + g_{np}(s, \omega))^{-1}\right), \tag{12}$$

and mapping $H^n(D)$ onto $(S \setminus \{0\})^n$. This and a result on the support of the random element (2) show that the support of the random elements (12) contains the set $(S \setminus \{0\})^n$. However, the support of (12) is a closed set. Since, by Lemma 10, the closure of $(S \setminus \{0\})^n$ is S^n, the support of the random element (12) contains the set S^n.

The products

$$\prod_p (1 + g_{jp}(s, \omega))^{-1}, \quad j = 1, ..., n,$$

consist of non-zero factors and converge uniformly on compact subsets of D for almost all $\omega \in \Omega$. Therefore, in view of Lemma 10, the set S^n contains the support of the random element (12). Combining this with the opposite inclusion, we obtain the assertion of the lemma. □

In the same way as Lemma 12, the following statement follows.

LEMMA **13.** *Suppose that the characters $\chi_1, \dots, \chi_n$ are pairwise non-equivalent. Then the support of the measure $P_{\underline{L},V}$ is the set S_V^n.*

Lemma **14.** *Suppose that $\chi_1, \dots, \chi_n$ are pairwise non-equivalent Dirichlet character, and that $F \in U$. Then the support of the random element $F(L(s,\omega,\chi))$ is the whole of $H(D)$.*

Proof. Let x be an element of $H(D)$, and G be any open neighbourhood of x. Since the function F is continuous, we have that $F^{-1}G$ is an open set, too. By the definition of the class U,

$$(F^{-1}G) \cup S^n \neq \varnothing,$$

therefore, there exists an element y which belongs to S^n and $F^{-1}G$ simultaneously. This means that $F^{-1}G$ is an open neighbourhood of the element y. Consequently, in virtue of Lemma 13,

$$m_H(\omega \in \Omega : F(\underline{L}(s,\omega,\underline{\chi})) \in G) = m_H(\omega \in \Omega : \underline{L}(s,\omega,\underline{\chi}) \in F^{-1}G) > 0.$$

This and properties of a support prove the lemma. □

The investigation of the support of the random element $F(\underline{L}_V(s,\omega,\underline{\chi}))$ leans essentially on the famous Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma **15.** *Let $K \subset \mathbb{C}$ be compact set with connected complement. Then every function $g(s)$ which is continuous on K and analytic inside K can be uniformly approximated on K by polynomials in s.*

The lemma is the Mergelyan theorem, see [10], [15].

Lemma **16.** *Suppose that $\chi_1, \dots, \chi_n$ are pairwise non-equivalent Dirichlet character, and that $F \in U_V$. Then the support of the random element $F(\underline{L}_V(s,\omega,\underline{\chi}))$ is the whole of $H(D_V)$.*

Proof. Let G be any open neighbourhood of x. Then $F^{-1}G$ also is an open set of $H^n(D_V)$. We have to prove that $(F^{-1}G) \cap S_V^n \neq \varnothing$. For this, we apply Lemma 15.

From the definition of the metric ρ in $H(D_V)$, it follows that in $H(D_V)$ it suffices to consider an approximation on compact subsets of D_V. Indeed, let $\varepsilon > 0$ be an arbitrary fixed number.Then there exists $l_0 \in \mathbb{N}$ such that

$$\sum_{l>l_0} 2^{-l} < \frac{\varepsilon}{2}. \tag{13}$$

Suppose that

$$\sup_{s \in K_{l_0}} |f(s) - p_(s)| < \frac{\varepsilon}{2}, \quad f, g \in H(D_V).$$

Since $K_l \subset K_{l+1}$, $l \in \mathbb{N}$, this implies the inequality

$$\sup_{s \in K_{l_0}} |f(s) - p_(s)| < \frac{\varepsilon}{2}$$

for all $l = 1, \dots, l_0$. Therefore, in view of (13),

$$\rho(f, g) < \varepsilon.$$

This shows that if the functions f and g are near one from another on K_l with sufficiently large l, then they are near in the sense of $H(D_V)$. We note that the sets K_l can be chosen with connected complements.

So, let $K \subset D_V$ be a compact subset having a connected complement, and $f \in H(D_V)$. Then, by Lemma 20, we can find a polynomial $p = p(s)$ which approximate the function f with a given accuracy uniformly on K. Hence, we have that if $f \in G$, then $p \in G$, too. By the definition of the class U_V,

$$(F^{-1}\{S\}) \cap S_V^n \neq \varnothing.$$

Since $p \in G$, hence we obtain that

$$(F^{-1}G) \cap S_V^n \neq \varnothing.$$

Thus, by Lemma 13,

$$m_H\big(\omega \in \Omega : F(\underline{L}_V(s, \omega, \underline{\chi})) \in G\big) = m_H\big(\omega \in \Omega : \underline{L}_V(s, \omega, \underline{\chi}) \in F^{-1}G\big) > 0.$$

This proves the lemma. □

4 Proof of Theorems 4 and 5

A proof uses limit theorems in the space of analytic functions as well as supports of the limit measures, and is standard.

Proof.[Proof of Theorem 4] By Lemma 15, we can find a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{14}$$

Define

$$G = \{g \in H(D) : \sup_{s \in K} |f(s) - g(s)| < \frac{\varepsilon}{2}\}.$$

Then the G is open. Therefore, Lemma 4 and the equivalent of weak convergence of probability measures in terms of open sets yield

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : F(L(s + i\tau, \chi_1), \dots, L(s + i\tau, \chi_n)) \in G\} \geq P_{F(\underline{L})}(G) > 0 \tag{15}$$

where $P_{F(\underline{L})}$ is the distribution of the random element $F(\underline{L}(s, \omega, \underline{\chi}))$. In view of Lemma 14, the polynomial $p(s)$ is an element of the support of the measure $P_{F(\underline{L})}$.

Therefore, the definition of G and properties of a support show that $P_{F(\underline{L})}(G) > 0$. This and (15) imply

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |F(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_n)) - f(s)| < \varepsilon\right\} > 0.$$

Combining this with (14), we obtain the theorem. □

PROOF.[Proof of Theorem 5] The theorem follows in the same way as Theorem 4 by using Lemmas 5 and 16 in place of Lemmas 4 and 14. □

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