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ON A GENERALIZATION OF THE BERNSTEIN–MARKOV INEQUALITY

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Abstract. It is shown that

$$\|P'Q\|_{L_p(I)} \leqslant c^{1+1/p}(N+M)\log(\min(N, M+1)+1)\|PQ\|_{L_p(I)}$$

for all real trigonometric polynomials P and Q of degree N and M, respectively, where $0 < p \leqslant \infty$, $I := (-\pi, \pi]$, and $c > 0$ is a suitable absolute constant. Also, it is shown that

$$\|f'g\|_{L_p(J)} \leqslant c^{1+1/p}(N+M)^2\|fg\|_{L_p(J)}$$

for all algebraic polynomials f and g of degree N and M, respectively, where $0 < p \leqslant \infty$, $J := [-1, 1]$, and $c > 0$ is a suitable absolute constant. Both of the above trigonometric and algebraic results are sharp up to the factor $c^{1+1/p}$. In fact, the results are proved for the much wider classes of generalized trigonometric and algebraic polynomials.

§1. Introduction

The function

$$P(x) := a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \quad a_k, b_k \in \mathbb{R}, \ \ a_n b_n \neq 0,$$

is called a real trigonometric polynomial of degree n. It is well known that every real trigonometric polynomial P of degree n can be written as

$$P(x) = \omega \prod_{j=1}^{2n} \sin((x - z_j)/2),$$

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where $\omega \in \mathbb{R}$, $z_j \in \mathbb{C}$, and the nonreal zeros z_j of P form conjugate pairs. The function

$$P(x) := \omega \prod_{j=1}^{s} |\sin((x - z_j)/2)|^{r_j}, \quad x \in \mathbb{R}, \tag{1.1}$$

where $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$ are distinct (mod 2π), and $0 < \omega \in \mathbb{R}$, is called a generalized trigonometric polynomial of degree $N := \frac{1}{2}\sum_{j=1}^{s} r_j$. If P is a constant identically, then its degree is defined to be 0. Note that the absolute value of a real trigonometric polynomial of degree n may be viewed as a generalized trigonometric polynomial of degree n. Let GTP_N denote the set of all generalized trigonometric polynomials of degree at most N. Observe that if $P \in \mathrm{GTP}_N$ is of the form (1.1), then

$$P(x) := \omega \prod_{j=1}^{s} (\sin((x - z_j)/2) \sin((x - \bar{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}, \quad x \in \mathbb{R},$$

where each T_j is a real trigonometric polynomial of degree 1 and nonnegative on the real line. For a $P \in \mathrm{GTP}_N$ of the form (1.1), the numbers z_j are called the zeros of P, while the exponent r_j is called the multiplicity of the zero z_j in P.

The problem arises how to define P' for $P \in \mathrm{GTP}_N$. Observe that if $r_j \geqslant 1$ for each $j = 1, 2, \ldots, s$ in (1.1), then, although P' may fail to exist at the zeros of P, the one-sided derivatives P'_- and P'_+ exist, and their absolute values are equal. This means that $|P'|$ is well defined on the real line by either $|P'_-|$ or $|P'_+|$. It is a simple exercise to check that if $f \in \mathrm{GTP}_N$ has only real zeros with multiplicities at least 1, then $|f'| \in \mathrm{GTP}_N$ has only real zeros as well, and at least one of any two adjacent zeros of $|f'|$ has multiplicity exactly 1.

It is well known that these generalized trigonometric polynomials satisfy the following Bernstein-type inequality on $I := (-\pi, \pi]$. Here and in what follows, $c, c_1, c_2, \ldots$ will always denote suitable positive absolute constants, not necessarily the same at each occurrence.

Theorem A ([1, Theorem A.4.12 and Corollary A.4.13]). *Let χ be a nonnegative, monotone nondecreasing, and convex function defined on $[0, \infty)$. Then*

$$\int_I \chi(N^{-q}|P'(t)|^q)\, dt \leqslant \int_I \chi(cP(t)^q)\, dt \tag{1.2}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) with all $r_j \geqslant 1$, and for every $0 < q \leqslant 1$. In particular,

$$\|P'\|_{L_p(I)} \leqslant c^{1+1/p} N \|P\|_{L_p(I)}, \quad 0 < p \leqslant \infty, \tag{1.3}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) with all $r_j \geqslant 1$.

Inequality (1.3) can be generalized to the following (cf. [6, Theorem 10.4]).

Theorem B. *We have*

$$\|P'Q\|_{L_p(I)} \leqslant A_{N,M}\|PQ\|_{L_p(I)}, \tag{1.4}$$

where

$$A_{N,M} := c^{1+1/p}(N + M\min\{p,1\})(M+1), \quad 0 < p \leqslant \infty, \tag{1.5}$$

for every $P \in \mathrm{GTP}_N$ *of the form* (1.1) *with all* $r_j \geqslant 1$, *and for every* $Q \in \mathrm{GTP}_M$.

The function

$$f(x) := \sum_{k=0}^{n} a_k x^k, \quad a_k \in \mathbb{R}, \ a_n \neq 0,$$

is called a real algebraic polynomial of degree n. It is well known that every real algebraic polynomial f can be written as

$$f(x) = \omega \prod_{j=1}^{n} (x - z_j),$$

where $\omega \in \mathbb{R}$, $z_j \in \mathbb{C}$, and the nonreal zeros z_j of f form conjugate pairs. The function

$$f(x) := \omega \prod_{j=1}^{s} |x - z_j|^{r_j}, \quad x \in \mathbb{R}, \tag{1.6}$$

where $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$ are distinct, and $0 < \omega \in \mathbb{R}$, is called a generalized algebraic polynomial of degree $N := \sum_{j=1}^{s} r_j$. If f is a constant identically, then its degree is defined to be 0. Note that the absolute value of a real algebraic polynomial of degree n may be viewed as a generalized algebraic polynomial of degree n. Let GAP_N be the set of all generalized algebraic polynomials of degree at most N. Observe that if $f \in \mathrm{GAP}_N$ is of the form (1.6), then

$$f(x) := \omega \prod_{j=1}^{s} ((x - z_j)(x - \overline{z}_j))^{r_j/2} = \prod_{j=1}^{s} g_j(x)^{r_j/2}, \quad x \in \mathbb{R},$$

where each g_j is a real algebraic polynomial of degree 2 and nonnegative on the real line. For a $P \in \mathrm{GAP}_N$ of the form (1.6), the numbers z_j are called the zeros of P, while the exponent r_j is called the multiplicity of the zero z_j in f.

The problem arises how to define f' for $f \in \mathrm{GAP}_N$. Observe that if $r_j \geqslant 1$ for each $j = 1, 2, \dots, s$ in (1.6), then, although f' may fail to exist at the zeros of f, the one-sided derivatives f'_- and f'_+ exist, and their absolute values are equal. This means that $|f'|$ is well defined on the real line by either $|f'_-|$ or $|f'_+|$. It is a simple exercise to check that if $f \in \mathrm{GAP}_N$ has only real zeros with multiplicities at least 1, then $|f'| \in \mathrm{GAP}_{N-1}$ has only real zeros as well, and at least one of any two adjacent zeros of $|f'| \in \mathrm{GAP}_{N-1}$ has multiplicity exactly 1.

These generalized algebraic polynomials satisfy the following Bernstein and Markov type inequalities, respectively, on $J := [-1, 1]$.

Theorem C. *We have*

$$\|\sqrt{1-x^2}f'(x)g(x)\|_{L_p(J)} \leqslant A_{N+1,M+1/p}\|fg\|_{L_p(J)}, \quad 0 < p \leqslant \infty, \tag{1.7}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with all* $r_j \geqslant 1$, *and for every* $g \in \mathrm{GAP}_M$, *where* $A_{N,M}$ *is defined in* (1.5). *In particular,*

$$\|\sqrt{1-x^2}f'(x)\|_{L_p(J)} \leqslant c^{1+1/p}N\|f\|_{L_p(J)}, \quad 0 < p \leqslant \infty, \tag{1.8}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with all* $r_j \geqslant 1$.

This can be obtained from Theorem B by the substitution

$$P(t) = f(\cos t) \in \mathrm{GTP}_N \quad \text{and} \quad Q(t) = g(\cos t)|\sin t|^{1/p} \in \mathrm{GTP}_{M+1/p} \tag{1.9}$$

(cf. [1, p. 409, E.10 a]).

Theorem D (cf. [6, Theorems 10.3 and 10.6], as well as [4, Theorem 1]). *We have*

$$\|f'g\|_{L_p(J)} \leqslant \begin{cases} A_{N,M}^2\|fg\|_{L_p(J)} & \text{if } 0 < p < \infty, \\ c(N+M)^2\|fg\|_{L_\infty(J)} & \text{if } p = \infty \end{cases} \tag{1.10}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with all* $r_j \geqslant 1$, *and for every* $g \in \mathrm{GAP}_M$. *In particular,*

$$\|f'\|_{L_p(J)} \leqslant c^{1+1/p}N^2\|f\|_{L_p(J)}, \quad 0 < p \leqslant \infty, \tag{1.11}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with all* $r_j \geqslant 1$.

While the usual Bernstein and Markov type inequalities (1.3), (1.8), and (1.11) are known to be sharp as for the order of magnitude, the same is not known for (1.4), (1.7), and (1.10). In fact, it is our purpose in the present paper to improve these inequalities under certain conditions.

Generalized trigonometric and algebraic polynomials are studied in a number of papers [2–8], and most of these results may be found in the book [1] with complete proofs. We formulate four more results about generalized polynomials, which are needed in the proof of the main results of this paper. In each of these, as before, $I := (-\pi, \pi]$ and $J := [-1, 1]$. The following Nikolskiĭ-type inequalities for the classes GTP_N and GAP_N were proved in [7] (cf. Theorems 5 and 6) as well as in [1] (cf. Theorems A.4.3 and A.4.4).

Theorem E. *Let χ be a nonnegative, monotone nondecreasing function defined on $[0,\infty)$ and such that $\chi(x)/x$ is monotone nonincreasing on $[0,\infty)$. Then for $0<q<p\leqslant\infty$ we have*

$$\|\chi(P)\|_{L_p(I)} \leqslant (c(1+qN))^{1/q-1/p}\|\chi(P)\|_{L_q(I)}$$

for every $P\in \mathrm{GTP}_N$. If $\chi(x)=x$, then $c=e(4\pi)^{-1}$ is a suitable choice.

Theorem F. *Let χ be a nonnegative, monotone nondecreasing function defined on $[0,\infty)$ and such that $\chi(x)/x$ is monotone nonincreasing on $[0,\infty)$. Then for $0<q<p\leqslant\infty$ we have*

$$\|\chi(f)\|_{L_p(J)} \leqslant (c(2+qN))^{2/q-2/p}\|\chi(f)\|_{L_q(J)}$$

for every $f\in \mathrm{GAP}_N$. If $\chi(x)=x$, then $c=e^2(2\pi)^{-1}$ is a suitable choice.

The following Remez-type inequalities for the classes GTP_N and GAP_N were proved in [5]. The Lebesgue measure of a set $A\subset\mathbb{R}$ is denoted by $m(A)$.

Theorem G. *We have*

$$\|P\|_{L_\infty(I)} \leqslant \exp(cNs), \quad 0<s<\pi/2,$$

for every $P\in\mathrm{GTP}_N$ satisfying $m\{t\in I : P(t)\leqslant 1\}\geqslant 2\pi - s$.

Theorem H. *We have*

$$\|f\|_{L_\infty(J)} \leqslant \exp(cNs^{1/2}), \quad 0<s<1,$$

for every $f\in\mathrm{GAP}_N$ satisfying $m\{t\in J : f(t)\leqslant 1\}\geqslant 2 - s$.

§2. The case of generalized trigonometric polynomials

Theorem 1. *We have*

$$\|P'Q\|_{L_p(I)} \leqslant c^{1+1/p}(N+M)\log(\min(N,M+1)+1)\|PQ\|_{L_p(I)}, \quad 0<p\leqslant\infty, \tag{2.1}$$

for all $P\in\mathrm{GTP}_N$ and $Q\in\mathrm{GTP}_M$ such that the roots of P and Q have multiplicities at least 1. Moreover, (2.1) is sharp apart from the constant $c^{1+1/p}$.

This improves considerably Theorem B (under the stronger condition that the multiplicities in Q are at least 1). In fact, we shall prove slightly more; this is formulated as a lemma, and it is a generalization of Theorem A.

Lemma 1. *Let χ be a nonnegative, monotone nondecreasing, convex function defined on $[0,\infty)$. Then*

$$\int_{-\pi}^{\pi} \chi\left(\left(\frac{|P'(t)|Q(t)}{(N+M)\log(N+1)}\right)^q\right) dt \leqslant \int_{-\pi}^{\pi} \chi(c(P(t)Q(t))^q)\, dt \qquad (2.2)$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) *with all $r_j \geqslant 1$, for every $Q \in \mathrm{GTP}_M$, and for every $0 < q \leqslant 1$. In particular,*

$$\|P'Q\|_{L_p(I)} \leqslant c^{1+1/p}(N+M)\log(N+1)\|PQ\|_{L_p(I)}, \quad 0 < p \leqslant \infty, \qquad (2.3)$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) *with all $r_j \geqslant 1$, and for every $Q \in \mathrm{GTP}_M$.*

Inequality (2.3) follows from (2.2) with $q = \min(1,p)$ and $\chi(x) = x^{\max(1,p)}$. On the other hand, (2.3) implies (2.1). Indeed, we can apply (2.3) with the roles of P and Q interchanged, to get

$$\|Q'P\|_{L_p(I)} \leqslant c^{1+1/p}(N+M)\log(M+1)\|PQ\|_{L_p(I)}, \quad 0 < p \leqslant \infty.$$

Coupled with the Bernstein-type inequality (cf. Theorem A)

$$\|(PQ)'\|_{L_p(I)} \leqslant c^{1+1/p}(N+M)\|PQ\|_{L_p(I)}, \quad 0 < p \leqslant \infty,$$

(cf. (1.3)), this yields the statement of Theorem 1.

In order to prove Lemma 1, we need several auxiliary statements.

Lemma 2. *Let $P \in \mathrm{GTP}_N$, and let Δ be an arbitrary interval with midpoint t. Put*

$$\mathcal{M}(P,\Delta) := \sum_{z_j \in \Delta} r_j,$$

where r_j is the multiplicity of the root z_j of P, and the sum is taken over all the zeros z_j of P lying in Δ. Then

$$\mathcal{M}(P,\Delta) \leqslant \left(\frac{e}{2}N|\Delta| + 1\right)\frac{\|P\|_{L_\infty(I)}}{P(t)} \quad \textit{for all } \Delta.$$

Proof. In the proof of E.11 on pp. 236–237 of [1] it was shown that if S is a usual trigonometric polynomial of degree at most n, then

$$\left(\frac{2\mathcal{M}(S,\Delta)}{e|\Delta|n}\right)^{\mathcal{M}(S,\Delta)} \leqslant \frac{\|S\|_{L_\infty(I)}}{|S(t)|}. \qquad (2.4)$$

To prove the lemma, first we assume that each r_j in the representation of $P \in \mathrm{GTP}_N$ is rational with a common denominator $q \in \mathbb{N}$, and apply (2.4) to the usual trigonometric polynomial $S = P^{2q}$ of degree at most $n = 2qN$. Since evidently $\mathcal{M}(S, \Delta) = 2q\mathcal{M}(P, \Delta)$, we get

$$\left(\frac{4q\mathcal{M}(P,\Delta)}{e|\Delta|2qN}\right)^{2q\mathcal{M}(P,\Delta)} \leqslant \left(\frac{\|P\|_{L_\infty(I)}}{|P(t)|}\right)^{2q},$$

i.e.,

$$\left(\frac{2\mathcal{M}(P,\Delta)}{e|\Delta|N}\right)^{\mathcal{M}(P,\Delta)} \leqslant \frac{\|P\|_{L_\infty(I)}}{|P(t)|}. \tag{2.5}$$

We may assume that $\mathcal{M}(P, \Delta) \geqslant (e/2)|\Delta|N+1$ (otherwise there is nothing to prove), and hence we may replace the left-hand side exponent in (2.5) by 1. This proves the lemma when each r_j is rational. Since (2.5) is independent of the common denominator q of the rational exponents, an obvious limit procedure yields the result for arbitrary exponents.

Lemma 3 ([1, p. 409, E.7 b]; cf. also [5]). *The inequality*

$$m(\{t \in I : P(t) \geqslant \lambda\|P\|_{L_\infty(I)}\}) \geqslant \frac{\mu(\lambda)}{N+1}$$

is fulfilled for every $P \in \mathrm{GTP}_N$, where $0 < \lambda < 1$ is arbitrary, and $\mu(\lambda) > 0$ depends only on λ.

We use the notation $\mathbb{T} := \mathbb{R} \pmod{2\pi}$.

Lemma 4. *Suppose $N \geqslant 1$ and $P \in \mathrm{GTP}_N$. Let $a \in I$ be such that*

$$P(a) = \|P\|_{L_\infty(I)}. \tag{2.6}$$

Then, with some suitable constants $c_1 > 0$ and $c_2 > 0$, we have

$$m\{t \in [a - 1/N, a + 1/N] : P(t) \geqslant c_2^{-1}\|P\|_{L_\infty(I)}\} \geqslant \frac{c_1}{N}.$$

Proof of Lemma 4. With $\omega := \pi - 1/N$ and $n := \lfloor N \rfloor$, let

$$R_N(t) := \left|T_n\left(\frac{\sin((t-\pi-a)/2)}{\sin(\omega/2)}\right)\right| \in \mathrm{GTP}_N,$$

where $T_n(x) = \cos(nt)$, $x = \cos t$, is the Chebyshev polynomial of degree n. Then there is an absolute constant $c_2 > 1$ such that

$$c_2 \leqslant |R_N(a)| = \|R_N\|_{L_\infty(I)} \tag{2.7}$$

and

$$|R_N(t)| \leqslant 1, \quad t \in \mathbb{T} \setminus [a - 1/N, a + 1/N]. \tag{2.8}$$

Assume (2.6) is fulfilled. Let

$$A := \{t \in [a - 1/N, a + 1/N] : P(t) \leqslant c_2^{-1} \|P\|_{L_\infty(I)}\}, \tag{2.9}$$

and let $Q := PR_N \in \mathrm{GTP}_{2N}$. Then, using (2.6)–(2.9), we obtain

$$Q(t) \leqslant c_2^{-1} \|Q\|_{L_\infty(I)}, \quad t \in A \cup (\mathbb{T} \setminus [a - 1/N, a + 1/N]). \tag{2.10}$$

Let

$$B := [a - 1/N, a + 1/N] \setminus A. \tag{2.11}$$

Observe that (2.10), (2.11), the relation $N \geqslant 1$, and Lemma 3 applied to $Q \in \mathrm{GTP}_{2N}$ imply that

$$m(B) \geqslant m\{t \in \mathbb{T} : Q(t) \geqslant c_2^{-1} \|Q\|_{L_\infty(I)}\} \geqslant \frac{c_1}{N}.$$

Hence

$$m(B) = m\{t \in [a - 1/N, a + 1/N] : P(t) \geqslant c_2^{-1} \|P\|_{L_\infty(I)}\} \geqslant \frac{c_1}{N},$$

and the lemma is proved.

Proof of Lemma 1. First we prove (2.3) for $p = \infty$. By considering a shift, if it is necessary, we need to prove only that

$$|(P'Q)(\pi)| \leqslant c(N + M) \log(N + 1) \|PQ\|_{L_\infty(I)} \tag{2.12}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) with all $r_j \geqslant 1$, and for every $Q \in \mathrm{GTP}_M$. To prove (2.12), it suffices to handle the case where P has only real zeros. This can be seen by Lemmas 5.1–5.3 in [4]. However, for the sake of completeness, we present the arguments. Let $Q \in \mathrm{GTP}_M$ be fixed, and let each $r_j \geqslant 1$ in the representation (1.1) of P be fixed. Let

$$P(x) := \omega \prod_{j=1}^{s} (\sin((x - z_j)/2) \sin((x - \bar{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}, \quad x \in \mathbb{R},$$

where each T_j is a real trigonometric polynomial of degree 1 and nonnegative on the real line. For a fixed $\delta \in (0, \pi)$, we put $I_\delta := [-\pi + \delta, \pi - \delta]$. A simple compactness argument shows that for the fixed exponents $r_1, r_2, \dots, r_s$, and for the fixed parameter $\delta \in (0, \pi)$ there are real trigonometric polynomials $\widetilde{T}_j$ of degree 1 that are nonnegative on the real line and such that with

$$\widetilde{S}(x) := \prod_{j=1}^{s} \widetilde{T}_j(x)^{r_j/2}, \quad x \in \mathbb{R},$$

we have

$$\max_S \left\{ \frac{|(S'Q)(\pi)|}{\|SQ\|_{L_\infty(I_\delta)}} \right\} = \frac{|(\widetilde{S}'Q)(\pi)|}{\|\widetilde{S}Q\|_{L_\infty(I_\delta)}},$$

where the maximum is taken over all $S \in \mathrm{GTP}_N$ of the form

$$S(x) := \omega \prod_{j=1}^{s} (\sin((x - z_j)/2) \sin((x - \overline{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}, \quad x \in \mathbb{R}, \quad (2.13)$$

where each T_j is a real trigonometric polynomial of degree 1 and nonnegative on the real line. To see this, we can normalize S so that

$$\|T_j\|_{L_\infty(I_\delta)} = 1, \quad j = 1, 2, \dots, s,$$

in (2.13); hence, the existence of the above real trigonometric polynomials $\widetilde{T}_j$ of degree 1 follows by a simple compactness argument. Now we can easily show that each $\widetilde{T}_j$ has only real zeros. Suppose to the contrary that $\widetilde{T}_k(x) \geqslant \eta > 0$ on the real line for some k. Consider

$$S_\varepsilon(x) := \prod_{j=1}^{s} T_{\varepsilon,j}(x)^{r_j/2},$$

where $T_{\varepsilon,j} := \widetilde{T}_j$ if $j \neq k$, and

$$T_{\varepsilon,k}(x) := \widetilde{T}_k(x) - \varepsilon \sin^2 \frac{x - \pi}{2}.$$

If $\varepsilon > 0$ is sufficiently small, then S_ε contradicts the maximality of $\widetilde{S}$. This contradiction implies that each $\widetilde{T}_j$, and hence $\widetilde{S}$, has only real zeros. Also, by definition,

$$\frac{|(P'Q)(\pi)|}{\|PQ\|_{L_\infty(I)}} \leqslant \frac{|(P'Q)(\pi)|}{\|PQ\|_{L_\infty(I_\delta)}} \leqslant \frac{|(\widetilde{S}'Q)(\pi)|}{\|\widetilde{S}Q\|_{L_\infty(I_\delta)}} \leqslant (1+\eta) \frac{|(\widetilde{S}'Q)(\pi)|}{\|\widetilde{S}Q\|_{L_\infty(I)}},$$

where, as it can be seen by the Remez-type inequality of Theorem G, the numbers $\eta > 0$ tend to 0 as the numbers $\delta > 0$ tend to 0. This finishes the proof of our claim that it suffices to prove (2.12) only for $P \in \mathrm{GTP}_N$ of the form (1.1) with all $r_j \geqslant 1$ and with only real zeros.

To prove (2.12) for $P \in \mathrm{GTP}_N$ of the form (1.1) with all $r_j \geqslant 1$ and with only real zeros, without loss of generality we may assume that $|P'(\pi)|Q(\pi) = \|P'Q\|_{L_\infty(I)}$. We apply Lemma 4 with $|P'|Q \in \mathrm{GTP}_{N+M}$ and $a = \pi$ to obtain

$$|P'(t)|Q(t) \geqslant \frac{1}{c_2}\|P'Q\|_{L_\infty(I)}, \quad t \in B, \tag{2.14}$$

where

$$B \subset K := \left[\pi - \frac{1}{N+M}, \pi + \frac{1}{N+M}\right], \quad |B| \geqslant \frac{c_1}{N+M}. \tag{2.15}$$

By Lemma 2 with $t = \pi$, we get $\mathcal{M}(|P'|Q, B) \leqslant \mathcal{M}(|P'|Q, K) \leqslant e + 1 < 4$. Thus, $\mathcal{M}(|P'|, B) < 4$, which implies $\mathcal{M}(P, B) < 5$. Denote the different zeros of P in I by α_j with respective multiplicities $r_j \geqslant 1$, $j = 1, 2, \ldots, m$ (then, of course, $2N = \sum_{j=1}^m r_j$). Thus (2.15), $\mathcal{M}(P, B) < 5$, and $r_j \geqslant 1$, $j = 1, 2, \ldots, m$, yield the existence of a $t \in B$ such that

$$|t - \alpha_j| > \frac{c}{N+M}, \quad j = 1, 2, \ldots, m.$$

Fixing this t, we introduce the intervals

$$I_k := \left[t - \frac{2^k c}{N+M}, t + \frac{2^k c}{N+M}\right), \quad k = 0, 1, \ldots, [\log_2 N],$$

and put

$$I_{[\log_2 N]+1} := [t - \pi, t + \pi).$$

Using (2.14) for our t, we can easily deduce that

$$\begin{aligned}\mathcal{M}(P, I_k) &\leqslant \mathcal{M}(|P'|, I_k) + 1 \leqslant \mathcal{M}(|P'|Q, I_k) + 1\\ &\leqslant \left(\frac{e}{2}(N+M)|I_k| + 1\right)\frac{\|P'Q\|_{L_\infty(I)}}{|P'(t)|Q(t)} + 1\\ &= ecc_2 2^k + c_2 + 1 \leqslant c_3 2^k, \quad k = 0, 1, \ldots, [\log_2 N].\end{aligned}$$

Also, because of the choice of t, P does not have a zero in I_0. Therefore we can estimate as follows:

$$\begin{aligned}\frac{|P'(\pi)|Q(\pi)}{\|PQ\|_{L_\infty(I)}} &\leqslant c_2\frac{|P'(t)Q(t)|}{|P(t)Q(t)|} = \frac{c_2}{2}\left|\sum_{j=1}^{m} r_j \operatorname{ctg}\frac{t-\alpha_j}{2}\right| \\ &\leqslant \sum_{k=1}^{[\log_2 N]+1} \frac{c_2}{2} \sum_{j\in I_k\setminus I_{k-1}} r_j \left|\operatorname{ctg}\frac{t-\alpha_j}{2}\right| \leqslant c_2 \sum_{k=1}^{[\log_2 N]+1} \frac{\mathcal{M}(P, I_k)}{c_4|I_{k-1}|} \\ &\leqslant c_2 \sum_{k=1}^{[\log_2 N]+1} \frac{c_3(N+M)}{cc_4} \leqslant c_5(N+M)\log(N+1).\end{aligned}$$

This proves (2.3) for $p=\infty$.

Now we turn to the proof of (2.2). Applying (2.3) with $p=\infty$ to the generalized trigonometric polynomials P and QR with

$$R(t) := \left|\frac{\sin(N+M)t}{\sin t}\right|^{2/q},$$

instead of P and Q, respectively, then using the Nikolskiĭ type inequality

$$\|\chi(P)\|_{L_p(I)} \leqslant (c(1+qN))^{1/q-1/p}\|\chi(P)\|_{L_p(I)}, \quad P\in \mathrm{GTP}_N,\ 0<q<p\leqslant\infty,$$

of Theorem E with $\chi(x)=x$, $p=\infty$ and with PQR instead of P, we obtain

$$\begin{aligned}\|P'QR\|^q_{L_\infty(I)} &\leqslant c_1^q(N+M)^q\log^q(N+1)\|PQR\|^q_{L_\infty(I)} \\ &\leqslant c(N+M)^{q+1}\log^q(N+1)\|PQR\|^q_{L_q(I)}.\end{aligned}$$

Since $R(0)^q=(N+M)^2$, the latter inequality implies

$$|P'(0)|^q Q(0)^q \leqslant c(N+M)^{q-1}\log^q(N+1)\|PQR\|^q_{L_q(I)}.$$

Using this with $P(\cdot+\tau)$ and $Q(\cdot+\tau)$ instead of $P(\cdot)$ and $Q(\cdot)$, respectively (τ is a fixed parameter), we see that

$$\left(\frac{|P'(\tau)|Q(\tau)}{(N+M)\log(N+1)}\right)^q \leqslant \int_{-\pi}^{\pi} c(P(t)Q(t))^q\frac{R(t-\tau)^q}{\|R^q\|_{L_1(I)}}\,dt,$$

because evidently $\|R^q\|_{L_1(I)} \geqslant c_2(N+M)$. Hence, by Jensen's inequality,

$$\chi\left(\int_a^b S(t)w(t)\,dt\right) \leqslant \int_a^b \chi(S(t))w(t)\,dt \tag{2.16}$$

(cf. [1, p. 414, E.20]) applied with $[a,b] = [-\pi,\pi]$, $S = c(PQ)^q$, and

$$w(t) = \frac{R(t)^q}{\|R^q\|_{L_1(I)}},$$

we obtain

$$\chi\left(\left(\frac{|P'(\tau)|Q(\tau)}{(N+M)\log(N+1)}\right)^q\right) \leqslant \int_{-\pi}^{\pi} \chi(c(P(t)Q(t))^q)\frac{R(t-\tau)^q}{\|R^q\|_{L_1(I)}}\,dt.$$

Integrating with respect to τ and using Fubini's theorem yields the inequality of the lemma. This finishes the proof.

Proof of Theorem 1. By the remark made after Lemma 1, it remains to show that (2.1) is sharp. To see this, we may assume that N and M are positive integers. Let $q = \min(1,p)$, and let

$$S_{N+M}(t) := \left|\frac{\sin((N+M+1)t)}{\sin t}\right|^{2/q} = C_{N+M}\left|\prod_{k=1}^{N+M}\sin\frac{t-\alpha_k}{2}\prod_{k=1}^{N+M}\sin\frac{t+\alpha_k}{2}\right|^{2/q},$$

where

$$\alpha_k := \frac{k\pi}{N+M+1}, \quad k = 1,2,\ldots,N+M.$$

Let P_N and Q_M be generalized trigonometric polynomials of degree $2N/q$ and $2M/q$, respectively, defined by

$$P_N(t) := \begin{cases} \left|\prod_{k=1}^{2N}\sin\frac{t-\alpha_k}{2}\right|^{2/q} & \text{if } N \leqslant M, \\ \left|\prod_{k=1}^{N+M}\sin\frac{t-\alpha_k}{2}\prod_{k=2M+1}^{N+M}\sin\frac{t+\alpha_k}{2}\right|^{2/q} & \text{if } N > M, \end{cases}$$

and

$$Q_M(t) := \frac{S_{N+M}(t)}{P_N(t)}.$$

It is easy to check that

$$S_{N+M}(0) = \max_{t\in I} S_{N+M}(t).$$

Hence,

$$\frac{\max_{t\in I}|P_N'(t)|Q_M(t)}{\max_{t\in I}P_N(t)Q_M(t)} \geqslant \frac{|P_N'(0)|Q_M(0)}{P_N(0)Q_M(0)} = \begin{cases} \frac{1}{q}\sum_{k=1}^{2N}\operatorname{ctg}\frac{\alpha_k}{2} \geqslant c(N+M)\log(N+1) & \text{if } N \leqslant M, \\ \frac{1}{q}\sum_{k=1}^{2M}\operatorname{ctg}\frac{\alpha_k}{2} \geqslant c(N+M)\log(M+1) & \text{if } N > M. \end{cases}$$

This proves the sharpness of (2.1) for $p=\infty$.

Now, let $0<p<\infty$. We have

$$P_N'(t)Q_M(t)=\begin{cases}\frac{1}{q}S_{N+M}(t)\sum_{k=1}^{2N}\operatorname{ctg}\frac{t-\alpha_k}{2} & \text{if } N\leqslant M,\\ \frac{1}{q}S_{N+M}(t)\left(\sum_{k=1}^{N+M}\operatorname{ctg}\frac{t-\alpha_k}{2}+\sum_{k=2M+1}^{N+M}\operatorname{ctg}\frac{t+\alpha_k}{2}\right) & \text{if } N>M.\end{cases}$$

We shall give a lower estimate of the L_p norm of this polynomial over the interval $[0,\alpha_1/2]$. Evidently,

$$S_{N+M}(t)\geqslant (c(N+M))^{2/q},\quad 0\leqslant t\leqslant \alpha_1/2.$$

On the other hand, if $t\in[0,\alpha_1/2]$ and $N\leqslant M$, then

$$\left|\sum_{k=1}^{2N}\operatorname{ctg}\frac{t-\alpha_k}{2}\right|\geqslant\sum_{k=1}^{N}\operatorname{ctg}\frac{\alpha_k}{2}\geqslant c(N+M)\log(N+1),$$

while if $t\in[0,\alpha_1/2]$ and $N>M$, then

$$\begin{aligned}&\left|\sum_{k=1}^{N+M}\operatorname{ctg}\frac{t-\alpha_k}{2}+\sum_{k=2M+1}^{N+M}\operatorname{ctg}\frac{t+\alpha_k}{2}\right|\\ &\quad\geqslant\sum_{k=1}^{2M}\operatorname{ctg}\frac{\alpha_k}{2}-\sum_{k=2M+1}^{N+M}\left|\operatorname{ctg}\frac{t-\alpha_k}{2}+\operatorname{ctg}\frac{t+\alpha_k}{2}\right|\\ &\quad\geqslant c_1(N+M)\log(M+1)-c_2\sum_{k=2M+1}^{N+M}\frac{\sin t}{\sin^2\frac{\alpha_k}{2}}\\ &\quad\geqslant c_1(N+M)\log(M+1)-c_3\frac{N+M}{M}\\ &\quad\geqslant c_4(N+M)\log(M+1).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_0^{\alpha_1/2}|P_N'(t)Q_M(t)|^p\,dt&\geqslant\frac{c^{1+p}}{N+M}((N+M)^{1+2/q}\log(\min(N,M)+1))^p\\ &\geqslant c^{1+p}(N+M)^{p-1+2p/q}\log^p(\min(N,M)+1).\end{aligned}$$

Compared with

$$\|P_NQ_M\|_{L_p}^p=\|S_{N+M}\|_{L_p}^p\leqslant c^{1+p}(N+M)^{2p/q-1},$$

this proves that

$$\|P_N'Q_M\|_{L_p}\geqslant c^{1+p}(N+M)\log(\min(N,M)+1)\|P_NQ_M\|_{L_p}.$$

§3. The case of generalized algebraic polynomials

In this section we improve estimates (1.7) and (1.10).

Theorem 2. *Let* $0 < p \leqslant \infty$. *We have*

$$\|\sqrt{1-x^2} f'(x) g(x)\|_{L_p(J)} \leqslant c^{1+1/p}(N+M)\log(\min(N, M+1)+1)\|fg\|_{L_p(J)} \quad (3.1)$$

for any $f \in \mathrm{GAP}_N$ *and any* $g \in \mathrm{GAP}_M$ *such that the roots of* f *and* g *have multiplicities at least* 1. *Moreover,* (3.1) *is sharp apart from the constant* $c^{1+1/p}$.

Proof. Using the substitution (1.9) in (2.3), we obtain

$$\|\sqrt{1-x^2} f'(x) g(x)\|_{L_p(J)} \leqslant c^{1+1/p}(N+M)\log(N+1)\|fg\|_{L_p(J)}, \quad 0 < p \leqslant \infty.$$

Interchanging the roles of f and g, and using the Bernstein type inequality (1.8) with fg instead of f, we obtain (3.1). The sharpness follows from the sharpness of the trigonometric analog (2.1).

Theorem 3. *Let* χ *be a nonnegative, monotone nondecreasing, convex function defined on* $[0,\infty)$. *Then*

$$\int_{-1}^{1} \chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) dx \leqslant 2\int_{-1}^{1} \chi(c(f(x)g(x))^q)\, dx \quad (3.2)$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with all* $r_j \geqslant 1$, *for every* $g \in \mathrm{GAP}_M$, *and for every* $0 < q \leqslant 1$. *In particular,*

$$\|f'g\|_{L_p(J)} \leqslant c^{1+1/p}(N+M)^2\|fg\|_{L_p(J)}, \quad 0 < p \leqslant \infty, \quad (3.3)$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with all* $r_j \geqslant 1$, *and for every* $g \in \mathrm{GAP}_M$. *The latter inequality is sharp up to the factor* $c^{1+1/p}$ *for all* $N, M \geqslant 1$.

Proof. Inequality (3.3) readily follows from (3.2) by putting $q = \min(1,p)$ and $\chi(x) = x^{\max(1,p)}$. In order to prove (3.2), we mention that (3.3) for $p = \infty$, i.e.,

$$\|f'g\|_{L_\infty(J)} \leqslant c(N+M)^2\|fg\|_{L_\infty(J)} \quad (3.4)$$

is nothing but the corresponding inequality in (1.10) proved in [4].

Now consider an ordinary algebraic polynomial $h \geqslant 0$ on $[-1,1]$ of degree at most $N+M$ such that

$$\int_{-1}^{1} h(t)\,dt \leqslant \frac{c}{(N+M)^2} \quad \text{and} \quad h(1)=1. \tag{3.5}$$

The existence of such a polynomial is guaranteed by known estimates for the Christoffel functions of the orthogonal Legendre polynomials (cf., e.g., Freud [9, Problem 10 on p. 132]). Let $0 \leqslant x \leqslant 1$, and let y be a fixed parameter to be specified later. We apply (3.4) on the interval $[x-1,x]$ instead of $[-1,1]$ for the generalized polynomials $f(x)$ and $g(x)h(x-y)^{1/q}$ (instead of $g(x)$):

$$|f'(x)|g(x)h(x-y)^{1/q} \leqslant c_2 q^{-2}(N+M)^2 \max_{x-1\leqslant t\leqslant x} f(t)g(t)h(t-y)^{1/q}, \quad 0 \leqslant x,y \leqslant 1.$$

Combining this with the Nikolskiĭ type inequality of Theorem F with $\chi(x)=x$, $p=\infty$, and with $f(t)g(t)h(t-y)^{1/q}$ instead of f on the interval $[x-1,x]$ instead of J, we get

$$(|f'(x)|g(x))^q h(x-y) \leqslant c_1(N+M)^{2q+2} \int_{x-1}^{x} (f(t)g(t))^q h(t-y)\,dt, \quad 0 \leqslant x,y \leqslant 1.$$

Putting $y=x-1$ and recalling (3.5), we deduce that

$$(|f'(x)|g(x))^q \leqslant c(N+M)^{2q+2} \int_{x-1}^{x} (f(t)g(t))^q h(t-x+1)\,dt$$

$$\leqslant c(N+M)^{2q} \int_{x-1}^{x} \frac{(f(t)g(t))^q h(t-x+1)}{\int_{x-1}^{x} h(u-x+1)\,du}\,dt, \quad 0 \leqslant x \leqslant 1.$$

Rearranging this and using (2.16) (Jensen's inequality) with $[a,b]=[x-1,x]$, $S=c(fg)^q$, and

$$w(t) := \frac{h(t-x+1)}{\int_{x-1}^{x} h(u-x+1)\,du} = \frac{h(t-x+1)}{\int_0^1 h(v)\,dv}$$

(note that $\int_{x-1}^{x} w(t)\,dt = 1$), we obtain

$$\chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) \leqslant \frac{\int_{x-1}^{x} \chi(c(f(t)g(t))^q)h(t-x+1)\,dt}{\int_0^1 h(v)\,dv} = \frac{\int_{-1}^{1} \chi(c(f(t)g(t))^q)\varphi_{[x-1,x]}(t)h(t-x+1)\,dt}{\int_0^1 h(v)\,dv},$$

where $\varphi_{[a,b]}(t)$ is the characteristic function of the interval $[a,b]$. Integrating both sides with respect to x on $[0,1]$, and using Fubini's theorem, we get

$$\int_0^1 \chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right)dx \leqslant \frac{\int_{-1}^1 \chi(c(f(t)g(t))^q)\int_0^1 \varphi_{[x-1,x]}(t)h(t-x+1)\,dx\,dt}{\int_0^1 h(v)\,dv}.$$

Here an easy calculation shows that

$$\int_0^1 \varphi_{[x-1,x]}(t)h(t-x+1)\,dx = \begin{cases} \int_0^{t+1} h(v)\,dv & \text{if } -1 \leqslant t \leqslant 0, \\ \int_t^1 h(v)\,dv & \text{if } 0 \leqslant t \leqslant 1, \end{cases}$$

which can be estimated by $\int_0^1 h(v)\,dv$ in both cases. Hence, (3.2) is proved without the factor 2 when the integral is taken over $[0,1]$ rather than $[-1,1]$ on the left-hand side. Similar arguments yield (3.2) without the factor 2 when the integral is taken over $[-1,0]$ rather than $[-1,1]$ on the left-hand side. In conclusion, (3.2) is true with the factor 2.

To prove the sharpness of (3.3), let $u_n^{(\lambda)}$ be the ultraspherical Jacobi polynomial of degree n with parameter $\lambda \geqslant 0$ normalized in such a way that $u_n^{(\lambda)}(1) = 1$. Then the absolute maximum of $u_n^{(\lambda)}$ on $[-1,1]$ is attained at ± 1 (cf. [10, p. 168]). Without loss of generality we may assume that $N \geqslant 1$ and $M \geqslant 1$ are integers. Let f_N be the monic polynomial of degree N that has N roots of the polynomial $u_{N+M}^{(\lambda)}(x)$ closest to 1, and let g_M be defined by $f_N g_M = u_{N+M}^{(\lambda)}$. Then for $p = \infty$ we get

$$f_N'(1)g_M(1) = \frac{f_N'(1)}{f_N(1)}u_{N+M}^{(\lambda)}(1) \geqslant \frac{\|f_N g_M\|_{L_p(J)}}{1-x_1} \geqslant c(N+M)^2\|f_N g_M\|_{L_p(J)},$$

where x_1 is the largest root of $u_{N+M}^{(\lambda)}$ (cf. [10, (6.6.6)]), where it was shown that $1 - x_1 \leqslant c(N+M)^{-2}$). We remark that in the rest of this proof $c, c_1, c_2, \ldots$ may depend on λ. Now, let $0 < p < \infty$, and let $\lambda > 2/p$. Using the estimates

$$|u_n^{(\lambda)}(\cos t)| \leqslant \begin{cases} c_2 & \text{if } 0 \leqslant t \leqslant c_1/n, \\ c_3(nt)^{-\lambda} & \text{if } c_1/n \leqslant t \leqslant \pi/2 \end{cases}$$

(cf. [10, (7.33.6)]), we get

$$\begin{aligned}\|f_N g_M\|_{L_p(J)} &= \|u_{N+M}^{(\lambda)}\|_{L_p(J)} = 2^{1/p}\|u_{N+M}^{(\lambda)}\|_{L_p([0,1])}\\ &= 2^{1/p}\left(\int_0^{\pi/2} |u_{N+M}^{(\lambda)}(\cos t)|^p \sin t\,dt\right)^{1/p}\\ &\leqslant c_4^{1/p}\left(\int_0^{c_1/(N+M)} t\,dt + (N+M)^{-p\lambda}\int_{c_1/(N+M)}^{\pi/2} t^{1-p\lambda}\,dt\right)^{1/p}\\ &\leqslant c_5^{1+1/p}(N+M)^{-2/p}.\end{aligned}$$

On the other hand, Markov's inequality (cf. Theorem D with $p := \infty$ and $g := 1$), together with the mean value theorem and the fact that the absolute maximum value 1 of u_n^λ on $[-1,1]$ is attained at 1, yields a constant c_6 such that

$$x_1 \leqslant 1 - \frac{2c_6}{(N+M)^2}$$

and

$$u_{N+M}^{(\lambda)}(x) \geqslant \frac{1}{2} \quad \text{if } 1 - \frac{c_6}{(N+M)^2} \leqslant x \leqslant 1.$$

Thus,

$$\begin{aligned}\|f_N' g_M\|_{L_p(J)} &\geqslant \left(\int_{1-c_6/(N+M)^2}^{1} \left|\frac{f_N'(x)}{f_N(x)} u_{N+M}^{(\lambda)}(x)\right|^p dx\right)^{1/p}\\ &\geqslant \left(\int_{1-c_6/(N+M)^2}^{1} \left|\frac{u_{N+M}^{(\lambda)}(x)}{x-x_1}\right|^p dx\right)^{1/p}\\ &\geqslant \left(\frac{c_6}{(N+M)^2}\left(\frac{(N+M)^2}{c_6}\right)^p\left(\frac{1}{2}\right)^p\right)^{1/p} = \frac{1}{2}{c_6}^{1/p-1}(N+M)^{2-2/p}\\ &= \frac{1}{2}{c_6}^{1/p-1}c_5^{-1-1/p}(N+M)^2\|f_N g_M\|_{L_p(J)},\end{aligned}$$

which proves the sharpness of (3.3).

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