H. D. Alber, Равномерные в L_2 асимптотики амплитуды рассеяния в области высоких частот,
Зап. научн. сем. ЛОМИ, 1991, том 195, 5–13

H.D.Alber

UNIFORM L_2-ASYMPTOTICS OF THE SCATTERING AMPLITUDE IN THE HIGH FREQUENCY LIMIT

In this article we study scattering of waves at a bounded domain B in $\mathbb{R}^3$ with smooth boundary ∂B and derive an expression which is uniformly asymptotic to the scattering amplitude belonging to this problem. More precisely, the difference between the scattering amplitude and this asymptotic expression tends to zero in the L_2-norm over the unit sphere. This asymptotic expression allows to read off clearly the form of the diffraction peak of the scattering amplitude in forward scattering direction.

This is an extended version of a seminar talk given at the Steklov Institute of Mathematics. Together with this talk, this article is based on the paper [3] written in cooperation with A.G.Ramm. We sketch some proofs here. Complete proofs can be found in [3]. Other articles of review character, which contain some extensions of the results discussed here, are [1, 2].

For $\Omega = \mathbb{R}^3 \setminus B$ let $u(x,k)$ be the solution of

$$\begin{aligned} &\Delta u(x,k) + k^2 u(x,k) = 0, \quad x \in \Omega, k > 0, \\ &u(x,k) + u_0(x, s_0, k) = 0, \quad x \in \partial\Omega, \\ &\partial_r u = iku + o(r^{-1}), \quad r = |x| \longrightarrow \infty, \end{aligned} \tag{1}$$

where $s_0 \in \mathbb{R}^3$ is a unit vector and

$$u_0(x, s_0, k) = \exp(iks_0 \cdot x) \tag{2}$$

is an incoming plane wave. u represents the wave scattered at B. It is well known that the limit

$$f(s, s_0, k) = \lim_{r \to \infty} r e^{-ikr} u(rs, s_0, k)$$

exists uniformly with respect to $s \in S_2 = \{ \omega \in \mathbb{R}^3 \mid |\omega| = 1 \}$. The function $f : S_2 \times S_2 \times \mathbb{R}^+ \longrightarrow \mathbb{C}$ is called scattering amplitude. We construct an expression $f_\infty(s, s_0, k)$ which satisfies

$$\lim_{k\to\infty}\int_{|s|=1}|f(s,s_0,k)-f_\infty(s,s_0,k)|^2 dS_s=0, \tag{3}$$

if the domain B and the vector s_0 fulfill the following two conditions:

(C1) Every ray with incoming direction s_0 reflected at ∂B according to the laws of geometrical optics has at most one point in common with ∂B.

(C2) There exists a constant $\beta>0$ with

$$-(x\cdot n(x))\geq\beta$$

for $x\in\partial B$, where $n(x)$ denotes the normal to ∂B pointing from Ω to B.

To construct f_∞ and to sketch the proof of (3) we need some definitions and notations. We choose a Cartesian coordinate system (x_1,x_2,x_3) with the x_3-axis pointing into the direction of s_0, hence $s_0=(0,0,1)$. For $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ let $x'=(x_1,x_2)$ denote the projection of x to the x_1,x_2 plane. Let

$$S'=\{x'\in\mathbb{R}^2:\text{ there exists } x_3\in\mathbb{R} \text{ with } (x',x_3)\in B\}$$

be the shadow projection of B, and let

$$\partial B_+=\{(x',x_3)\in\mathbb{R}^3: x'\in S',\ x_3=\inf_{(x',t)\in B} t\}$$

$$\partial\partial B_+=\{x\in\partial B: s_0\cdot n(x)=0\}$$

$$\partial B_-=\partial B\setminus\overline{\partial B_+}.$$

$\overline{\partial B_+}=\partial B_+\cup\partial\partial B_+$ is the illuminated part of ∂B. Let

$$S=\{(x',x_3)\in\Omega: x'\in S',\ x_3>\inf_{(x',t)\in B} t\}.$$

S is the shadow region. We now define an approximation to u. Choose a family $\{\chi_\alpha\}_{\alpha>0}$ of functions $\chi_\alpha\in C_0^\infty(S')$ with $0\leq\chi_\alpha(y)\leq 1$, $\chi_\alpha(y)=1$ for $y\in S'$ with dist $(y,\partial S')\geq\alpha^{-1}$, and such that the function

$$\alpha\mapsto\chi_\alpha:(0,\infty)\longrightarrow C^1(\mathbb{R}^2)$$

is continuous, where $C^1(\mathbb{R}^2)$ is equipped with the norm

$\|\varphi\|_\infty + \|\nabla\varphi\|_\infty$. With this definition of χ_α the function $y \mapsto \chi_\alpha(y'): \partial B \to \mathbb{R}$ is equal to 1 outside a neighborhood o of the set $\partial\partial B_+$. The size of this neighborhood decreases if α tends to infinity. We approximate u by the function

$$u_\alpha(x,k) = \frac{1}{2\pi} \int_{\partial B_+} \frac{e^{ik|x-y|}}{|x-y|} \chi_\alpha(y') \partial_n u_0(y, s_0, k)\, dS_y, \qquad (4)$$

a modified Kirchhoff approximation. The following theorem shows in what sense u_α is an approximation to u. It is our basic result, and the results about the scattering amplitude follow from it.

THEOREM 1. There exists a function $k \mapsto \alpha(k)$ with $\alpha(k) \to \infty$ and a function $k \mapsto c(k) > 0$ with $c(k) \to 0$ for $k \to \infty$ such that

$$\|u(\cdot,k) - u_{\alpha(k)}(\cdot,k)\|_{m,\Omega_R} \leq R^{1/2}(1+|k|^m)c(k) \qquad (5)$$

for $R > 1$, $m = 0,1$. Here $\|\cdot\|_{m,\Omega_R}$ is the usual Sobolev norm with $\Omega_R = \{x \in \Omega : |x| < R\}$.

The proof uses the following two lemmas:

LEMMA 2. Let B satisfy (C2), and let $r = |x|$. If u solves the problem

$$(\Delta + k^2)u = f \quad \text{in } \Omega, k > 0,$$

$$u = \mu \quad \text{on } \partial B_-, \qquad (6)$$

$$\lim_{R\to\infty} \int_{|x|=R} r \left|\frac{\partial u}{\partial r} - iku\right|^2 dS = 0,$$

with $\|rf\|_{0,\Omega} < \infty$, then

$$\|\nabla u\|^2_{0,\Omega_R} + k^2\|u\|^2_{0,\Omega_R} \leq CR[\log R\|rf\|^2_{0,\Omega} + \|\nabla_T\mu\|^2_{\partial\Omega} + (1+k^2)\|\mu\|^2_{\partial\Omega}], \qquad (7)$$

for all $R > 1$, where $\nabla_T\mu$ denotes the tangential gradient on ∂B , and where the constant C only depends on Ω .

LEMMA 3. (i) The function u_α defined in (2) belongs to $C^\infty(\overline{\Omega})$. (ii) For $\alpha > 0$ and $x \in \partial B$ let

$$\tilde{u}_\alpha(x,k) = \chi_\alpha(x')\,u(x,k).$$

Then

$$\tilde{u}_\alpha(x,k) - u_\alpha(x,k) = O(k^{-1}),$$

$$\nabla_T(\tilde{u}_\alpha(x,k) - u_\alpha(x,k)) = O(1),$$

for $k \to \infty$, uniformly with respect to $x \in \partial B$ and with respect to all α in compact intervals of $[1,\infty)$.

The proofs of these lemmas are given in [3]. The proof of Lemma 2 uses an estimate of Morawetz-Ludwig [5]; the proof of Lemma 3 is based on asymptotic evaluation of the boundary integral appearing in the definition (4) of u_α.

SKETCH OF THE PROOF OF THEOREM 1. We show that there exists a function $k \mapsto \alpha(k)$ with $\alpha(k) \to \infty$ for $k \to \infty$ such that

$$\|u(\cdot,k) - u_{\alpha(k)}(\cdot,k)\|_{0,\partial B} = o(1) \tag{8}$$

$$\|\nabla_T[u(\cdot,k) - u_{\alpha(k)}(\cdot,k)]\|_{0,\partial B} = o(k) \tag{9}$$

for $k \to \infty$. Since the function $v = u - u_\alpha$ satisfies $\Delta v + k^2 v = 0$ in Ω and the radiation condition (6) of Lemma 2, the assumptions of this lemma are satisfied for v, and therefore Theorem 1 follows from the estimate (7) and from (8), (9).

With the function $\tilde{u}_\alpha$ defined in Lemma 3 we have

$$\|u - u_\alpha\|_{0,\partial B} \le \|u - \tilde{u}_\alpha\|_{0,\partial B} + \|\tilde{u}_\alpha - u_\alpha\|_{0,\partial B}. \tag{10}$$

Lemma 3 yields that for every positive integer n there exists $k_0 = k_0(n)$ with

$$\sup_{1 \le \alpha \le n+1} \|\tilde{u}_\alpha(\cdot,k) - u_\alpha(\cdot,k)\|_{0,\partial B} \le \frac{1}{n} \tag{11}$$

for $k \ge k_0$. Let $\alpha \mapsto k_1(\alpha)$ be a continuous, strictly increasing function with $k_1(n) \ge k_0(n)$ and with $k_1(\alpha) \to \infty$ for $\alpha \to \infty$, and let $\alpha \mapsto k_2(\alpha)$ be a continuous, strictly increasing function with $k_2(\alpha) \ge k_1(\alpha)$. For $2 \le n \le \alpha \le n+1$ we then have $k_2(\alpha) \ge k_2(n) \ge k_1(n) \ge k_0(n)$, and therefore the inequality (11) yields

$$\|\tilde{u}_\alpha(\cdot,k_2(\alpha))-u_\alpha(\cdot,k_2(\alpha))\|_{0,\partial B}\le\frac{1}{n}\le\frac{1}{\alpha-1}. \tag{12}$$

Let $k\mapsto\alpha(k)$ be the inverse of $\alpha\mapsto k_2(\alpha)$. From (12) we then obtain

$$\|\tilde{u}_{\alpha(k)}(\cdot,k)-u_{\alpha(k)}(\cdot,k)\|_{0,\partial B}\le\frac{1}{\alpha(k)-1}=o(1),\ k\to\infty.$$

Note that $\alpha(k)$ in this formula can be any continuous, strictly increasing function with $\alpha(k)\longrightarrow\infty$ and with $\alpha(k)\le\alpha_1(k)$, where $k\mapsto\alpha_1(k)$ is the inverse of $\alpha\mapsto k_1(\alpha)$. Thus, (10) implies that for the proof of (8) it suffices to show that

$$\|u-\tilde{u}_\alpha\|_{0,\partial B}=o(1),\ \alpha\to\infty.$$

To prove this relation, observe that (1) and (2) imply

$$u(x,k)=-u_0(x,s_0,k)=-e^{ikx_3}\quad\text{for } x\in\partial B.$$

Therefore

$$\|u-\tilde{u}_\alpha\|^2_{\partial B}=\|u-\chi_\alpha u\|^2_{\partial B}=\int_{\partial B}|1-\chi_\alpha(x')|^2 dS_x\to 0$$

for $\alpha\mapsto\infty$, since the definition of χ_α yields $0\le\chi_\alpha\le 1$ and

$$\text{meas}\left[\operatorname{supp}(x\mapsto(1-\chi_\alpha(x')))\right]\longrightarrow 0$$

for $\alpha\longrightarrow\infty$. This proves (8). To finish the proof of Theorem 1 it remains to show that (9) holds, which is proved analogously.

We use Theorem 1 to compute the high frequency asymptotics of the scattering amplitude $f(s,s_0,k)$. Let

$$f_\alpha(s,s_0,k)=\lim_{r\to\infty} r e^{-ikr} u_\alpha(rs,k). \tag{13}$$

From the definition of u_α in (4) we obtain by a symple calculation that

$$f_\alpha(s,s_0,k)=\frac{1}{2\pi}\int_{\partial B_+} e^{-iks\cdot y}\chi_\alpha(y')\partial_n u_0(y,s_0,k)dS_y$$

$$= \frac{ik}{2\pi} \int_{\partial B_+} e^{ik(s_0 - s)\cdot y} s_0 \cdot n(y) \chi_\alpha(y') dS_y . \tag{14}$$

The limit in (13) is uniform with respect to $s \in S_2$.

THEOREM 4. Let $k \mapsto \alpha(k)$ and $k \mapsto c(k)$ be the functions defined in Theorem 1. Then

$$\int_{|s|=1} |f(s,s_0,k) - f_{\alpha(k)}(s,s_0,k)|^2 dS_s \le 2c(k)^2 \to 0$$

for $k \to \infty$.

PROOF. Fix $k > 0$. From the estimate in Theorem 1 it follows that there exists a sequence $\{r_m\}_{m=1}^{\infty}$ with $r_m \to \infty$ and with

$$\lim_{m\to\infty} \int_{|s|=1} |r_m e^{-ikr_m} (u(r_m s,k) - u_{\alpha(k)}(r_m s,k))|^2 dS_s$$

$$= \lim_{m\to\infty} \int_{|x|=r_m} |u(x,k) - u_{\alpha(k)}(x,k)|^2 dS_x \le 2c(k)^2 .$$

The statement of the theorem follows from this estimate and from (13), since $r e^{-ikr} u(rs,k)$ tends to $f(s,s_0,k)$ for $r \to \infty$; uniformly with respect to $s \in S_2$.

THEOREM 5. We have

(i) $$\lim_{k\to\infty} \int_{|s|=1} |f(s,s_0,k)|^2 dS_s = 2 \,\mathrm{meas}\, S'$$

(ii) $$\mathrm{Im}\, f(s_0,s_0,k) = \frac{k}{2\pi} \,\mathrm{meas}\, S' + o(k), \quad k \to \infty .$$

The proof of (i) is given in [3] and is based on Theorem 4, whereas (ii) is an immediate consequence of (i) and of the well known relation

$$f(s,s_0,k) - \overline{f(s_0,s,k)} = \frac{ik}{2\pi} \int_{|\omega|=1} \overline{f(\omega,s,k)} f(\omega,s_0,k) dS_\omega ,$$

cf. [4, 6].

Theorem 4 shows that the scattering amplitude behaves asymptotically like the expression $f_\alpha(s, s_0, k)$.

In the next theorem it is shown that f_α can be replaced by another term, which gives moze information about the structure of the scatterint amplitude. In particular, the form of the diffraction peak of the scattering amplitude in forward scattering direction can be read off clearly from this term.

To formulate this theorem, we need some notations. Let

$$B(s_0)=\left\{x=(x_1,x_2,x_3)\in\mathbb{R}^3 : x'\in S',\ x_3>\inf_{(x',t)\in B} t\right\},$$

and let $\chi_{B(s_0)}$ be the caracteristic function of this set. Choose $R>0$ such that $\overline{B}\subseteq\{|x|<R\}$ and let $\chi_R\in C^\infty(\mathbb{R})$ satisfy $0\le\chi_R(t)\le 1$ and

$$\chi_R(t)=\begin{cases}1, & t<R\\ 0, & t>R+1.\end{cases}$$

By partial integration it follows from (14) that

$$\begin{aligned}f_\alpha(s,s_0,k)&=\frac{ik}{2\pi}\int_{\partial B_+} e^{ik(s_0-s)\cdot y}\chi_\alpha(y')\chi_R(y_3)\,s_0\cdot n(y)\,dS_y\\ &=\frac{1}{2\pi}k^2(1-s\cdot s_0)\int_{B(s_0)} e^{ik(s_0-s)\cdot y}\chi_\alpha(y')\chi_R(y_3)\,dy\\ &\quad-\frac{ik}{2\pi}\int_{B(s_0)} e^{ik(s_0-s)\cdot y}\chi_\alpha(y')\chi_R'(y_3)\,dy.\end{aligned}$$

The second term can be replaced by a simpler term:

THEOREM 6. Let $\chi_{S'}$ denote the characteristic function of the shadow projection S', and let

$$\hat{\chi}_{S'}(\xi)=\frac{1}{2\pi}\int_{R^2} e^{-i\xi\cdot y}\chi_{S'}(y)\,dy,\quad \xi\in\mathbb{R}^2$$

$$\hat{\chi}_{\alpha,R}(\xi)=\frac{1}{2\pi}\int_{R^3} e^{-i\xi\cdot y}\chi_{B(s_0)}(y)\chi_\alpha(y')\chi_R(y_3)\,dy,\quad \xi\in\mathbb{R}^3$$

be the Fourier transforms of $\chi_{S'}$ and of $\chi_{B(s_0)}\chi_\alpha\chi_R$.
Then

$$\lim_{k\to\infty}\int_{|s|=1}\Big|f(s,s_0,k)-k^2(1-s\cdot s_0)\hat{\chi}_{\alpha(k),R}(k(s-s_0))$$

$$-ik\hat{\chi}_{S'}(ks')\chi_{[1/2,1]}(s\cdot s_0)\Big|^2 dS_s=0,$$

where $\chi_{[1/2,1]}$ is the characteristic function of the interval $[1/2,1]$, and $s'=(s_1,s_2)\in\mathbb{R}^2$ if $s=(s_1,s_2,s_3)\in S_2$.

Observe that for all sufficiently large k the function

$$s\mapsto ik\hat{\chi}_{S'}(ks')\chi_{[1/2,1]}(s\cdot s_0)$$

defined on the unit sphere S_2 represents a "peak" concentrated near to the point $s_0=(0,0,1)$ on the unit sphere, and describes the Diffraction peak of the scattering amplitude in forward scattering direction.

The proof of this theorem is given in [3].

ACKNOWLEDGEMENT. The author wants to thank the members of the group of mathematicians of the Steklov Institute working in inverse problems and in particular Yu.Kurylev for their hospitality during the stay at the Steklov Institute.

References

1. Alber H.D. The highest order singularities of the scattering kernel. To appear in: Lazarov, Petkov (ed.): Integral equations and inverse problems. Proceedings of a conference held in Varna. Longman, Harlow.
2. Alber H.D., Leis R. Initial boundary value and scattering problems in mathematical physics. In: Hildebrandt S. Leis R. (ed.): Partial differential equations and calculus of variations, Lecture Notes in Mathematics 1988, 1357, Springer, Berlin.
3. Alber H.D., Ramm A.G. Scattering amplitude and algorithm for solving the inverse scattering problem for a class of non-convex obstacles, J.Math.Anal.Appl. 1986, 117, 570-597.

4. Lax P.D., Phillips R.S. Scattering theory. Academic Press, New York 1967.
5. Morawetz C.S., Ludwig D. An inequality for the reduced wave operator and the justification of geometrical optics. Comm. Pure Appl. Math. 1968, 21, 187-203.
6. Ramm A.G. Scattering by obstacles. Reidel, Dordrecht, 1986.