A. M. Abylaeva, Boundedness, compactness for a class of fractional integration operators of Weyl type,
Eurasian Math. J., 2016, том 7, номер 1, 9–27

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 1 (2016), 9 – 27

BOUNDEDNESS, COMPACTNESS FOR A CLASS OF FRACTIONAL INTEGRATION OPERATORS OF WEYL TYPE

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Communicated by E.D. Nursultanov

Key words: fractional integration operator, Weyl operator, Riemann-Liouville operator, Hadamard operator, Erdelyi-Kober operator, boundedness, compactness.

AMS Mathematics Subject Classification: 26A33, 26D10, 47G10.

Abstract. We establish criteria for the boundedness and compactness for a class of operators of fractional integration involving the Weyl operator.

1 Introduction

Let $I=(a,b)$, $0\le a<b\le\infty$, $0<q,p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$. Let u, v be almost everywhere positive and locally integrable functions on I. By $L_{p,u}\equiv L_p(u,I)$ we denote the set of all measurable functions f on I such that

$$\|f\|_{p,u}=\left(\int\limits_a^b|f(x)|^p u(x)dx\right)^{\frac{1}{p}}<\infty.$$

In the case $u\equiv 1$ we write $L_p\equiv L_p(I)$. Let W be a positive strictly increasing and locally absolutely continuous function on I. Suppose $\frac{dW(x)}{dx}\equiv w(x)$ for almost everywhere $x\in I$.

Let $1>\alpha>0$. We consider the operator

$$K_{\alpha,\beta}f(x)=\int\limits_x^b\frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}},\ \ x\in I. \tag{1.1}$$

In the case $\beta=0,\ u\equiv 1$ the dual operator to operator (1.1) has the form

$$K^*_{\alpha,\beta}f(x)=\int\limits_a^x\frac{f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}},\ \ x\in I. \tag{1.2}$$

Operator (1.2) is called [12] the operator of fractional integration of the function f of the function W. Weighted estimates for operator (1.2) were previously considered in [9], [1].

When $W(x) = x$, $u \equiv 1$, $\beta = 0$ operator (1.1) is the Weyl operator

$$I_\alpha^* f(x) = \int_x^b \frac{f(s)ds}{(s-x)^{1-\alpha}}, \quad x \in I, \tag{1.3}$$

which is dual to the Riemann-Liouville operator

$$I_\alpha g(s) = \int_a^s \frac{g(x)dx}{(s-x)^{1-\alpha}}, \quad s \in I. \tag{1.4}$$

Operators (1.3) and (1.4) acting from the weighted space $L_{p,u}$ to the weighted space $L_{q,v}$ are investigated in papers [2], [3], [4], [8], [10], [11] and others, where necessary and sufficient conditions for their boundedness, compactness are obtained for various relations between the parameters α, p, q and under various assumptions regarding the weight functions u and v. Two-sided estimates of their norms are also obtained.

We investigate operator (1.1) acting from the space $L_{p,w}$ to $L_{q,v}$. From the obtained results new assertions follow, in simple terms, for operators (1.3) and (1.4), generalizing the results of [4], [8], [10].

The positivity and monotonicity of W implies the existence of the non-negative limit $\lim\limits_{x\to a^+} W(x) \equiv W(a)$. Futher, we assume $W(a) = 0$ and otherwise, we consider the operator $K_{\alpha,\beta}$ in the form, where function $W(x)$ is replaced by the function $W_0(x) = W(x) - W(a), x \in I$.

Further, the norm of the linear operator T from a normed space to another one is denoted briefly by $\|T\|$. Which spaces are meant will be clear from the context.

Throughout the paper the products of the form $0 \cdot \infty$ are supposed be equal to zero. Relations $A \ll B$, $A \gg B$ mean $A \le cB$ with a constant c depending only on p, q, α which can be different in different places. If $A \ll B$ and $A \gg B$ then we write $A \approx B$. By $\mathbb{Z}$ we denote the set of all integer numbers, χ_E denotes the characteristic function of the set E.

2 Auxiliary assertions

To prove the main results we need some well-known assertions.

Along with operator (1.1) we consider the Hardy operator

$$H_{\alpha,\beta} f(x) = \int_x^b u(s) W^{\beta+\alpha-1}(s) f(s) w(s) ds. \tag{2.1}$$

It is easy to see that for $f \ge 0$

$$K_{\alpha,\beta} f(x) \ge H_{\alpha,\beta} f(x), \ \ \forall x \in I. \tag{2.2}$$

Issues of boundedness and compactness of operator (2.1) in weighted Lebesgue spaces were studied quite completely. A summary of the results can be found in [7]. The

following Theorem A and Theorem B are corollaries of Theorem 5 and Theorem 6 in [7].

Theorem A. *Let $1 < p \le q < \infty$. The operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$A_{\alpha,\beta} = \sup_{z\in I} \left(\int_a^z v(x)dx\right)^{\frac{1}{q}} \left(\int_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $\|H_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.

Theorem B. *Let $0 < q < p < \infty$, $p > 1$. The operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$B_{\alpha,\beta} = \left(\int_a^b \left(\int_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}w(s)ds\right)^{\frac{q(p-1)}{p-q}} \times \left(\int_a^z v(x)dx\right)^{\frac{q}{p-q}} v(z)dz\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.

Remark 1. *In the case $1 < q < p < \infty$, $p > 1$ the value $B_{\alpha,\beta}$ is equivalent to the value*

$$\widetilde{B}_{\alpha,\beta}(a,b) = \left(\int_a^b \left(\int_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds\right)^{\frac{p(q-1)}{p-q}} \times \left(\int_a^z v(x)dx\right)^{\frac{p}{p-q}} u^{p'}(z)W^{p'(\alpha+\beta-1)}(z)w(z)dz\right)^{\frac{p-q}{pq}}.$$

Remark 2. *Note that a function u non-decreasing on I and such that $uW^{\beta+\alpha-1} \in L_{p',w}(z,b)$, for all $z \in I$, exists if and only if $W^{\beta+\alpha-1} \in L_{p',w}(z,b)$ for all $z \in I$.*

3 Boundedness of the operator $K_{\alpha,\beta}$

Theorem 3.1. *Let $0 < \alpha < 1$, $\frac{1}{\alpha} < p \le q < \infty$ and $\beta \le 0$ ($\beta < \frac{1}{p} - \alpha$ when $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $A_{\alpha,\beta} < \infty$. Moreover, $\|K_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.*

Proof. Necessity. Let the operator $K_{\alpha,\beta}$ be bounded from $L_{p,w}$ to $L_{q,v}$. Then, in view of (2.2), the operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ and $\|K_{\alpha,\beta}\| \ge \|H_{\alpha,\beta}\|$, therefore by Theorem A the value $A_{\alpha,\beta} < \infty$ and

$$\|K_{\alpha,\beta}\| \gg A_{\alpha,\beta}. \tag{3.1}$$

Sufficiency. Since the function W is continuous and strictly increasing on I and $W(a) = 0$, then for any $k \in \mathbb{Z}$ the set $\left\{x : W(x) \le 2^k, x \in I\right\}$ is non-empty. Denoting $x_k = \sup\left\{x : W(x) \le 2^k, x \in I\right\}$ we obtain a sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ such that $0 < x_k \le x_{k+1}$, $\forall k \in \mathbb{Z}$, and if $x_k < b$, then $W(x_k) = 2^k$, $2^k \le W(x) \le 2^{k+1}$ for $x_k \le x \le x_{k+1}$, $\int\limits_{x_{k-1}}^{x_k} w(s)ds = 2^{k-1}$, and if $x_{k+1} = b$, then $\int\limits_{x_k}^{x_{k+1}} w(s)ds \le 2^k$. These facts will be used below without reminders. We assume that $I_k = [x_k, x_{k+1})$, $k \in \mathbb{Z}$, $\mathbb{Z}_0 = \{k : k \in \mathbb{Z},\ I_k \ne \emptyset\}$. Then $\mathbb{Z}_0 \subseteq \mathbb{Z}$ and $I = \bigcup\limits_{k\in\mathbb{Z}} I_k = \bigcup\limits_{k\in\mathbb{Z}_0} I_k$. Since $I_k = \emptyset$, $\forall k \in \mathbb{Z} \setminus \mathbb{Z}_0$, and integrals over these intervals are equal to zero, then in the sequel, without loss of generality, we suppose that $\mathbb{Z} = \mathbb{Z}_0$.

Let $A_{\alpha,\beta} < \infty$. We need to prove that the inequality

$$\|T_{\alpha,\beta}f\|_{q,v} \ll A_{\alpha,\beta}\|f\|_{p,w}, \quad f \in L_{p,w}, \tag{3.2}$$

holds, which means $\|T_{\alpha,\beta}\| \ll A_{\alpha,\beta}$ and, together with (3.1), gives

$$\|T_{\alpha,\beta}\| \approx A_{\alpha,\beta}.$$

It suffices to prove inequality (3.2) for $f \ge 0$. So let $f \ge 0$. Using the relation $I = \bigcup\limits_k I_k$, we have

$$\|K_{\alpha,\beta}f\|_{q,v}^q = \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$= \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left[\left(\int\limits_x^{x_{k+1}} + \int\limits_{x_{k+1}}^b\right)\frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right]^q dx$$

$$\ll \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_x^{x_{k+1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$+ \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_{x_{k+1}}^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx = J_1 + J_2. \tag{3.3}$$

We estimate the values J_1 and J_2 separately. Using Hölder's inequality, nondecreasing of the function u and $\beta \le 0$ and in view of change of variables $W(s) = W(x)t$ we have

$$J_1 = \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_x^{x_{k+1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\le \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_x^{x_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}}\left(\int\limits_x^{x_{k+1}} \frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{(1-\alpha)p'}}\right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} u^q(x_{k+1}) \int_{x_{k-1}}^{x_k} v(x) W^{\frac{q}{p'}(p'\beta+p'(\alpha-1))}(x)$$

$$\times W^{\frac{q}{p'}}(x) \left(\int_1^{\frac{W(x_{k+1})}{W(x_{k-1})}} t^{p'\beta}(t-1)^{p'(\alpha-1)} dt \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} u^q(x_{k+1})$$

$$\times 2^{\frac{q}{p'}(p'(\beta+\alpha-1))(k-1)} 2^{\frac{q}{p'}k} \left(\int_1^4 t^{p'\beta}(t-1)^{p'(\alpha-1)} dt \right)^{\frac{q}{p'}} \int_{x_{k-1}}^{x_k} v(x) dx. \quad (3.4)$$

By the assumptions of the theorem $\alpha > \frac{1}{p}$, therefore $\int_1^4 t^{p'\beta}(t-1)^{p'(\alpha-1)} dt < \infty$.

The expression $F = u^q(x_{k+1}) 2^{q(\beta+\alpha-1)(k-1)} 2^{\frac{q}{p'}k}$ is estimated as follows. Since $\beta + \alpha - 1 < 0$ then

$$F = u^q(x_{k+1}) 2^{3q|\beta+\alpha-1|} 2^{q(\beta+\alpha-1)(k+2)} 2^{-\frac{q}{p'}} 2^{\frac{q}{p'}(k+1)}$$

$$= 2^{3q|\beta+\alpha-1|-\frac{q}{p'}} u^q(x_{k+1}) 2^{q(\beta+\alpha-1)(k+2)} \left(\int_{x_{k+1}}^{x_{k+2}} w(s) ds \right)^{\frac{q}{p'}}$$

$$\leq 2^{3q|\beta+\alpha-1|-\frac{q}{p'}} \left(\int_{x_{k+1}}^{x_{k+2}} W^{p'(\beta+\alpha-1)}(s) u^{p'}(s) w(s) ds \right)^{\frac{q}{p'}}.$$

Substituting this estimate in (3.4) we obtain

$$J_1 \ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \int_{x_{k-1}}^{x_k} v(x) dx$$

$$\times \left(\int_{x_{k+1}}^{x_{k+2}} u^{p'}(s) W^{p'(\beta+\alpha-1)} w(s) ds \right)^{\frac{q}{p'}}$$

$$\leq A_{\alpha,\beta}^q \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \leq A_{\alpha,\beta}^q \left(\sum_k \int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}}$$

$$\ll A^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{3.5}$$

Now, we estimate J_2.

$$J_2 = \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx$$

$$\le \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x_k))^{1-\alpha}} \right)^q dx$$

$$\le \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-\frac{1}{2}W(x_{k+1}))^{1-\alpha}} \right)^q dx$$

$$\le 2^{q(1-\alpha)} \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^\beta(s)f(s)w(s)}{(W(s))^{1-\alpha}} \right)^q dx$$

$$\ll \int_a^b v(x) \left(\int_x^b u(s)W^{\beta+\alpha-1}(s)f(s)w(s)ds \right)^q dx = \|H_{\alpha,\beta}f\|^q_{q,v}. \tag{3.6}$$

Then, by Theorem A

$$J_2 \ll A^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{3.7}$$

Inequalities (3.3), (3.5) and (3.7) imply inequality (3.2). □

Theorem 3.2. *Let $0 < \alpha < 1$, $0 < q < p < \infty$, $p > \frac{1}{\alpha}$ and $\beta \le 0$ ($\beta < \frac{1}{p} - \alpha$ in the case $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $B_{\alpha,\beta} < \infty$. Moreover, $\|K_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.*

Proof. Necessity and the estimate

$$\|K_{\alpha,\beta}\| \gg B_{\alpha,\beta} \tag{3.8}$$

follows by relation (2.2) and Theorem B.
Sufficiency. Let $B_{\alpha,\beta} < \infty$. If the inequality

$$\|K_{\alpha,\beta}f\|_{q,v} \ll B_{\alpha,\beta}\|f\|_{p,w}, \tag{3.9}$$

holds then by (3.8) and (3.9) we obtain $\|K_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.

To prove (3.9) we use relation (3.3) of Theorem 3.1. Estimate for J_2 directly follows by (3.6) and Theorem B:

$$J_2 \ll B^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{3.10}$$

By (3.5) we have

$$J_1 \ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \int_{x_{k-1}}^{x_k} v(x) dx \times \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q}{p'}}$$

$$\left(\text{applying the Hölder inequality with the exponents } \frac{p}{q}, \ \frac{p}{p-q} \right)$$

$$\leq \left(\sum_k \left(\int_{x_{k-1}}^{x_k} v(x) dx \right)^{\frac{p}{p-q}} \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}} \times \left(\sum_k \int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \leq G \|f\|_{p,w}^q, \quad (3.11)$$

where

$$G = \left(\sum_k \left(\int_{x_{k-1}}^{x_k} v(x) dx \right)^{\frac{p}{p-q}} \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}.$$

Using the relation

$$\left(\int_{x_{k-1}}^{x_k} v(x) dx \right)^{\frac{p}{p-q}} = \frac{p}{p-q} \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k-1}}^{x} v(t) dt \right)^{\frac{q}{p-q}} dx$$

we estimate G:

$$G \ll \left(\sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k-1}}^{x} v(t) dt \right)^{\frac{q}{p-q}} dx \times \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\leq \left(\sum_k \int_{x_{k-1}}^{x_k} \left(\int_a^x v(t)dt \right)^{\frac{q}{p-q}} \right.$$

$$\left. \times \left(\int_x^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} v(x)dx \right)^{\frac{p-q}{p}}$$

$$\leq B_{\alpha,\beta}^q. \tag{3.12}$$

By (3.11) and (3.12) it follows that

$$J_1 \ll B_{\alpha,\beta}^q \|f\|_{p,w}^q. \tag{3.13}$$

Therefore, by (3.3), (3.10) and (3.13) it follows that inequality (3.9) holds. □

4 The compactness of the operator $K_{\alpha,\beta}$

Theorem 4.1. *Let $0 < \alpha < 1$, $\frac{1}{\alpha} < p \leq q < \infty$ and $\beta \leq 0$ ($\beta < \frac{1}{p} - \alpha$ if $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $A_{\alpha,\beta} < \infty$ and*

$$\lim_{z \to a^+} A_{\alpha,\beta}(z) = \lim_{z \to b^-} A_{\alpha,\beta}(z) = 0,$$

where

$$A_{\alpha,\beta}(z) = \left(\int_a^z v(x)dx \right)^{\frac{1}{q}} \left(\int_z^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{1}{p'}}.$$

Proof. Necessity. Let the operator $K_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded and therefore, by Theorem 3.1, $A_{\alpha,\beta} < \infty$. First, we prove that $\lim\limits_{z \to b^-} A_{\alpha,\beta}(z) = 0$.

Let $F(t) = \int\limits_t^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds$. Since $A_{\alpha,\beta} < \infty$ and function u non-decreasing then $0 < F(t) < \infty$ for $t \in I$. Consider the family of functions $\{f_t\}_{t \in I}$, where

$$f_t(x) = \chi_{(t,b)}(x) u^{p'-1}(x) W^{(p'-1)(\beta+\alpha-1)}(x) (F(t))^{-\frac{1}{p}}. \tag{4.1}$$

Then

$$\int_a^b |f_t(x)|^p w(x)dx = (F(t))^{-1} \int_t^b u^{p'}(x) W^{p'(\beta+\alpha-1)}(x) w(x)dx \equiv 1. \tag{4.2}$$

We show that the family of functions $\{f_t\}$ weakly converges to zero in $L_{p,w}$. Let $g \in L_{p',w^{1-p'}} = (L_{p,w})^*$.

Applying the Holder inequality and using (4.2) we have

$$\int_a^b f_t(x)g(x)dx \le \left(\int_t^b |f_t(x)|^p w(x)dx\right)^{\frac{1}{p}} \left(\int_t^b |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_t^b |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}}.$$

Since $g \in L_{p',w^{1-p'}}$ then the last integral converges to zero as $t \to b$, which means the weak convergence to zero the family of function $\{f_t\}$. Then, by the compactness of the operator $K_{\alpha,\beta}$ from $L_{p,w}$ to $L_{q,v}$

$$\lim_{z\to b^-} \|K_{\alpha,\beta} f_t\|_{q,v} = 0. \tag{4.3}$$

We have

$$\|K_{\alpha,\beta} f_t\|_{q,v}^q = \int_a^b v(x) \left(\int_x^b \frac{u(s)W^\beta(s)f_t(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\ge \int_a^t v(x) \left(\int_t^b \frac{u(s)W^\beta(s)f_t(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\ge \int_a^t v(x)dx \left(\int_t^b u(s)W^{\beta+\alpha-1}(s)f_t(s)w(s)ds\right)^q$$

$$= (F(t))^{-\frac{q}{p}} \left(\int_t^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\right)^q \int_a^t v(x)dx = (A_{\alpha,\beta}(t))^q. \tag{4.4}$$

By (4.3) and (4.4) we obtain that $\lim\limits_{t\to b^-} A_{\alpha,\beta}(t) = 0$.

Now, we show $\lim\limits_{t\to a^+} A_{\alpha,\beta}(t) = 0$.

The compactness of the operator $K_{\alpha,\beta} : L_{p,w} \to L_{q,v}$ implies the compactness of the adjoint operator

$$K^*_{\alpha,\beta} g(x) = u(s)W^\beta(s)w(s) \int_a^s \frac{g(x)dx}{(W(s)-W(x))^{1-\alpha}}$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$.

We introduce the family of functions $\{g_t\}_{t\in I}$, where

$$g_t(x) = \chi_{(a,t)}(x) \left(\int_a^t v(x)dx\right)^{-\frac{1}{q'}} v(x).$$

Since almost everywhere $v>0$ and $A_{\alpha,\beta}<\infty$ then the function g_t is well defined. In view of the equality

$$\int_a^b |g_t(x)|^{q'} v^{1-q'}(x)dx = \left(\int_a^t v(x)dx\right)^{-1}\left(\int_a^t v(x)dx\right)=1$$

for $f\in L_{q,v}=(L_{q',v^{1-q'}})^*$ we have

$$\int_a^b f(x)g_t(x)dx \le \left(\int_a^t |f(x)|^q v(x)dx\right)^{\frac{1}{q}}\left(\int_a^t |g_t(x)|^{q'}v^{1-q'}(x)dx\right)^{\frac{1}{q'}}$$

$$=\left(\int_a^t |f(x)|^q v(x)dx\right)^{\frac{1}{q}}.$$

Consequently $\lim\limits_{t\to a^+}\int\limits_a^b f(x)g_t(x)dx=0$ for any $f\in L_{q,v}$, which means the weak convergence to zero the family of functions g_t. Then by the compactness of the operator $K^*_{\alpha,\beta}$ from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$

$$\lim_{t\to a^+}\|K^*_{\alpha,\beta}g_t\|_{p',w^{1-p'}}=0. \tag{4.5}$$

We have

$$\|K^*_{\alpha,\beta}g_t\|^{p'}_{p',w^{1-p'}} \ge \int_t^b |u(s)W^\beta(s)w(s)|^{p'}\left(\int_a^t \frac{g_t(x)dx}{(W(s)-W(x))^{1-\alpha}}\right)^{p'} w^{1-p'}(s)ds$$

$$\ge \int_t^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\left(\int_a^t v(x)dx\right)^{-\frac{p'}{q'}}\left(\int_a^t v(x)dx\right)^{p'}=(A_{\alpha,\beta}(t))^{p'}. \tag{4.6}$$

By (4.5) and (4.6) it follows that $\lim\limits_{t\to a^+}A_{\alpha,\beta}(t)=0$. The necessity is proved.

Sufficiency. Let $A_{\alpha,\beta}<\infty$ and $\lim\limits_{z\to a^+}A_{\alpha,\beta}(z)=\lim\limits_{z\to b^-}A_{\alpha,\beta}(z)=0$.

Yet for $a<c<d<b$

$$P_cf=\chi_{(a,c]}f,\ \ P_{cd}f=\chi_{(c,d]}f,\ \ Q_df=\chi_{(d,b)}f.$$

Then $f=P_cf+P_{cd}f+Q_df$ and by the equalities $P_{cd}K_{\alpha,\beta}Q_d\equiv 0,\ \ P_{cd}K_{\alpha,\beta}P_c\equiv 0$, $Q_dK_{\alpha,\beta}P_c\equiv 0$, we have

$$K_{\alpha,\beta}f=P_{cd}K_{\alpha,\beta}P_{cd}f+Q_dK_{\alpha,\beta}Q_df+P_{cd}K_{\alpha,\beta}Q_df+P_cK_{\alpha,\beta}f. \tag{4.7}$$

We show that the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}$ to $L_{q,v}$. Since $P_{cd}K_{\alpha,\beta}P_{cd}f(x)=0$ when $x\in I\backslash(c,d]$ then it suffices to show that the operator

$P_{cd}K_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$ and this is equivalent to the compactness of the operator $Kf(x)=\int\limits_c^d K(x,s)f(s)ds$ with the kernel

$$K(x,s)=\frac{u(s)W^{\beta}(s)v^{\frac{1}{q}}(x)\chi_{(c,d)}(s-x)w^{\frac{1}{p'}}(s)}{(W(s)-W(x))^{1-\alpha}}$$

from L_p to L_q.

Let $\{x_k\}_{k\in\mathbb{Z}}$ be a sequence of points introduces in the proof of Theorem 3.1. There are the points x_{i-1}, x_n, $x_{i-1}<x_n$ such that $x_{i-1}\le c<x_i$, $x_{n-1}<d\le x_n$. We assume that the number c,d are chosen so that $x_i<x_{n-1}$. Similarly to obtaining estimates of J_1, J_2 in Theorem 3.1, we have

$$\int\limits_c^d\left(\int\limits_c^d|K(x,s)|^{p'}ds\right)^{\frac{q}{p'}}dx=\int\limits_c^d v(x)\left(\int\limits_x^d\frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}}dx$$

$$\le\sum_{k=i}^{n}\int\limits_{x_{k-1}}^{x_k}v(x)\left[\left(\int\limits_{x_{k+1}}^{b}+\int\limits_x^{x_{k+1}}\right)\frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{p'(1-\alpha)}}\right]^{\frac{q}{p'}}dx$$

$$\le\mu(n-i+1)A^q_{\alpha,\beta}<\infty,$$

where the constant μ does not depend on i,n. Therefore, on the basis of the theorem in Kantorovich and Akilov [5] (page 420), the operator K is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ from $L_{p,w}$ to $L_{q,v}$.

By (4.7) we have

$$\|K_{\alpha,\beta}-P_{cd}K_{\alpha,\beta}P_{cd}\|\le\|Q_dK_{\alpha,\beta}Q_d\|+\|P_{cd}K_{\alpha,\beta}Q_d\|+\|P_cK_{\alpha,\beta}\|. \tag{4.8}$$

We shall show that the right-hand side of (4.8) tends to zero as $c\to a^+$, $d\to b^-$. This will imply that the operator $K_{\alpha,\beta}$ being a uniform limit of compact operators, is compact from $L_{p,w}$ to $L_{q,v}$.

On the basis of Theorem 3.1, we have:

$$\|Q_dK_{\alpha,\beta}Q_df\|_{q,v}=\left(\int\limits_d^b v(x)\left(\int\limits_x^b\frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\ll\sup_{d<z<b}\left(\int\limits_d^z v(x)dx\right)^{\frac{1}{q}}\left(\int\limits_z^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\right)^{\frac{1}{p'}}\|f\|_{p,w}$$

$$\le\sup_{d<z<b}A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Hence

$$\lim_{d\to b^-}\|Q_dK_{\alpha,\beta}Q_df\|\ll\lim_{d\to b^-}\sup_{d<z<b}A_{\alpha,\beta}(z)=\lim_{z\to b^-}A_{\alpha,\beta}(z)=0; \tag{4.9}$$

Let $1 > \varepsilon > 0$. To estimate $\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v}$ we introduce the functions v_ε, u_ε defined by $v_\varepsilon(x) = v(x)$ for $x \in (a,d]$ and $v_\varepsilon(x) = \varepsilon^q v(x)$ for $x \in I \setminus (a,d]$, $u_\varepsilon(s) = u(s)$ for $s \in (d,b)$ and $u_\varepsilon(s) = \varepsilon u(s)$ for $s \in I \setminus (d,b)$. Obviously, the function u_ε is non-decreasing on I. Then by Theorem 3.1

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v} = \left(\int_c^d v(x)\left(\int_d^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b v_\varepsilon(x)\left(\int_x^b \frac{u_\varepsilon(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}} \ll A^\varepsilon_{\alpha,\beta}\|f\|_{p,w}, \tag{4.10}$$

where

$$A^\varepsilon_{\alpha,\beta} = \sup_{z\in I}\left(\int_a^z v_\varepsilon(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u_\varepsilon^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}.$$

We estimate $A^\varepsilon_{\alpha,\beta}$.

$$A^\varepsilon_{\alpha,\beta} \leq \sup_{a<z<d}\left(\int_a^z v(x)dx\right)^{\frac{1}{q}}\left(\varepsilon^{p'}\int_z^d u^{p'}(s)W^{p'\beta}(s)w(s)ds + \int_d^b u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}$$

$$+ \sup_{d<z<b}\left(\int_a^d v(x)dx + \varepsilon^q\int_d^z v(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}$$

$$\ll 2(\varepsilon A_{\alpha,\beta} + A_{\alpha,\beta}(d)).$$

Hence, by (4.10) we have

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v} \ll (\varepsilon A_{\alpha,\beta} + A_{\alpha,\beta}(d))\|f\|_{p,w}. \tag{4.11}$$

Where, due to the independence of the left-hand side of (28) of $\varepsilon > 0$, by letting $\varepsilon \to 0^+$, we obtain

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v} \ll A_{\alpha,\beta}(d)\|f\|_{p,\omega}.$$

Then

$$\lim_{d\to b^-}\|P_{cd}K_{\alpha,\beta}Q_d\| \ll \lim_{d\to b^-} A_{\alpha,\beta}(d) = 0. \tag{4.12}$$

Similarly, we obtain

$$\|P_cK_{\alpha,\beta}\|_{q,v} = \left(\int_a^c v(x)\left(\int_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{a<z<c} A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Therefore

$$\lim_{c\to a^+}\|P_cK_{\alpha,\beta}f\| \ll \lim_{c\to a^+}\sup_{a<z<c} A_{\alpha,\beta}(z) = \lim_{z\to a^+} A_{\alpha,\beta}(z) = 0. \tag{4.13}$$

By (4.8), (4.9), (4.12) and (4.13) it follows that $\lim\limits_{c\to a^+,d\to b^-}\|K_{\alpha,\beta}-P_{cd}K_{\alpha,\beta}P_{cd}\| = 0$. □

Theorem 4.2. *Let $0<\alpha<1$, $p>\frac{1}{\alpha}$ and $\beta\le 0$ ($\beta<\frac{1}{p}-\alpha$ in the case $W(b)=\infty$). Let u be a non-decreasing function on I. If $b<\infty$ and $0<q<p<\infty$ or $a=0, b=\infty$ and $1<q<p<\infty$, then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $B_{\alpha,\beta}<\infty$.*

Proof. In the case $b<\infty$ and $0<q<p<\infty$ the statement of Theorem 4.2 follows by Ando Theorem and its generalizations [6]. Therefore, we prove Theorem 4.2 in the case $a=0, b=\infty$ and $1<q<p<\infty$.
Necessity. Let the operator $K_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded. Hence, by Theorem 3.2 $B_{\alpha,\beta}<\infty$.
Sufficiency. Let $B_{\alpha,\beta}<\infty$. Here $K_{\alpha,\beta}f = P_dK_{\alpha,\beta}P_df + P_dK_{\alpha,\beta}Q_dF + Q_dK_{\alpha,\beta}f$. Therefore

$$\|K_{\alpha,\beta}-P_dK_{\alpha,\beta}P_d\| \le \|P_dK_{\alpha,\beta}Q_d\| + \|Q_dK_{\alpha,\beta}\|. \tag{4.14}$$

Since $d<\infty$ then the operator $P_dK_{\alpha,\beta}P_d$ is compact from $L_{p,w}(0,d)$ to $L_{q,v}(0,d)$, which is equivalent to its compactness from $L_{p,w}$ to $L_{q,v}$. We show that the right-hand side of (4.14) tends to zero as $d\to\infty$. Then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ as the uniform limit of compact operators. On the basis of Theorem 3.2

$$\|Q_dK_{\alpha,\beta}\| \le \left(\int_d^\infty\left(\int_z^\infty u^{p'}(s)W^{p'(\alpha+\beta-1)}w(s)ds\right)^{\frac{q(p-1)}{p-q}}\right. \times \left.\left(\int_d^z v(x)dx\right)^{\frac{q}{p-q}}v(z)dz\right)^{\frac{(p-q)}{pq}}.$$

Hence, since $B_{\alpha,\beta}<\infty$, it follows that

$$\lim_{d\to\infty}\|Q_dK_{\alpha,\beta}\| = 0. \tag{4.15}$$

Let $1>\varepsilon>0$. To estimate $\|P_dK_{\alpha,\beta}Q_df\|$ we suppose as above, that $v_\varepsilon(x)=v(x)$ for $x\in(0,d]$ and $v_\varepsilon(x)=\varepsilon^q v(x)$ for $x\in(d,\infty)$, $u_\varepsilon(s)=u(s)$ for $s\in(d,\infty)$ and $u_\varepsilon(s)=\varepsilon u(s)$ for $s\in(0,d]$. Obviously, the function u_ε is non-decreasing on $I=(0,\infty)$. Now, by Theorem 3.2, estimating the norm $\|P_dK_{\alpha,\beta}Q_d\|$ as in (4.10), and then passing to the limit as $\varepsilon\to 0^+$, we obtain

$$\|P_dK_{\alpha,\beta}Q_d\| \ll \left(\int_0^d v(x)dx\right)^{\frac{1}{q}}\left(\int_d^\infty u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds\right)^{\frac{1}{p'}} = A_{\alpha,\beta}(d). \tag{4.16}$$

By Remark 1 $B_{\alpha,\beta} \approx \widetilde{B}_{\alpha,\beta}(0,\infty)$. Since $A_{\alpha,\beta}(d) \ll \widetilde{B}_{\alpha,\beta}(d,\infty)$ then by (4.16) it follows that $\lim\limits_{d\to\infty} \|P_d K_{\alpha,\beta} Q_d\| = 0$. Hence by (4.15) it follows that the right-hand side of (4.14) tends to zero as $d \to \infty$. □

5 Dual case

We consider the operator

$$T_{\alpha,\beta}f(x) = u(x)W^{\beta}(x)\int\limits_a^x \frac{v(s)f(s)ds}{(W(x)-W(s))^{1-\alpha}}$$

acting from $L_{p,v}$ to $L_{q,w}$.

Assume that

$$A^*_{\alpha,\beta}(z) = \left(\int\limits_a^z v(x)dx\right)^{\frac{1}{p'}} \left(\int\limits_z^b u^q(x)W^{q(\beta+\alpha-1)}(x)w(x)dx\right)^{\frac{1}{q}},$$

$$A^*_{\alpha,\beta} = \sup_{z\in I} A^*_{\alpha,\beta}(z).$$

Theorem 5.1. *Let $0<\alpha<1$, $1<p\le q<\frac{1}{1-\alpha}$ and $\beta\le 0$ ($\beta<1-\frac{1}{q}-\alpha$ in the case $W(b)=\infty$). Let u be a non-decreasing function on I. Then the operator $T_{\alpha,\beta}$*

i) *is bounded from $L_{p,v}$ to $L_{q,w}$ if and only if $A^*_{\alpha,\beta}(z)<\infty$, moreover, $\|T_{\alpha,\beta}\| \approx A^*_{\alpha,\beta}$,*
ii) *is compact from $L_{p,v}$ to $L_{q,w}$ if and only if $A^*_{\alpha,\beta}(z)<\infty$ and*

$$\lim_{z\to a} A^*_{\alpha,\beta}(z) = \lim_{z\to b} A^*_{\alpha,\beta}(z) = 0.$$

Proof. The operator $T_{\alpha,\beta}$ acting from $L_{p,v}$ to $L_{q,w}$ is adjoint to the operator

$$\widetilde{K}_{\alpha,\beta}f(x) = v(x)\int\limits_x^b \frac{u(s)W^{\beta}(s)f(s)ds}{(W(x)-W(s))^{1-\alpha}}$$

acting from $L_{q',w^{1-q'}}$ to $L_{p',v^{1-p'}}$, which is equivalent to the action of the operator $K_{\alpha,\beta}$ from $L_{q',\omega}$ to $L_{p',v}$. Consequently, the operator $T_{\alpha,\beta}$ is bounded and compact from $L_{p,v}$ to $L_{q,\omega}$ if and only if the operator $K_{\alpha,\beta}$ is bounded and compact from $L_{q',\omega}$ to $L_{p',v}$ respectively. Since by the assumptions of Theorem 5.1 it follows that $\frac{1}{\alpha} < q' \le p' < \infty$ then on the basis of Theorems 3.1 and 4.1 the validity of the Statements i) and ii) of Theorem 5.1 follows. □

Similarly, on the basis of Theorem 4.2, we have

Theorem 5.2. *Yet* $0<\alpha<1$, $1<q<\min\{p,\frac{1}{1-\alpha}\}$, $p>1$ *and* $\beta\le 0$ *(*$\beta<1-\frac{1}{q}-\alpha$ *in the case* $W(b)=\infty$*). Let* u *be a non-decreasing function on* I*. Then the operator* $T_{\alpha,\beta}$ *is bounded and compact from* $L_{p,v}$ *to* $L_{q,w}$ *if and only if* $B^*_{\alpha,\beta}(z)<\infty$*, where*

$$B^*_{\alpha,\beta}=\left(\int\limits_a^b\left(\int\limits_a^x v(x)dx\right)^{\frac{p(q-1)}{p-q}}\left(\int\limits_x^b u^q(s)W^{q(\beta+\alpha-1)}w(s)ds\right)^{\frac{p}{p-q}}v(x)dx\right)^{\frac{p-q}{p}}.$$

6 Applications

We consider the weighted Weyl operator

$$\widetilde{I}^*_\alpha g(s)=\omega(s)\int\limits_s^\infty\frac{\rho(x)g(x)dx}{(x-s)^{1-\alpha}},\ \ s>0$$

and the weighted Riemann-Liouville operator

$$\widetilde{I}_\alpha f(x)=\rho(x)\int\limits_0^x\frac{\omega(s)f(s)ds}{(x-s)^{1-\alpha}},\ \ x>0$$

acting from L_p to L_q, where the weight functions ρ and ω are almost everywhere positive and locally integrable on $I=(0,\infty)$. The actions of the operator $K_{\alpha,\beta}$ from $L_{p,\omega}$ to $L_{q,v}$ and the operator $T_{\alpha,\beta}$ from $L_{p,v}$ to $L_{q,\omega}$ are equivalent to the actions of the operators

$$\widetilde{K}_{\alpha,\beta}g(s)=v^{\frac{1}{q}}(s)\int\limits_s^b\frac{u(x)W^\beta(x)w^{\frac{1}{p'}}(x)g(x)dx}{(W(x)-W(s))^{1-\alpha}},\tag{6.1}$$

$$\widetilde{T}_{\alpha,\beta}f(x)=w^{\frac{1}{q}}(x)u(x)W^\beta(x)\int\limits_a^x\frac{v^{\frac{1}{p'}}(s)f(s)ds}{(W(x)-W(s))^{1-\alpha}},\tag{6.2}$$

from L_p to L_q, respectively.

Let $\omega(s)=v^{\frac{1}{q}}(s)$ in (6.1) and $\omega(s)=v^{\frac{1}{p'}}(s)$ in (6.2). If $W(x)=x$, $a=0,b=\infty$ and $\rho(x)=u(x)x^\beta$ then the operators (6.1) and (6.2) coincide with the operators I^*_α and I_α, respectively. Therefore, by Theorems 3.1- 4.2 we have

Corollary 6.1. *Let* $0<\alpha<1$, $\beta<\frac{1}{p}-\alpha$ *and* $\rho(x)=u(x)x^\beta$*, where* u *is a non-decreasing function on* $I=(0,\infty)$*. Then the operator* $\widetilde{I}^*_\alpha$

i) for $\frac{1}{\alpha}<p\le q<\infty$ *is bounded from* L_p *to* L_q *if and only if* $\widetilde{A}_\alpha<\infty$*, moreover,* $\|\widetilde{I}^*_\alpha\|\approx\widetilde{A}_\alpha$*, and is compact from* L_p *to* L_q *if and only if* $\widetilde{A}_\alpha<\infty$ *and* $\lim\limits_{z\to0^+}\widetilde{A}_\alpha(z)=\lim\limits_{z\to\infty}\widetilde{A}_\alpha(z)=0$*, where*

$$\widetilde{A}_\alpha(z)=\left(\int\limits_z^\infty\rho^{p'}(x)x^{p'(\alpha-1)}dx\right)^{\frac{1}{p'}}\left(\int\limits_0^z\omega^q(s)ds\right)^{\frac{1}{q}},\ \widetilde{A}_\alpha=\sup_{z\in I}\widetilde{A}_\alpha(z);$$

ii) for $0 < \max\{q, \frac{1}{\alpha}\} < p < \infty$ *is bounded (compact) from* L_p *to* L_q *if* $b < \infty$ *(for* $1 < \max\{q, \frac{1}{\alpha}\} < p < \infty$ *if* $b = \infty$*) if and only if* $\widetilde{B}_\alpha < \infty$*, moreover,* $\|\widetilde{I}^*_\alpha\| \approx \widetilde{B}_\alpha$*, where*

$$\widetilde{B}_\alpha = \left(\int_0^\infty \left(\int_0^z \omega^q(s)ds\right)^{\frac{q}{p-q}} \left(\int_z^\infty \rho^{p'}(x)x^{p'(\alpha-1)}dx\right)^{\frac{q(p-1)}{p-q}} \omega^q(z)dz\right)^{\frac{p-q}{qp}}.$$

Remark 3. In the case $\beta = 0$, $0 < \max\left\{q, \frac{1}{\alpha}\right\} < p < \infty$ the boundedness and compactness of the operator $\widetilde{I}^*_\alpha$ fram L_p to L_q was also studied in [4]. However, the assertions of Theorems 4.1 and 4.2 and Theorems 7 and 8 in [4] are not correct, because the given there criteria involves the integral $\int_{2t}^\infty (u(t)t^{\alpha-1})^{p'}\,dt$ for non-decreasing functions u which for $\frac{1}{p} < \alpha$ diverges for any such function.

Theorem 5.1 and 5.2 imply

Corollary 6.2. *Let* $0 < \alpha < 1$, $\beta < 1 - \frac{1}{q} - \alpha$ *and* $\rho(x) = u(x)x^\beta$*, where* u *is a non-decreasing function on* $I = (0, \infty)$*. Then the operator* $\widetilde{I}_\alpha$

i) for $1 < p \le q < \frac{1}{1-\alpha}$ *is bounded from* L_p *to* L_q *if and only if* $\widetilde{A}^*_\alpha < \infty$*, moreover,* $\|\widetilde{I}_\alpha\| \approx \widetilde{A}^*_\alpha$*, and compact from* L_p *to* L_q *if and only if* $\widetilde{A}^*_\alpha < \infty$ *and* $\lim\limits_{z\to 0^+} \widetilde{A}^*_\alpha(z) = \lim\limits_{z\to\infty} \widetilde{A}^*_\alpha(z) = 0$*, where*

$$\widetilde{A}^*_\alpha(z) = \left(\int_z^\infty \rho^q(x)x^{q(\alpha-1)}dx\right)^{\frac{1}{q}} \left(\int_0^z \omega^{p'}(s)ds\right)^{\frac{1}{p'}}, \quad \widetilde{A}^*_\alpha = \sup_{z\in I} \widetilde{A}^*_\alpha(z);$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ *is bounded(compact) if and only if* $\widetilde{B}^*_\alpha < \infty$*, moreover,* $\|\widetilde{I}_\alpha\| \approx \widetilde{B}^*_\alpha$*, where*

$$\widetilde{B}^*_\alpha = \left(\int_0^\infty \left(\int_z^\infty \rho^q(x)x^{q(\alpha-1)}dx\right)^{\frac{p}{p-q}} \left(\int_0^z \omega^{p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z)dz\right)^{\frac{p-q}{pq}}.$$

From (6.2) for $W(x) = x^\sigma$, $\sigma > 0$ $a = 0$, $b = \infty$, $\rho(x) = u(x)x^{\sigma\beta+\frac{\sigma-1}{q}}$, $v^{\frac{1}{p'}}(s) = \omega(s)s^{\sigma\gamma+\sigma-1}$ *we obtain the weighted Erdelyi-Kober operator*

$$E_{\alpha,\gamma}f(x) = \rho(x)\int_0^x \frac{\omega(s)s^{\sigma\gamma+\sigma-1}f(s)ds}{(x^\sigma - s^\sigma)^{1-\alpha}},$$

where γ *is a real number, and for* $W(x) = \ln\frac{x}{a}$, $a > 0$, $\rho(x) = u(x)x^{-\frac{1}{q}}(\ln\frac{x}{a})^\beta$, $v^{\frac{1}{p'}}(s) = \omega(s)\frac{1}{s}$ *we obtain the weighted Hadamar operator*

$$\mathcal{H}_\alpha f(x) = \rho(x)\int_a^x \frac{\omega(s)f(s)ds}{s(\ln\frac{x}{s})^{1-\alpha}}.$$

Corollary 6.3. *Let $0 < \alpha < 1$, $\beta < 1 - \frac{1}{q} - \alpha$ and $\rho(x) = u(x)x^{\sigma\beta+\frac{\sigma-1}{q}}$, where u is a non-decreasing function on $I = (0, \infty)$. Then the operator $E_{\alpha,\gamma}$*

i) for $1 < p \le q < \frac{1}{1-\alpha}$ is bounded from L_p to L_q if and only if $A^{\circ}_{\alpha,\gamma} < \infty$, moreover, $\|E_{\alpha,\gamma}\| \approx A^{\circ}_{\alpha,\gamma}$ and compact from L_p to L_q if and only if $A^{\circ}_{\alpha,\gamma} < \infty$ and $\lim\limits_{z\to 0^+} A^{\circ}_{\alpha,\gamma}(z) = \lim\limits_{z\to\infty} A^{\circ}_{\alpha,\gamma}(z) = 0$, where $A^{\circ}_{\alpha,\gamma} = \sup\limits_{z\in I} A^{\circ}_{\alpha,\gamma}(z)$,

$$A^{\circ}_{\alpha,\gamma}(z) = \left(\int_z^{\infty} \rho^q(x)x^{q\sigma(\alpha-1)}dx\right)^{\frac{1}{q}} \left(\int_0^z \omega^{p'}(s)s^{p'(\sigma\gamma+\sigma-1)}ds\right)^{\frac{1}{p'}}.$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ is bounded (compact) from L_p to L_q if and only if $B^{\circ}_{\alpha,\gamma} < \infty$, moreover, $\|E_{\alpha,\gamma}\| \approx B^{\circ}_{\alpha,\gamma}$, where

$$B^{\circ}_{\alpha,\gamma} = \left(\int_0^{\infty} \left(\int_z^{\infty} \rho^q(x)x^{q\sigma(\alpha-1)}dx\right)^{\frac{p}{p-q}} \left(\int_0^z \omega^{p'}(s)s^{p'(\sigma\gamma+\sigma-1)}ds\right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z)z^{p'(\sigma\gamma+\sigma-1)}dz\right)^{\frac{p-q}{pq}}.$$

Corollary 6.4. *Let $a > 0, 0 < \alpha < 1$, $\beta \le 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ in the case $b = \infty$) and $\rho(x) = u(x)x^{-\frac{1}{q}}(ln\frac{x}{a})^{\beta}$, where u is non-decreasing function on $I = (a, b)$. Then the operator $\mathcal{H}_{\alpha}$*

i) for $1 < p \le q < \frac{1}{1-\alpha}$ is bounded from L_p to L_q if and only if $A^1_{\alpha} < \infty$, moreover, $\|H_{\alpha}\| \approx A^1_{\alpha}$, and compact from L_p to L_q if and only if $A^1_{\alpha} < \infty$ and $\lim\limits_{z\to a^+} A^1_{\alpha}(z) = \lim\limits_{z\to b^-} A^1_{\alpha}(z) = 0$, where $A^1_{\alpha} = \sup\limits_{z\in I} A^1_{\alpha}(z)$,

$$A^1_{\alpha}(z) = \left(\int_z^b \rho^q(x)\left(\ln\frac{x}{a}\right)^{q(\alpha-1)}dx\right)^{\frac{1}{q}} \left(\int_a^z \omega^{p'}(s)s^{-p'}ds\right)^{\frac{1}{p'}};$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ is bounded (compact) from L_p to L_q if and only if $B^1_{\alpha} < \infty$, moreover, $\|H_{\alpha}\| \approx B^1_{\alpha}$, where

$$B^1_{\alpha} = \left(\int_a^b \left(\int_z^b \rho^q(x)\left(ln\frac{x}{a}\right)^{q(\alpha-1)}dx\right)^{\frac{p}{p-q}} \left(\int_a^z \omega^{p'}(s)s^{-p'}ds\right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z)z^{-p'}dz\right)^{\frac{p-q}{pq}}.$$

Acknowledgments

The work on the paper was supported by the Science Committee of the Ministry of Education and Science of Kazakhstan, project 5499/GF4 (priority direction "Intellectual potential of the country").

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Received: 11.01.2016