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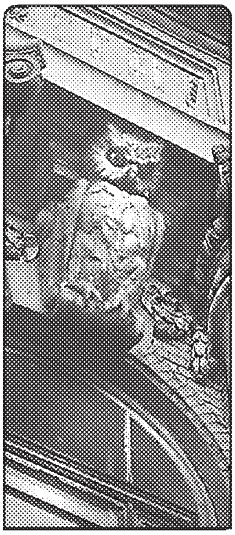

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Article

Reconstruction formula for differential systems with a singularity

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Abstract. Our studies concern some aspects of scattering theory of the singular differential systems $y' - x^{-1}Ay - q(x)y = \rho By$, $x > 0$ with $n \times n$ matrices $A, B, q(x), x \in (0, \infty)$, where A, B are constant and ρ is a spectral parameter. We concentrate on the important special case when $q(\cdot)$ is smooth and $q(0) = 0$ and derive a formula that express such $q(\cdot)$ in the form of some special contour integral, where the kernel can be written in terms of the Weyl-type solutions of the considered differential system. Formulas of such a type play an important role in constructive solution of inverse scattering problems: use of such formulas, where the terms in their right-hand sides are previously found from the so-called main equation, provides a final step of the solution procedure. In order to obtain the above-mentioned reconstruction formula, we establish first the asymptotical expansions for the Weyl-type solutions as $\rho \to \infty$ with $o\left(\rho^{-1}\right)$ rate remainder estimate.

Keywords: differential systems, singularity, integral equations, asymptotical expansions

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Формула восстановления для систем дифференциальных уравнений с особенностью

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Аннотация. В работе изучаются некоторые аспекты теории рассеяния для сингулярных систем дифференциальных уравнений $y' - x^{-1}Ay - q(x)y = \rho By$, $x > 0$ со спектральным параметром ρ, где $A, B, q(x), x \in (0, \infty)$ — $n \times n$ матрицы, причем матрицы A, B постоянны. В данной работе мы рассматриваем важный частный случай, когда матрица-функция $q(\cdot)$ является гладкой и $q(0) = 0$. В этом случае для $q(\cdot)$ получено выражение в виде контурного интеграла, где ядро записывается в терминах решений типа Вейля рассматриваемой системы. Формулы такого типа играют важную роль в конструктивном решении обратных задач рассеяния: применение формул, где величины, стоящие в правой части, предварительно найдены из так называемого основного уравнения, является завершающим шагом процедуры решения. Для вывода указанных формул восстановления мы предварительно устанавливаем асимптотики решений типа Вейля при $\rho \to \infty$ с оценкой остаточного члена $o\left(\rho^{-1}\right)$.
Ключевые слова: системы дифференциальных уравнений, сингулярность, интегральные уравнения, асимптотические разложения
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Introduction

Our studies concern some aspects of scattering theory of the differential systems

$$y' - x^{-1}Ay - q(x)y = \rho By, \quad x > 0 \tag{1}$$

with $n \times n$ matrices $A, B, q(x), x \in (0, \infty)$, where A, B are constant and ρ is a spectral parameter.

Differential equations with coefficients having non-integrable singularities at the end or inside the interval often appear in various areas of natural sciences and engineering. For $n = 2$, there exists an extensive literature devoted to different aspects of spectral theory of the radial Dirac operators, see, for instance [1–5].

Systems of the form (1) with $n > 2$ and arbitrary complex eigenvalues of the matrix B appear to be considerably more difficult for investigation even in the "regular" case of $A = 0$ [6]. Some difficulties of principal matter also appear due to the presence of the singularity. Whereas the "regular" case of $A = 0$ has been studied fairly completely to date [6–8], for system (1) with $A \neq 0$ there are no similar general results.

In this paper, we consider the important special case when $q(\cdot)$ is smooth and $q(0) = 0$ and, provided also that the discrete spectrum is empty, derive a formula that expresses such $q(\cdot)$ in the form of some special contour integral, where the kernel can be written in terms of the Weyl-type solutions of system (1). Formulas of such a type play an important role in constructive solution of inverse scattering problems: use of such formulas, where the terms in their right-hand sides are previously found from the so-called *main equation* (see, for instance, [9, 10]), provides a final step of the solution procedure. In order to obtain the above-mentioned reconstruction formula we establish first the asymptotical expansions for the Weyl-type solutions as $\rho \to \infty$ with $o(\rho^{-1})$ rate remainder estimate.

1. Preliminary remarks

Consider first the following unperturbed system

$$y' - x^{-1}Ay = \rho By \tag{2}$$

and its particular case corresponding to the value $\rho = 1$ of the spectral parameter

$$y' - x^{-1}Ay = By, \tag{3}$$

but to *complex* (in general) values of x.

Assumption 1. Matrix A is off-diagonal. The eigenvalues $\{\mu_j\}_{j=1}^n$ of the matrix A are distinct and such that $\mu_j - \mu_k \notin \mathbb{Z}$ for $j \neq k$, moreover, $\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2 < \dots < \mathrm{Re}\mu_n$, $\mathrm{Re}\mu_k \neq 0$, $k = \overline{1, n}$.

Assumption 2. $B = diag(b_1, \dots, b_n)$, the entries $b_1, \dots, b_n$ are nonzero distinct points on a complex plane such that $\sum\limits_{j=1}^{n} b_j = 0$ and such that any three points are noncolinear.

Under Assumption 1 system (3) has the fundamental matrix $c(x) = (c_1(x), \dots, c_n(x))$, where

$$c_k(x) = x^{\mu_k}\hat{c}_k(x),$$

$\det c(x) \equiv 1$ and all $\hat{c}_k(\cdot)$ are entire functions, $\hat{c}_k(0) = \mathfrak{h}_k$, $\mathfrak{h}_k$ is an eigenvector of the matrix A corresponding to the eigenvalue μ_k. We define $C_k(x, \rho) := c_k(\rho x)$, $x \in (0, \infty)$, $\rho \in \mathbb{C}$. We note that the matrix $C(x, \rho)$ is a solution of unperturbed system (2) (with respect to x for given spectral parameter ρ).

Let Σ be the following union of lines through the origin in $\mathbb{C}$:

$$\Sigma = \bigcup_{(k,j):j\neq k} \{z : \mathrm{Re}(zb_j) = \mathrm{Re}(zb_k)\}.$$

By virtue of Assumption 2 for any $z \in \mathbb{C} \setminus \Sigma$ there exists the ordering $R_1, \dots, R_n$ of the numbers $b_1, \dots, b_n$ such that $\mathrm{Re}(R_1 z) < \mathrm{Re}(R_2 z) \dots < \mathrm{Re}(R_n z)$. Let $\mathscr{S}$ be a sector $\{z = r\exp(i\gamma), r \in (0, \infty), \gamma \in (\gamma_1, \gamma_2)\}$ lying in $\mathbb{C} \setminus \Sigma$. Then, according to [11], system (3) has the fundamental matrix $e(x) = (e_1(x), \dots, e_n(x))$ which is analytic in $\mathscr{S}$, continuous in $\overline{\mathscr{S}} \setminus \{0\}$, and admits the asymptotics

$$e_k(x) = \mathrm{e}^{xR_k}(\mathfrak{f}_k + x^{-1}\eta_k(x)), \quad \eta_k(x) = O(1), \quad x \to \infty, \quad x \in \overline{\mathscr{S}},$$

where $(\mathfrak{f}_1, \dots, \mathfrak{f}_n) = \mathfrak{f}$ is a permutation matrix such that $(R_1, \dots, R_n) = (b_1, \dots, b_n)\mathfrak{f}$. We define $E(x, \rho) := e(\rho x)$.

Everywhere below we assume that the following additional condition is satisfied.

Condition 1. For all $k = \overline{2, n}$ the numbers

$$\Delta_{0k} := \det(e_1(x), \dots, e_{k-1}(x), c_k(x), \dots, c_n(x))$$

are not equal to 0.

Under Condition 1 system (3) has the fundamental matrix $\psi_0(x) = (\psi_{01}(x), \dots, \psi_{0n}(x))$ which is analytic in $\mathscr{S}$, continuous in $\overline{\mathscr{S}} \setminus \{0\}$ and admits the asymptotics:

$$\psi_{0k}(xt) = \exp(xtR_k)(\mathfrak{f}_k + o(1)), \quad t \to \infty, \quad x \in \mathscr{S}, \quad \psi_{0k}(x) = O(x^{\mu_k}), \quad x \to 0.$$

We define $\Psi_0(x, \rho) := \psi_0(\rho x)$. As above, we note that the matrices $E(x, \rho)$, $\Psi_0(x, \rho)$ solve (2).

In the sequel we use the following notations:

- $\{\mathfrak{e}_k\}_{k=1}^n$ is the standard basis in $\mathbb{C}^n$;
- $\mathscr{A}_m$ is the set of all ordered multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_1 < \alpha_2 < \dots < \alpha_m$, $\alpha_j \in \{1, 2, \dots, n\}$;
- for a sequence $\{u_j\}$ of vectors and a multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ we define $u_\alpha := u_{\alpha_1} \wedge \dots \wedge u_{\alpha_m}$;
- for a numerical sequence $\{a_j\}$ and a multi-index α we define
$$a_\alpha := \sum_{j \in \alpha} a_j, \quad a^\alpha := \prod_{j \in \alpha} a_j;$$
- for a multi-index α the symbol α' denotes the ordered multi-index that complements α to $(1, 2, \dots, n)$;
- for $k = \overline{1, n}$ we denote
$$\overrightarrow{a}_k := \sum_{j=1}^k a_j, \quad \overleftarrow{a}_k := \sum_{j=k}^n a_j, \quad \overrightarrow{a}^k := \prod_{j=1}^k a_j, \quad \overleftarrow{a}^k := \prod_{j=k}^n a_j.$$
We note that Assumptions 1,2 imply, in particular, $\sum\limits_{k=1}^n \mu_k = \sum\limits_{k=1}^n R_k = 0$ and therefore for any multi-index α one has $R_{\alpha'} = -R_\alpha$ and $\mu_{\alpha'} = -\mu_\alpha$.
- the symbol $V^{(m)}$, where V is an $n \times n$ matrix, denotes the operator acting in $\wedge^m \mathbb{C}^n$ so that for any vectors $u_1, \dots, u_m$ the following identity holds:
$$V^{(m)}(u_1 \wedge u_2 \wedge \dots \wedge u_m) = \sum_{j=1}^m u_1 \wedge u_2 \wedge \dots \wedge u_{j-1} \wedge Vu_j \wedge u_{j+1} \wedge \dots \wedge u_m;$$
- if $h \in \wedge^n \mathbb{C}^n$, then $|h|$ is a number such that $h = |h| \mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \dots \wedge \mathfrak{e}_n$;
- for $h \in \wedge^m \mathbb{C}^n$ we set: $\|h\| := \sum\limits_{\alpha \in \mathscr{A}_m} |h_\alpha|$, where $\{h_\alpha\}$ are the coefficients from the expansion $h = \sum\limits_{\alpha \in \mathscr{A}_m} h_\alpha \mathfrak{e}_\alpha$.

2. Asymptotics of the Weyl-type solutions

Let $\mathscr{S} \subset \mathbb{C} \setminus \Sigma$ be an open sector with vertex at the origin. For arbitrary $\rho \in \mathscr{S}$ and $k \in \{1, \dots, n\}$ we define the *k-th Weyl-type solution* $\Psi_k(x, \rho)$ as a solution of (1) normalized with the asymptotic conditions:

$$\Psi_k(x, \rho) = O\left(x^{\mu_k}\right), \quad x \to 0, \qquad \Psi_k(x, \rho) = \exp(\rho x R_k)(\mathfrak{f}_k + o(1)), \quad x \to \infty. \tag{4}$$

If $q(\cdot)$ is off-diagonal matrix function summable on the semi-axis $(0, \infty)$, then for arbitrary given $\rho \in \mathscr{S}$ k-th Weyl-type solution exists and is unique provided that the characteristic function

$$\Delta_k(\rho) = |F_{k-1}(x, \rho) \wedge T_k(x, \rho)|$$

does not vanish at this ρ. Here $\{F_k(x, \rho)\}_{k=1}^n$, $\{T_k(x, \rho)\}_{k=1}^n$ are certain tensor-valued functions (*fundamental tensors*) defined as solutions of certain Volterra integral equations, see [12, 13] for details.

For arbitrary fixed arguments x, ρ (where $\Delta_k(\rho) \neq 0$) the value $\Psi_k = \Psi_k(x, \rho)$ is the unique solution of the following linear system:

$$F_{k-1} \wedge \Psi_k = F_k, \quad \Psi_k \wedge T_k = 0. \tag{5}$$

This fact and also some properties of the Weyl-type solutions were established in works [12, 14], in particular, the following asymtotics for $\rho \to \infty$ was obtained:

$$\Psi_k(x, \rho) = \Psi_{0k}(x, \rho) + o\left(\exp(\rho x R_k)\right). \tag{6}$$

For our purposes we need more detailed asymptotics that can be obtained provided that the potential $q(\cdot)$ is smooth enough and vanishes as $x \to 0$.

We denote by $\mathscr{P}(\mathscr{S})$ the set of functions $F(\rho)$, $\rho \in \mathscr{S}$ admitting the representation:

$$F(\rho) = \sum_{\lambda \in \Lambda} f(\lambda) \exp(\lambda \rho).$$

Here the set Λ (depending on $F(\cdot) \in \mathscr{P}(\mathscr{S})$) is such that $\mathrm{Re}(\lambda\rho) < 0$ for all $\lambda \in \Lambda$, $\rho \in \mathscr{S}$. We note that the set of scalar functions belonging to $\mathscr{P}(\mathscr{S})$ is an algebra with respect to pointwise multiplication.

Theorem 1. *Suppose that $q(\cdot)$ is an absolutely continuous off-diagonal matrix function such that $q(0) = 0$. Denote by $\hat{q}_o(\cdot)$ the off-diagonal matrix function such that $[B, \hat{q}_o(x)] = -q(x)$ for all $x > 0$ (here $[\cdot, \cdot]$ denotes the matrix commutator). Define the diagonal matrix $d(x) = diag(d_1(x), \dots, d_n(x))$, where*

$$d_k(x) := \int\limits_x^\infty t^{-1} \left([\hat{q}_o(t), A]\right)_{kk} \, dt$$

and set $\hat{q}(x) := \hat{q}_o(x) + d(x)$.

Suppose that all the functions $q_{ij}(\cdot), q'_{ij}(\cdot)$ and $\tilde{q}_{ij}(\cdot)$, where $\tilde{q}(x) := \hat{q}'(x) + x^{-1}[\hat{q}(x), A]$, belong to $X_p := L_1(0, \infty) \cap L_p(0, \infty)$, $p > 2$.

Then for each fixed $x > 0$ and $\rho \to \infty$, $\rho \in \overline{\mathscr{S}}$ the following asymptotics holds:

$$\rho(\Psi(q, x, \rho) - \Psi_0(x, \rho)) \exp(-\rho x R) = \mathfrak{f}\Gamma(x) + \hat{q}(x)\mathfrak{f} + \mathscr{E}(x, \rho) + o(1),$$

where $\Gamma(x)$ is some diagonal matrix, $\mathscr{E}(x, \cdot) \in \mathscr{P}(\mathscr{S})$.

Proof. Denote

$$\tilde F_k(x,\rho) := \exp\left(-\rho x \overrightarrow{R}_k\right) F_k(x,\rho), \quad \tilde T_k(x,\rho) := \exp\left(-\rho x \overleftarrow{R}_k\right) T_k(x,\rho).$$

By virtue of [13, Theorem 1] the following asymptotics hold

$$\rho\tilde F_k(q,x,\rho) = \rho\tilde F_{0k}(x,\rho) + \sum_{\alpha\in\mathscr{A}_k} f_{k,\alpha}(x)\mathfrak{f}_\alpha + \mathscr{E}(x,\rho) + o(1),$$
$$\rho\tilde T_k(q,x,\rho) = \rho\tilde T_{0k}(x,\rho) + d_{0k}\tilde T_{0k}(x,\rho) + \tag{7}$$
$$+\sum_{\alpha\in\mathscr{A}_{n-k+1}} T^0_{k,\alpha^*(k)} g_{k,\alpha,\alpha^*(k)}(x)\mathfrak{f}_\alpha + \mathscr{E}(x,\rho) + o(1),$$

where $\alpha^*(k) := (k,\dots,n)$, $\alpha_*(k) := (1,\dots,k)$; $f_{k,\alpha}(x)$, $g_{k,\alpha,\alpha^*(k)}(x)$ are some scalars that can be written explicitly in terms of $q(\cdot)$.

For the Weyl-type solutions of the unperturbed system we have the asymptotics (following directly from their definition)

$$\tilde\Psi_{0k}(x,\rho) = \mathfrak{f}_k + \mathscr{E}(x,\rho) + O\left(\rho^{-1}\right), \tag{8}$$

where $\tilde\Psi_{0k}(x,\rho) := \exp(-\rho x R_k)\Psi_{0k}(x,\rho)$. Here and below we use the same symbol $\mathscr{E}(\cdot,\cdot)$ for different functions such that $\mathscr{E}(x,\cdot)\in\mathscr{P}(\mathscr{S})$ for each fixed x.

We rewrite relations (5) in the form of the following linear system with respect to value $\tilde\Psi_k = \tilde\Psi_k(x,\rho)$ of the function $\tilde\Psi_k(x,\rho) := \exp(-\rho x R_k)\Psi_k(x,\rho)$:

$$\tilde F_{k-1}\wedge\tilde\Psi_k = \tilde F_k, \quad \tilde\Psi_k\wedge\tilde T_k = 0.$$

By making the substitution

$$\tilde\Psi_k = \tilde\Psi_{0k} + \hat\Psi_k, \tag{9}$$

we obtain

$$\tilde F_{k-1}\wedge\hat\Psi_k = \tilde F_k - \tilde F_{k-1}\wedge\tilde\Psi_{0k}, \quad \hat\Psi_k\wedge\tilde T_k = -\tilde\Psi_{0k}\wedge\tilde T_k.$$

The obtained relations we transform into the following system of linear algebraic equations

$$\sum_{j=1}^n m_{ij}\gamma_{jk} = u_i, \qquad i=\overline{1,n} \tag{10}$$

with respect to coefficients $\{\gamma_{jk}\}$ of the expansion

$$\hat\Psi_k(x,\rho) = \sum_{j=1}^n \gamma_{jk}(x,\rho)\mathfrak{f}_j. \tag{11}$$

Coefficients $\{m_{ij}\}$, $\{u_i\}$ can be calculated as follows:

$$m_{ij} = \left|\tilde F_{k-1}\wedge\mathfrak{f}_j\wedge\mathfrak{f}_\alpha\right|,$$
$$u_i = \left|(\tilde F_k - \tilde F_{k-1}\wedge\tilde\Psi_{0k})\wedge\mathfrak{f}_\alpha\right|, \quad \alpha=\alpha^*(k)\setminus i, \quad i=\overline{k,n},$$
$$m_{ij} = \left|\mathfrak{f}_\alpha\wedge\mathfrak{f}_j\wedge\tilde T_k\right|, \quad u_i = -\left|\mathfrak{f}_\alpha\wedge\tilde\Psi_{0k}\wedge\tilde T_k\right|, \quad \alpha=\alpha_*(k-1)\setminus i, \quad i=\overline{1,k-1}.$$

Using (7), (8) and taking into account that

$$\tilde{F}^0_{k-1} \wedge \tilde{\Psi}_{0k} = \tilde{F}^0_k, \quad \tilde{\Psi}_{0k} \wedge \tilde{T}^0_k = 0,$$

we obtain the following asymptotics for the coefficients of SLAE (10) as $\rho \to \infty$

$$m_{ij}(x,\rho) = O\left(\rho^{-1}\right), \ j \neq i, \ m_{ii}(x,\rho) = m^0_{ii} + O\left(\rho^{-1}\right), \ m^0_{ii} = (-1)^{k-i}|\mathfrak{f}|, \ i = \overline{k,n}, \quad (12)$$

and

$$m_{ij}(x,\rho) = m^0_{ij} + O\left(\rho^{-1}\right), \quad i = \overline{1,k-1}, \quad j = \overline{1,k-1},$$
$$m_{ij}(x,\rho) = m^0_{ij} + \mathscr{E}(x,\rho) + O\left(\rho^{-1}\right), \quad i = \overline{1,k-1}, \quad j = \overline{k,n},$$

where

$$m^0_{ij} = T^0_{k,\alpha^*(k)}|\mathfrak{f}_\alpha \wedge \mathfrak{f}_j \wedge \mathfrak{f}_{\alpha^*(k)}|, \quad \alpha = \alpha_*(k-1) \setminus i,$$

and therefore

$$m_{ij}(x,\rho) = O\left(\rho^{-1}\right), \ j \neq i, \ j < k, \qquad m_{ij}(x,\rho) = \mathscr{E}(x,\rho) + O\left(\rho^{-1}\right), \ j = \overline{k,n}, \quad (13)$$

$$m_{ii}(x,\rho) = m^0_{ii} + O\left(\rho^{-1}\right), \quad m^0_{ii} = (-1)^{k-1-i}|\mathfrak{f}|T^0_{k,\alpha^*(k)} \quad (14)$$

for $i = \overline{1,k-1}$.

Proceeding in a similar way we obtain

$$\rho u_i(x,\rho) = u^1_i(x) + \mathscr{E}(x,\rho) + o(1), \quad (15)$$

$$u^1_i(x) = (-1)^{k-i}|\mathfrak{f}|f_{k,\alpha}(x) - \delta_{i,k}|\mathfrak{f}|f_{(k-1),\alpha_*(k-1)}(x), \ \alpha = \alpha_*(k-1) \cup \{i\}, \ i = \overline{k,n}, \quad (16)$$

where $\delta_{i,k}$ is a Kroeneker delta,

$$u^1_i(x) = -(-1)^{k-i}\,|\mathfrak{f}|\,T^0_{k,\alpha^*(k)}g_{k,\beta,\alpha^*(k)}(x), \ \beta = \alpha' \setminus k, \ \alpha = \alpha_*(k-1) \setminus i, \ i = \overline{1,k-1}. \quad (17)$$

Using the obtained asymptotics we obtain from (10) the auxiliary estimate $\gamma_{ik}(x,\rho) = O\left(\rho^{-1}\right)$.

Then, using in (10) the substitution $\gamma_{ik}(x,\rho) = \rho^{-1}\hat{\gamma}_{ik}(x,\rho)$ (where, as it was shown above, $\hat{\gamma}_{ik}(x,\rho) = O(1)$) we obtain for $i = \overline{k,n}$

$$m_{ii}(x,\rho)\hat{\gamma}_{ik}(x,\rho) = u^1_i(x) + \mathscr{E}(x,\rho) - \sum_{j \neq i} m_{ij}(x,\rho)\hat{\gamma}_{jk}(x,\rho) + o(1).$$

In view of (12), (15) this yields

$$\hat{\gamma}_{ik}(x,\rho) = \gamma^1_{ik}(x) + \mathscr{E}(x,\rho) + o(1), \quad \gamma^1_{ik} = \frac{u^1_i(x)}{m^0_{ii}}, \quad (18)$$

$i = \overline{k,n}$.

Similarly, for $i < k$ we have

$$m_{ii}(x,\rho)\hat{\gamma}_{ik}(x,\rho) = u^1_i(x) + \mathscr{E}(x,\rho) - \sum_{j \geqslant k} m_{ij}(x,\rho)\hat{\gamma}_{jk}(x,\rho) - \sum_{j<k, j\neq i} m_{ij}(x,\rho)\hat{\gamma}_{jk}(x,\rho) + o(1).$$

Using (13), (14) the obtained relation can be transformed as follows:

$$m^0_{ii}\hat{\gamma}_{ik}(x,\rho) = u^1_i(x) + \mathscr{E}(x,\rho) - \sum_{j \geqslant k} m_{ij}(x,\rho)\hat{\gamma}_{jk}(x,\rho) + o(1).$$

Now, using in the right hand side of the obtained formula (13), (14) for $m_{ij}(x,\rho)$ and (18) for $\hat{\gamma}_{jk}(x,\rho)$ with $j=\overline{k,n}$ we conclude that formulas (18) are true for $i<k$ as well.

In our further calculations we use particular form of the coefficients $f_{k,\alpha}(x)$ and $g_{k,\alpha,\beta}(x)$ given by [13, Theorem 1].

For $i=\overline{k,n}$ from (18), (16), (12) we get

$$\gamma^1_{ik}(x)=\delta_{i,k}\tilde{\gamma}^1_{ik}(x)+f_{k,\alpha}(x),\quad \alpha=\alpha_*(k-1)\cup i. \tag{19}$$

Theorem 1 [13] yields

$$f_{k,\alpha}(x)=\chi_\alpha\left|\left(\hat{q}^{(k)}(x)\mathfrak{f}_{\alpha_*(k)}\right)\wedge\mathfrak{f}_{\alpha'}\right|,\quad \chi_\alpha:=|\mathfrak{f}_\alpha\wedge\mathfrak{f}_{\alpha'}|.$$

Recall that any arbitrary linear operator V acting in $\mathbb{C}^n$ can be expanded onto the wedge algebra $\wedge\mathbb{C}^n$ so that the identity

$$V(h_1\wedge\cdots\wedge h_m)=(Vh_1)\wedge\cdots\wedge(Vh_m)$$

remains true for any set of vectors $h_1,\ldots,h_m$, $m\leqslant n$; moreover, for any $h\in\wedge^n\mathbb{C}^n$ one has $Vh=|V|h$ (here $|V|$ denotes determinant of matrix of the operator V in the standard coordinate basis $\{\mathfrak{e}_1,\ldots,\mathfrak{e}_n\}$). In what follows, the symbol $\mathfrak{f}$ denotes the above mentioned expansion of the operator corresponding to the transmutation matrix $\mathfrak{f}$. We should note also that the relation $(\mathfrak{f}^{-1}V\mathfrak{f})^{(k)}=\mathfrak{f}^{-1}V^{(k)}\mathfrak{f}$ is true for any $n\times n$ matrix V. Taking this into account we obtain

$$f_{k,\alpha}(x)=\chi_\alpha\left|\left(\mathfrak{f}\left(\mathfrak{f}^{-1}\hat{q}^{(k)}(x)\mathfrak{f}\mathfrak{e}_{\alpha_*(k)}\right)\right)\wedge(\mathfrak{f}\mathfrak{e}_{\alpha'})\right|=$$
$$=|\mathfrak{f}_\alpha\wedge\mathfrak{f}_{\alpha'}||\mathfrak{f}|\left|\left(\mathfrak{f}^{-1}\hat{q}^{(k)}(x)\mathfrak{f}\mathfrak{e}_{\alpha_*(k)}\right)\wedge\mathfrak{e}_{\alpha'}\right|=|\mathfrak{e}_\alpha\wedge\mathfrak{e}_{\alpha'}|\left|\left(\left(\mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f}\right)^{(k)}\mathfrak{e}_{\alpha_*(k)}\right)\wedge\mathfrak{e}_{\alpha'}\right|.$$

For the particular multi-index $\alpha=\alpha_*(k-1)\cup i$ arising at (19) and arbitrary $n\times n$ matrix V we have

$$|\mathfrak{e}_\alpha\wedge\mathfrak{e}_{\alpha'}|\left|\left(V^{(k)}\mathfrak{e}_{\alpha_*(k)}\right)\wedge\mathfrak{e}_{\alpha'}\right|=V_{ik}.$$

Substituting the obtained relations into (19) we arrive at

$$\gamma^1_{ik}(x)=\delta_{i,k}\tilde{\gamma}^1_{ik}(x)+\left(\mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f}\right)_{ik},\quad i=\overline{k,n}. \tag{20}$$

Proceeding in a similar way in the case $i<k$, using (13), (14), (17) we obtain:

$$\gamma^1_{ik}(x)=g_{k,\beta,\alpha^*(k)}(x),\quad \beta=\alpha'\setminus k,\alpha=\alpha_*(k-1)\setminus i. \tag{21}$$

Theorem 1 [13] yields

$$g_{k,\alpha,\beta}(x)=\chi_\alpha\left|\left(\hat{q}^{(n-k+1)}(x)\mathfrak{f}_\beta\right)\wedge\mathfrak{f}_{\alpha'}\right|$$

for $\beta\neq\alpha$. Repeating the arguments above we obtain

$$g_{k,\alpha,\beta}(x)=|\mathfrak{e}_\alpha\wedge\mathfrak{e}_{\alpha'}|\left|\left(\left(\mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f}\right)^{(n-k+1)}\mathfrak{e}_\beta\right)\wedge\mathfrak{e}_{\alpha'}\right|.$$

In particular, one gets

$$g_{k,\beta,\alpha^*(k)}=|\mathfrak{e}_\beta\wedge\mathfrak{e}_{\beta'}|\left|\left(\left(\mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f}\right)^{(n-k+1)}\mathfrak{e}_{\alpha^*(k)}\right)\wedge\mathfrak{e}_{\beta'}\right|.$$

If $\beta = \alpha' \setminus k$, $\alpha = \alpha_*(k-1) \setminus i$, $i < k$, then for arbitrary $n \times n$ matrix V we have

$$|\mathfrak{e}_\beta \wedge \mathfrak{e}_{\beta'}| \left| \left(V^{(n-k+1)} \mathfrak{e}_{\alpha^*(k)}\right) \wedge \mathfrak{e}_{\beta'} \right| = V_{ik}.$$

Substituting the obtained relations into (21) we obtain

$$\gamma^1_{ik}(x) = \left(\mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f}\right)_{ik}, \quad i = \overline{1, k-1}. \tag{22}$$

From (22), (20), (18) we obtain

$$\rho\gamma_{ik}(x,\rho) = \hat{\gamma}_{ik}(x,\rho) = \delta_{i,k}\tilde{\gamma}^1_{ik}(x) + \left(\mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f}\right)_{ik} + \mathscr{E}(x,\rho) + o(1).$$

In terms of the matrix $\gamma = (\gamma_{ik})_{i,k=\overline{1,n}}$ this is equivalent to

$$\rho\gamma(x,\rho) = \Gamma(x) + \mathfrak{f}^{-1}\hat{q}(x)\mathfrak{f} + \mathscr{E}(x,\rho) + o(1),$$

where the matrix $\Gamma(x)$ is diagonal. Finally, using (11) in the form $\hat{\Psi}(x,\rho) = \mathfrak{f}\gamma(x,\rho)$ we obtain the required relation. □

3. Reconstruction formula

Let $\mathscr{S}_\nu$, $\nu = \overline{1, N}$ be the open pairwise nonintersecting sectors such that $\mathbb{C} \setminus \Sigma = \bigcup\limits_{\nu=1}^{N} \mathscr{S}_\nu$. Suppose that the sectors are enumerated in counterclockwise order. We denote by Σ_ν the open ray dividing $\mathscr{S}_\nu$ and $\mathscr{S}_{\nu+1}$ (assuming $\mathscr{S}_{N+1} := \mathscr{S}_1$). We agree that the rays Σ_ν are oriented from 0 to ∞. Denote by Σ^+_ν and Σ^-_ν the edges of the cut (along Σ_ν) belonging to $\mathscr{S}_{\nu+1}$ and $\mathscr{S}_\nu$ respectively. We agree that Σ^+_ν is oriented from 0 to ∞ while Σ^-_ν is oriented from ∞ to 0.

For a function $f(\rho)$, $\rho \in \mathscr{S}_\nu \cup \mathscr{S}_{\nu+1}$ and arbitrary $\rho_0 \in \Sigma_\nu$ we denote by $f^\pm(\rho_0)$ the limit values (if they exist)

$$f^-(\rho_0) := \lim_{\rho\to\rho_0, \rho\in\mathscr{S}_\nu} f(\rho), \quad f^+(\rho_0) := \lim_{\rho\to\rho_0, \rho\in\mathscr{S}_{\nu+1}} f(\rho).$$

We say that off-diagonal matrix function $q(\cdot) \in X_p$ belongs to the class G^p_0 if for any $\nu \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, n\}$ it is true that $\Delta_k(\rho) \neq 0$ for all $\rho \in \overline{\mathscr{S}_\nu}$. If $q(\cdot) \in G^p_0$ then the limit values $\Psi^\pm_k(x,\rho_0)$ exist for any $k \in \{1, \ldots, n\}$, $\rho_0 \in \Sigma_\nu$, $\nu \in \{1, \ldots, N\}$.

We denote by $\Psi(x,\rho)$ the matrix function $\Psi(x,\rho) = (\Psi_1(x,\rho), \ldots, \Psi_n(x,\rho))$ and introduce the following *spectral mappings matrix*

$$P(x,\rho) := \Psi(x,\rho)\Psi_0^{-1}(x,\rho).$$

If $q(\cdot) \in G^p_0$ then the limit values $P^\pm_k(x,\rho_0)$ exist for any $k \in \{1, \ldots, n\}$, $\rho_0 \in \Sigma_\nu$, $\nu \in \{1, \ldots, N\}$. We denote $\hat{P}(x,\rho) := P^+(x,\rho) - P^-(x,\rho)$. Following theorem contains the main result of the paper.

Theorem 2. *Suppose that the potential $q(\cdot) \in G^p_0$ satisfies the conditions of Theorem 1. Then the following relation (reconstruction formula) holds*

$$q(x) = \frac{1}{2\pi i}\int\limits_\Sigma \left[B, \hat{P}(x,\rho)\right] d\rho,$$

where (as above) the brackets $[\cdot,\cdot]$ *denote the matrix commutator and the integral is considered as the following limit (existing for each* $x>0$*)*

$$\frac{1}{2\pi i}\int\limits_{\Sigma}\left[B,\hat{P}(x,\rho)\right]d\rho:=\lim_{r\to\infty}\frac{1}{2\pi i}\int\limits_{\Sigma^r}\left[B,\hat{P}(x,\rho)\right]d\rho,$$

$\Sigma^r:=\Sigma\cap\{\rho:|\rho|\leqslant r\}$.

Proof. Consider the function

$$F(x,\rho):=\rho[B,P(x,\rho)]+q(x).$$

From Theorem 1 we have the asymptotics

$$\hat{\Psi}(x,\rho):=(\Psi(x,\rho)-\Psi_0(x,\rho))\exp(-\rho xR)=\rho^{-1}(\mathfrak{f}\Gamma_\nu(x)+\hat{q}(x)\mathfrak{f}+\mathscr{E}_\nu(x,\rho)+o(1))$$

as $\rho\to\infty$, $\rho\in\mathscr{S}_\nu$, where $R=diag(R_1,\dots,R_n)$, $\Gamma_\nu(x)$ are some diagonal matrices and $\mathscr{E}_\nu(x,\cdot)\in\mathscr{P}(\mathscr{S}_\nu)$.

For $\tilde{\Psi}_0(x,\rho)$ we have

$$\tilde{\Psi}_0(x,\rho)=\mathfrak{f}+\mathscr{E}_\nu(x,\rho)+o(1)$$

as $\rho\to\infty$, $\rho\in\mathscr{S}_\nu$ (we use the same symbol for denoting possibly different functions from $\mathscr{P}(\mathscr{S}_\nu)$).

Since $|\det\tilde{\Psi}_0|=1$ the following asymptotics is also valid

$$\tilde{\Psi}_0^{-1}(x,\rho)=\mathfrak{f}^{-1}+\mathscr{E}_\nu(x,\rho)+o(1),\quad \rho\to\infty,\quad \rho\in\mathscr{S}_\nu.$$

Therefore, for $\rho\to\infty$, $\rho\in\mathscr{S}_\nu$ we have

$$P(x,\rho)=I+\hat{\Psi}(x,\rho)\tilde{\Psi}_0^{-1}(x,\rho)=I+\rho^{-1}(\mathfrak{f}\Gamma_\nu(x)\mathfrak{f}^{-1}+\hat{q}(x)+\mathscr{E}_\nu(x,\rho)+o(1)). \tag{23}$$

Since the matrices $\Gamma_\nu(x)$ are diagonal the matrices $\mathfrak{f}\Gamma_\nu(x)\mathfrak{f}^{-1}$ are diagonal as well and we have $[B,\mathfrak{f}\Gamma_\nu(x)\mathfrak{f}^{-1}]=0$. Thus, from (23) we deduce

$$F(x,\rho)=\mathscr{E}_\nu(x,\rho)+o(1),\qquad \rho\to\infty,\quad \rho\in\mathscr{S}_\nu. \tag{24}$$

Define

$$\gamma=\bigcup_{\nu=1}^{N}\left(\Sigma_\nu^-\cup\Sigma_\nu^+\right),\quad \gamma_r:=\gamma\cap\{\rho:|\rho|\leqslant r\},\quad \Gamma_r:=\gamma_r\cup C_r,$$

where C_r is the circle $\{\rho:|\rho|=r\}$ with a counterclockwise orientation.

By virtue of the Jordan lemma from asymptotics (24) it follows that for any arbitrary fixed $\rho\in\mathbb{C}\setminus\Sigma$ we have

$$\lim_{r\to\infty}\int\limits_{C_r}\frac{d\zeta}{\zeta-\rho}F(x,\zeta)=0.$$

Therefore, the Cauchy integral formula for the closed contour Γ_r (where $r>|\rho|$)

$$F(x,\rho)=\frac{1}{2\pi i}\int\limits_{\Gamma_r}\frac{d\zeta}{\zeta-\rho}F(x,\zeta)$$

can be transformed as follows:

$$F(x,\rho) = \lim_{r\to\infty} \frac{1}{2\pi i} \int\limits_{\Sigma^r} \frac{d\zeta}{\zeta-\rho}(F^+(x,\zeta) - F^-(x,\zeta)).$$

Taking into account that $F^+(x,\zeta) - F^-(x,\zeta) = \zeta[B, \hat{P}(x,\zeta)]$ we obtain

$$F(x,\rho) = \lim_{r\to\infty} \frac{1}{2\pi i} \int\limits_{\Sigma^r} \frac{d\zeta}{\zeta-\rho}\zeta[B, \hat{P}(x,\zeta)]. \tag{25}$$

Moreover, we can proceed in a similar way applying the Cauchy formula to the function $P(x,\rho) - I$. Thus we obtain

$$P(x,\rho) - I = \frac{1}{2\pi i} \int\limits_{\Gamma_r} \frac{d\zeta}{\zeta-\rho}(P(x,\zeta) - I)$$

and since from (24) it follows that

$$\lim_{r\to\infty} \int\limits_{C_r} \frac{d\zeta}{\zeta-\rho}(P(x,\zeta) - I) = 0$$

we get the following representation:

$$P(x,\rho) = I + \lim_{r\to\infty} \frac{1}{2\pi i} \int\limits_{\Sigma^r} \frac{d\zeta}{\zeta-\rho}(P^+(x,\zeta) - P^-(x,\zeta)).$$

Substituting this to the definition of the function $F(x,\rho)$, we get the following representation:

$$F(x,\rho) = q(x) + \lim_{r\to\infty} \frac{1}{2\pi i} \int\limits_{\Sigma^r} \frac{d\zeta}{\zeta-\rho}\rho[B, \hat{P}(x,\zeta)].$$

Comparing it with (25) we obtain the desired relation. □

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