S. Shimorin, Branching points area theorems for univalent functions, *Алгебра и анализ*, 2006, том 18, выпуск 1, 187–240

Алгебра и анализ
Том 18 (2006), №1

BRANCHING POINTS AREA THEOREMS FOR UNIVALENT FUNCTIONS

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Area theorems of a new type are obtained by considering branching point compositions with univalent functions. They can be formulated both in the form of integral estimates and in the form of Grunsky and Goluzin-type inequalities.

§1. Introduction

Area methods are among the most important and powerful tools in the theory of conformal mappings. The monograph [8] was devoted entirely to these topics; a wide exposition of various forms of area principles can be found also in the classical monographs [5] or [10]. Recent progress in estimating integral means of the derivatives of univalent functions (see [12, 7]) was based on the use of area inequalities combined with techniques of Bergman spaces. This was an inspiration for taking a new look at area methods in the theory of conformal mappings.

The classical polynomial area theorem for univalent functions can be formulated as follows: if ψ is a function univalent in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, and p is a Laurent polynomial

$$p(z) = \sum_{n=0}^{m} \frac{p_n}{z^n},$$

then the composition $p(\psi)$ written as a Laurent series converging for z near $\mathbb{T} = \partial\mathbb{D}$, namely,

$$p(\psi(z)) = \sum_{k=-\infty}^{+\infty} c_k z^k,$$

satisfies

$$\big[p(\psi), p(\psi)\big] \leqslant 0, \tag{1.1}$$

where $[\cdot,\cdot]$ is the indefinite inner product defined as follows:

$$[g,g]=\sum_{k\in\mathbb{Z}} k|c_k|^2 \quad \text{for} \quad g(z)=\sum_{k\in\mathbb{Z}} c_k z^k. \tag{1.2}$$

Equivalently, one can rewrite (1.1) as

$$\sum_{k\geqslant 0} k|c_k|^2 \leqslant \sum_{l<0}(-l)|c_l|^2 \tag{1.3}$$

(and this corresponds to a canonical decomposition of the indefinite norm (1.2) into a sum of orthogonal positive and negative parts). The conclusion of the polynomial area theorem remains true if p is an arbitrary rational function with poles in $\psi(\mathbb{D})$ or even a function regular in a neighborhood of $\overline{\mathbb{C}}\setminus\psi(\mathbb{D})$ (where $\overline{\mathbb{C}}$ stands for the Riemann sphere $\mathbb{C}\cup\{\infty\}$).

It is well known that the polynomial area theorem is equivalent to various modifications of the Grunsky inequalities and Goluzin inequalities, which are usually more important for the applications than the area theorem itself. In fact, the derivation of Grunsky or Goluzin inequalities from the area theorem reduces to an *effective* (in an appropriate sense, say, for rational functions p written in the form of sums of partial fractions) calculation of the positive and negative parts of the decomposition of $p(\psi)$ with respect to the indefinite norm (1.2).

The present paper is devoted to area theorems valid for functions p that are no longer single-valued (but still analytic) in $\mathbb{C}$. More precisely, we consider functions of the form

$$p(z)=(z-\mu_1)^{\theta_1}\dots(z-\mu_n)^{\theta_n}q(z), \tag{1.4}$$

where $\mu_1,\dots,\mu_n\in\psi(\mathbb{D})$, "branching multiplicities" θ_k are in the interval $(0,1)$, and q is a rational function with poles in $\psi(\mathbb{D})$ (rigorously, such functions p can be thought of as defined on an appropriate subdomain of the universal covering surface of $\overline{\mathbb{C}}\setminus\{\mu_1,\dots,\mu_n\}$ and character-automorphic there).

In one particular case where $n=1$ and $\mu_1=0=\psi(0)$, area theorems of such a kind are known. Namely, if $p(z)=z^{\theta-1}$ with $\theta\in(0,1)$ and $\psi(0)=0$, then the area theorem of Prawitz [11] proved in 1927 asserts that

$$\sum_{n\geqslant 0}(n+\theta)|c_n|^2\leqslant(1-\theta)|c_{-1}|^2, \tag{1.5}$$

where the coefficients c_n are defined by

$$p(\psi(z))=(\psi(z))^{\theta-1}=\sum_{n\geqslant -1} c_n z^{n+\theta}$$

or (more rigorously)

$$\left(\frac{\psi(z)}{z}\right)^{\theta-1}=\sum_{n\geqslant 0}c_{n-1}z^n.$$

A polynomial version of Prawitz' theorem was obtained by Nehari [9] in 1969. He proved that for $\psi(0)=0$ and $p(z)=z^{\theta-1}q(z)$ (where q is regular and single-valued in $\overline{\mathbb{C}}\setminus\psi(\mathbb{D})$) we have

$$\sum_{n\in\mathbb{Z}}(n+\theta)|c_n|^2\leqslant 0,$$

where the coefficients c_n are defined by the Laurent type decomposition

$$p(\psi(z))=\sum_{n\in\mathbb{Z}}c_n z^{n+\theta},$$

In this context, like in the case of the classical polynomial area theorem, one can effectively find a decomposition of $p(\psi)$ into positive and negative parts with respect to the indefinite norm

$$\sum_{n\in\mathbb{Z}}(n+\theta)|c_n|^2 \tag{1.6}$$

and then derive analogs of Grunsky and Goluzin inequalities (for Grunsky-type inequalities, this was done by Nehari in the same paper [9]).

The recent paper [1] by N. Abuzyarova and H. Hedenmalm develops a geometrical approach to area theorems. Using analysis on certain doubly connected domain, they obtained an area-type integral inequality corresponding to the case where $\psi(0)=0$, $p(z)=z^{1/2}(z-\psi(\lambda))^{-1/2}$, $\lambda\in\mathbb{D}$.

In the context of general functions of the form (1.4), following questions arise very naturally. What is an appropriate analog of the indefinite norms (1.2) and (1.6) so that one still has the fundamental inequality $\left[p(\psi),p(\psi)\right]\leqslant 0$ (eventually, under some additional conditions on the function q)? Is there some natural decomposition of this norm into positive and negative parts which can be effectively calculated for functions $p(\psi)$? What are then appropriate analogs of Grunsky and Goluzin inequalities?

The answer to the first of these questions is given by the well-known indefinite inner product

$$\left[f,g\right]=\frac{i}{2\pi}\int_{\mathbb{T}}f(\zeta)\,d\left(\overline{g(\zeta)}\right), \tag{1.7}$$

where $\mathbb{T}=\partial\mathbb{D}$ is the positively oriented unit circle. The right hand side of this formula is well defined if both f and g are character-automorphic (with the same character) functions defined on the universal covering of $\mathbb{T}$, because the form $f\,d\overline{g}$ in this case does not depend on the choice of the branch of this covering. It is easy to see that the inner products (1.2) and (1.6) are special cases of (1.7). We shall call (1.7) *the Dirichlet form* because for functions f holomorphic in $\mathbb{D}$ the inner product $\big[f, f\big]$ reduces to the Dirichlet integral

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

Here, dA stands for the normlalized area measure: $dA(z) = dx\,dy/\pi$, $z = x + iy$. Clearly, any function p of the form (1.4) can be alternatively written as

$$p(z) = z^{\theta_1 + \cdots + \theta_n} q_1(z),$$

where q_1 is regular and single-valued in a neighborhood of $\overline{\mathbb{C}} \setminus \psi(\mathbb{D})$. If m is the integral part of $\theta_1 + \cdots + \theta_n$, and the function q has a zero of multiplicity at least $m + 1$ at infinity in the case where $\theta_1 + \cdots + \theta_n$ is not an integer, or of multiplicity at least m in the case where $\theta_1 + \cdots + \theta_n$ is an integer, then Nehari's result (or the classical polynomial area theorem) implies that

$$[p \circ \psi, p \circ \psi] \leqslant 0.$$

Our goal in the present paper is to answer two other of the above questions. First, it is well known from the theory of indefinite inner product spaces that the fundamental decompositions of such spaces into orthogonal sums of positive and negative subspaces are not unique (see [3, Section I.11]). §3 of the present paper is devoted to such decompsitions with respect to the Dirichlet form, which are most relevant for functions $p(\psi)$ with p of the form (1.4). Namely, in the case where $\theta_1 + \cdots + \theta_n \leqslant 1$, we find a decomposition of the Dirichlet form (see Theorem 3.4 in §3) such that one can *effectively* calculate the decomposition of $p \circ \psi$ where p is given by (1.4) and the rational function q with poles in $\psi(\mathbb{D})$ that occurs in that formula is written as a sum of partial fractions.

This decomposition allows us to obtain corresponding area theorems. In order to formulate them in an economical form, in §4 we introduce an abstract language of domination of kernel functions. This language gives a better understanding even of classical topics, such as the Goluzin-Lebedev inequalities for univalent functions.

Two main area theorems (Theorems 5.1 and 5.2) presented in the form of certain domination relations for kernel functions are formulated and proved in §5. Using the domination formalism, one can interpret them either as analogs of Goluzin-Lebedev inequalities or in the form of integral estimates. Finally, §6 is devoted to examples and special cases. In the cases of one and two branching points, we obtain explicit formulas for kernel functions involved in the formulation of area theorems.

One of applications of our techniques is the area-type proof of the pointwise inequality:

$$\Big|\zeta\frac{F''(\zeta)}{F'(\zeta)}+\frac{4|\zeta|^2-2}{|\zeta|^2-1}-\frac{4|\zeta|^2}{|\zeta|^2-1}\,\frac{E(1/|\zeta|)}{K(1/|\zeta|)}\Big|\leqslant\frac{4|\zeta|^2}{|\zeta|^2-1}\Big(1-\frac{E(1/|\zeta|)}{K(1/|\zeta|)}\Big),\quad |\zeta|>1, \tag{1.8}$$

valid for functions $F\in\Sigma$ (i.e., functions F univalent in $\bar{\mathbb{C}}\setminus\bar{\mathbb{D}}$ and normalized so that $F(\zeta)=\zeta+b_0+b_1\zeta^{-1}+\dots$). Here, K and E are standard complete elliptic integrals of the first and, respectively, second kind. This inequality was obtained by G. M. Goluzin (see [6, Ch. IV, Section 3]) by the variational method. Recently, N. Abuzyarova and H. Hedenmalm [1] found an integral inequality of area type leading to (1.8). Their method was based on an appropriate apllication of the Green formula in certain doubly connected domain sheeted over the Riemann sphere, and elliptic functions appear in explicit formulas for the Green function of this domain. In §6, we obtain an integral inequality equivalent to that obtained in [1] as a special case of the domination corresponding to two branching points. In our method, the elliptic integrals appear in explicit formulas for appropriate reproducing kernels.

§2. Character-automorphic functions and the Dirichlet form

Let Ω be a simply connected domain in $\mathbb{C}$ and let $\mathcal{M}=\{\mu_1,\dots,\mu_n\}$ be a finite sequence of distinct points in Ω. Let $\Theta=\{\theta_1,\dots,\theta_n\}$, where $\theta_k\in[0,1)$, $k=1,\dots,n$. We shall say that a function f analytic and multivalued in $\Omega\setminus\mathcal{M}$ is $(\mathcal{M},\Theta)$-character-automorphic there if the analytic continuation of f along any closed path γ lying in $\Omega\setminus\mathcal{M}$ and satisfying

$$\operatorname{ind}(\gamma,\mu_k)=m_k$$

results in multiplication of f by $\exp(2\pi i(\theta_1m_1+\dots+\theta_nm_n))$. A typical example of such a function is

$$f(z)=(z-\mu_1)^{\theta_1}\dots(z-\mu_n)^{\theta_n}$$

(writing such expressions, we assume always that the value of f at some fixed point of the universal covering surface of $\Omega\setminus\mathcal{M}$ is specified). Moreover, if f is an arbitrary $(\mathcal{M},\Theta)$-character-automorphic function in $\Omega\setminus\mathcal{M}$, then the ratio

$$\frac{f(z)}{(z-\mu_1)^{\theta_1}\dots(z-\mu_n)^{\theta_n}}$$

is $(\mathcal{M},0)$-character-automorphic and hence single-valued in $\Omega\setminus\mathcal{M}$. We shall say that f is a meromorphic $(\mathcal{M},\Theta)$-character-automorphic function in Ω if f has the form

$$f(z)=(z-\mu_1)^{\theta_1}\dots(z-\mu_n)^{\theta_n}g(z),$$

where g is single-valued and meromorphic in Ω. Let $\mathfrak{M}_{\mathcal{M},\Theta}$ denote the family of all meromorphic $(\mathcal{M},\Theta)$-character-automorphic functions in Ω. Clearly, the family $\mathfrak{M}_{\mathcal{M},\Theta}$ is linear and closed with respect to the differentiation operator d/dz.

The following lemma is almost obvious.

Lemma 2.1. *Suppose Ω is a simply connected domain, $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \Omega$, and ψ is a conformal mapping from Ω onto Ω'. Let also $\mathcal{M} = \{\mu_1, \dots, \mu_n\}$, where $\mu_k = \psi(\lambda_k)$, $k = 1, \dots, n$. As before, $\Theta = \{\theta_1, \dots, \theta_n\}$, with $\theta_k \in [0, 1)$, $k = 1, \dots, n$.*

If f is meromorphic $(\mathcal{M},\Theta)$-character-automorphic in Ω', then the composition $f \circ \psi$ is meromorphic (Λ,Θ)-character-automorphic in Ω.

More generally, let Ω be an arbitrary domain in $\mathbb{C}$, γ a closed path in Ω, and let $\theta \in [0, 1)$. We shall say that a function f analytic and multivalued in Ω is (γ,θ)-character-automorphic if the analytic continuation of f along γ results in multiplication of f by $\exp(2\pi i\theta)$.

In what follows, the letter $\mathbb{T}$ will denote the unit circle viewed either as a set or as a positively oriented closed path, depending on the context. We denote by dm the normalized arc length measure on $\mathbb{T}$.

Let f and g be functions analytic (and multivalued) in some neighborhood of $\mathbb{T}$ and $(\mathbb{T},\theta)$-character-automorphic there. The Dirichlet form $[f, g]$ is defined by

$$[f, g] := \frac{i}{2\pi} \int_{\mathbb{T}} f(\zeta)\, d\overline{g(\zeta)}. \tag{2.1}$$

The integral is well defined because the differential form $f d\bar{g}$ is $(\mathbb{T}, 0)$-character-automorphic. Here are some alternative formulas for the calculation of the Dirichlet form:

$$\begin{aligned} [f, g] &= \frac{i}{2\pi} \int_{\mathbb{T}} \partial_{\bar{z}}\Big(f(z)\overline{g(z)}\Big)\, d\bar{z} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \partial_z\Big(f(z)\overline{g(z)}\Big)\, dz = \int_{\mathbb{T}} z\, \partial_z f(z)\, \overline{g(z)}\, dm(z). \end{aligned} \tag{2.2}$$

Here,

$$\partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

are standard Wirtinger differential operators.

Clearly, any $(\mathbb{T},\theta)$-character-automorphic function f has a representation

$$f(z) = z^{\theta} f_1(z),$$

where f_1 is analytic and single-valued near $\mathbb{T}$ and hence

$$f(z) = \sum_{k\in\mathbb{Z}} c_k z^{k+\theta}. \tag{2.3}$$

Writing a similar formula

$$g(z) = \sum_{k\in\mathbb{Z}} d_k z^{k+\theta}, \tag{2.4}$$

for another $(\mathbb{T},\theta)$-character automorphic function g, we easily obtain

$$[f,g] = \sum_{k\in\mathbb{Z}} (k+\theta) c_k \overline{d_k}. \tag{2.5}$$

This formula can serve as a definition of the Dirichlet form for functions f and g defined almost everywhere on the universal covering of $\mathbb{T}$ and having representations (2.3) and (2.4) with coefficients satisfying

$$\sum_{k\in\mathbb{Z}} |k||c_k|^2 < +\infty \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |k||d_k|^2 < +\infty.$$

The class of such functions will be denoted by $\mathcal{G}_\theta$ (θ can be an arbitrary real number).

In the case where θ is not an integer, $\mathcal{G}_\theta$ supplied with the Dirichlet form is a Kreĭn space. A fundamental symmetry is

$$Jf(z) = \sum_{k\in\mathbb{Z}} \operatorname{sign}(k+\theta) c_k z^{k+\theta}$$

for f defined by (2.3), and the corresponding Hilbert J-norm is

$$\|f\|^2_{J,\theta} = \sum_{k\in\mathbb{Z}} |k+\theta||c_k|^2. \tag{2.6}$$

A natural positive (respectively, negative) subspace of $\mathcal{G}_\theta$ consisting of functions of the form

$$f(z) = \sum_{k+\theta\geqslant 0} f_k z^{k+\theta},$$

respectively,

$$g(z) = \sum_{k+\theta\leqslant 0} g_k z^{k+\theta},$$

will be denoted by $\mathcal{G}^+_\theta$ (respectively, $\mathcal{G}^-_\theta$). In the case where θ is an integer, the subspace $\{C\}$ consisting of constant functions is an isotropic subspace of $\mathcal{G}_\theta$ and the quotient-space $\mathcal{G}_\theta/\{C\}$ is a Kreĭn space. Positive and negative subspaces $\mathcal{G}^+_\theta$ and $\mathcal{G}^-_\theta$ are defined as before (obviously, $\mathcal{G}^+_\theta \cap \mathcal{G}^-_\theta = \{C\}$ in the case where θ is an integer).

The formula

$$[f,g]=\int_{\mathbb{T}} z\partial_z f(z)\,\overline{g(z)}\,dm(z) \tag{2.7}$$

remains valid for arbitrary $f,g\in\mathcal{G}_\theta$ if $z\partial_z f$ is interpreted as a distribution with Fourier series

$$h(z)=z\partial_z f(z)\sim\sum_{k\in\mathbb{Z}}(k+\theta)c_k z^{k+\theta} \tag{2.8}$$

and the integral

$$\int_{\mathbb{T}} h\bar g\,dm$$

is interpreted in the sense of the Parseval identity.

From (2.5) (or from (2.2)) it follows easily that

$$\big[z^\eta f(z),z^\eta g(z)\big]=\eta\int_{\mathbb{T}} f\bar g\,dm+[f,g]$$

for any $f,g\in\mathcal{G}_\theta$. It turns out that a similar formula is true if z^η is replaced by more general multivalued factors unimodular on $\mathbb{T}$.

Let $\Lambda=\{\lambda_1,\dots,\lambda_n\}\subset\mathbb{D}$ and $\Theta=\{\theta_1,\dots,\theta_n\}$ with $\theta_k\in\mathbb{R}$, $k=1,\dots,n$. We define

$$\omega_{\Lambda,\Theta}(z):=\prod_{k=1}^n\left(\omega_{\lambda_k}(z)\right)^{\theta_k}, \tag{2.9}$$

where

$$\omega_\lambda(z)=\frac{z-\lambda}{1-\bar\lambda z}. \tag{2.10}$$

Remark. The function $\omega_{\Lambda,\Theta}$ can be thought of as a multivalued finite Blaschke product. The function

$$\omega_{\Lambda,\mathbf{1}}(z)=\prod_{k=1}^n\omega_{\lambda_k}(z) \tag{2.11}$$

(usual finite Blaschke product) will be denoted simply by $\omega_\Lambda(z)$.

Proposition 2.1. *For any $\theta\in[0,1)$ and $f,g\in\mathcal{G}_\theta$, we have*

$$\big[\omega_{\Lambda,\Theta}f,\omega_{\Lambda,\Theta}g\big]=\int_{\mathbb{T}}\left(\sum_{k=1}^n\theta_k\frac{1-|\lambda_k|^2}{|1-\bar\lambda_k\zeta|^2}\right)f(\zeta)\overline{g(\zeta)}\,dm(\zeta)+[f,g]. \tag{2.12}$$

Proof. An explicit calculation shows that

$$\partial_z\omega_{\Lambda,\Theta}(z)=\left(\sum_{k=1}^n\theta_k\frac{1-|\lambda_k|^2}{(z-\lambda_k)(1-\bar\lambda_k z)}\right)\omega_{\Lambda,\Theta}(z)$$

and that

$$\frac{z(1-|\lambda|^2)}{(z-\lambda)(1-\bar{\lambda}z)} = \frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2}$$

for $z \in \mathbb{T}$. Using the last identity in (2.2), we find

$$\begin{aligned}\big[\omega_{\Lambda,\Theta} f, \omega_{\Lambda,\Theta} g\big] &= \int_{\mathbb{T}} z\, \partial_z \omega_{\Lambda,\Theta}(z)\, f(z) \overline{\omega_{\Lambda,\Theta}(z) g(z)}\, dm(z) \\ &\quad + \int_{\mathbb{T}} z\omega_{\Lambda,\Theta}(z)\, \partial_z f(z) \overline{\omega_{\Lambda,\Theta}(z) g(z)}\, dm(z) \\ &= \int_{\mathbb{T}} \bigg(\sum_{k=1}^{n} \theta_k \frac{1-|\lambda_k|^2}{|1-\bar{\lambda}_k z|^2} \bigg) f(z)\overline{g(z)}\, dm(z) + \int_{\mathbb{T}} z\partial_z f(z) \overline{g(z)}\, dm(z),\end{aligned}$$

which yields (2.12). •

Remark. If $f = g$ is a function holomorphic in $\mathbb{D}$ and all θ_k are positive integers, (2.12) is a special case of the well-known Carleson formula for the Dirichlet integral (see [4]).

Now, we turn to the properties of the Dirichlet form related to different composition operators. An easiest of them is the involution $\sharp$ defined by

$$f^\sharp(z) := \overline{f\Big(\frac{1}{\bar{z}}\Big)} \tag{2.13}$$

for f analytic near $\mathbb{T}$. Clearly, $f^\sharp = \bar{f}$ if we view f as a function defined only on (the universal covering of) $\mathbb{T}$. From (2.1), it follows immediately that

$$[f^\sharp, g^\sharp] = -[g, f]. \tag{2.14}$$

We also have

$$\omega^\sharp_{\Lambda,\Theta}(z) = (\omega_{\Lambda,\Theta}(z))^{-1} \tag{2.15}$$

and

$$(z\partial_z g)^\sharp\,(u) = -u\partial_u g^\sharp(u). \tag{2.16}$$

The following two propositions are sources of all area thorems derived in the present paper.

Proposition 2.2. *Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_n \in \mathbb{D}$, and let ψ be a smooth conformal map of $\mathbb{D}$ into itself. Let $\psi(\Lambda) = \{\psi(\lambda_1), \dots, \psi(\lambda_n)\}$. Let p be a meromorphic function of class $\mathfrak{M}_{\psi(\Lambda),\Theta}$ in some domain containing the closed unit disk. Assume also that all poles of p lie in the domain $\psi(\mathbb{D})$. Then*

$$[p,p] - [p\circ\psi, p\circ\psi] = \int_{\mathbb{D}\setminus\psi(\mathbb{D})} |p'(z)|^2\, dA(z). \tag{2.17}$$

Proof. The left-hand side of (2.17) can be written as

$$\frac{i}{2\pi}\int_{\mathbb{T}} p\,d\bar{p} - \frac{i}{2\pi}\int_{\mathbb{T}} p\circ\psi\, d\overline{p\circ\psi} = \frac{i}{2\pi}\bigg(\int_{\mathbb{T}} p\,d\bar{p} - \int_{\psi(\mathbb{T})} p\,d\bar{p}\bigg) = \frac{i}{2\pi}\int_{\Gamma} p\,d\bar{p}, \tag{2.18}$$

where Γ is the oriented boundary of $\mathbb{D}\setminus\psi(\mathbb{D})$. An application of the Green formula now implies the result. •

The calculation (2.18) is by no means new. A very similar argument appears, for example, in the paper [9] (see formula (2.3) of that paper).

Formula (2.17) remains true even without any smoothness assumption on ψ. Indeed, any p in the context of Proposition 2.2 has the form

$$p(z) = \omega_{\psi(\Lambda),\Theta}(z)\bigg(\sum_{k=1}^{m}\frac{c_k}{(z-\psi(a_k))^{\nu_k}} + q(z)\bigg)$$

with $a_k \in \mathbb{D}$ and q analytic in $\overline{\mathbb{D}}$. Clearly, for arbitrary ψ the composition $p\circ\psi$ belongs to the class $\mathcal{G}_\theta$ with $\theta = \theta_1 + \cdots + \theta_n$, which shows that $[p\circ\psi, p\circ\psi]$ is well defined. Moreover, the negative coefficients in the decomposition

$$p(\psi(z)) = \sum_{k\in\mathbb{Z}} c_k z^{k+\theta}, \quad z\in\mathbb{T},$$

decay as

$$|c_{-k}| = O(r_0^k), \quad k\to+\infty,$$

for some $r_0 \in (0,1)$. An approximation argument involving the functions $\psi_r(z) = \psi(rz)$, $r\in(0,1)$, now implies the claim.

Corollary 2.1. *Let $\Lambda = \{\lambda_1,\dots,\lambda_n\}\subset\mathbb{D}$, and let ψ be a conformal self-map of $\mathbb{D}$, $\Theta = \{\theta_1,\dots,\theta_n\}$, $\theta_k\in[0,1)$. If p is a meromorphic function of class $\mathfrak{M}_{\psi(\Lambda),\Theta}$ in the closed unit disk with poles in the domain $\psi(\mathbb{D})$, then*

$$[p\circ\psi, p\circ\psi] \leqslant [p,p], \tag{2.19}$$

and equality happens if and only if ψ is a full self-map of $\mathbb{D}$ (i.e., ψ maps $\mathbb{D}$ onto $\mathbb{D}\setminus E$, where E is an area-null set).

Proposition 2.3. *Let f be a univalent function defined in $\mathbb{D}$. If P is a function of the form*

$$P(\zeta) = \bigg[\prod_{k=1}^{n}(\zeta - f(\lambda_k))^{\theta_k}\bigg]\sum_{\nu=1}^{n}\frac{c_\nu}{\zeta - f(u_\nu)} \tag{2.20}$$

such that

$$\overline{\lim_{|\zeta|\to\infty}}\,|P(\zeta)| < +\infty, \tag{2.21}$$

then

$$-[P\circ f, P\circ f] = \int_{\mathbb{C}\setminus f(\mathbb{D})} |P'(z)|^2\, dA(z).$$

Proof. First, we consider the case where f is sufficiently smooth and bounded. The function $P(\zeta)$ can be written near infinity as

$$P(\zeta) = \zeta^{\theta_1+\cdots+\theta_n} P_1\Big(\frac{1}{\zeta}\Big),$$

where P_1 is a function regular near the origin. Condition (2.21) implies that P_1 has zero of order at least $\theta = \theta_1 + \cdots + \theta_n$ at the origin. This implies that

$$|P'(\zeta)| = O\Big(\frac{1}{|\zeta|}\Big) \tag{2.22}$$

near infinity. We may assume without loss of generality that

$$\lim_{|\zeta|\to\infty} P(\zeta) = 0, \tag{2.23}$$

because $\overline{\lim\limits_{|\zeta|\to\infty}} |P(\zeta)| > 0$ is possible only in the case where $\theta = \theta_1 + \cdots + \theta_n$ is an integer and P_1 has zero of order θ at the origin, and in this case one may replace $P(\zeta)$ by $P(\zeta) - C$, where $C = \lim\limits_{|\zeta|\to\infty} P(\zeta)$, without changing $[P\circ f, P\circ f]$. Relations (2.22) and (2.23) imply that

$$\lim_{R\to\infty} \int_{R\mathbb{T}} P\, d\bar{P} = 0.$$

The same argument as in the proof of Proposition 2.2 shows that for sufficiently large R (such that $R\mathbb{D}$ contains $f(\mathbb{D})$) we have

$$\frac{i}{2\pi}\int_{R\mathbb{T}} P\, d\bar{P} - [P\circ f, P\circ f] = \int_{R\mathbb{D}\setminus f(\mathbb{D})} |P'(z)|^2\, dA(z).$$

It remains to let $R\to\infty$.

The case of arbitrary f can be obtained by replacing first f by $f_r(z) := f(rz)$, $r\in(0,1)$, and then letting $r\to 1-0$. Passage to the limit is justified by the following lemma.

Lemma 2.2. *The function $P\circ f$ belongs to $\mathcal{G}_\theta$, $\theta = \theta_1 + \cdots + \theta_n$. Moreover, the negative coefficients in the decomposition*

$$P(f(\zeta)) = \sum_{k\in\mathbb{Z}} c_k \zeta^{k+\theta}$$

decay as

$$|c_{-k}| = O(r_0^k), \quad k\to+\infty,$$

where r_0 is some number in $(0,1)$.

Proof. The lemma will follow if we show that the single-valued function

$$Q(z) = \prod_{k=1}^{n} \Big(\frac{f(\zeta) - f(\lambda_k)}{\zeta - \lambda_k}\Big)^{\theta_k} \cdot \Big(\sum_{\nu=1}^{N} \frac{c_\nu}{f(\zeta) - f(u_\nu)}\Big) \cdot \prod_{\nu=1}^{N} (\zeta - u_\nu)$$

belongs to $\mathcal{G}_0^+$.

Let m be an integer such that $m < \theta \leqslant m+1$. Then condition (2.21) implies that

$$P(\zeta) = \frac{\prod_{k=1}^{n}(\zeta - f(\lambda_k))^{\theta_k} \cdot q(\zeta)}{\prod_{\nu=1}^{N}(\zeta - f(u_\nu))},$$

where q is a polynomial of degree at most $N - m - 1$. Hence

$$P(\zeta) = \prod_{k=1}^{n}(\zeta - f(\lambda_k))^{\theta_k} \sum_{\nu=m+1}^{N} \frac{d_\nu}{(\zeta - f(u_\nu)) \prod_{l=1}^{m}(\zeta - f(u_l))}.$$

We see that it suffices to show that for any choice of $u_1, \dots, u_{m+1} \in \mathbb{D}$ the function

$$R(\zeta) = \prod_{k=1}^{n} \Big(\frac{f(\zeta) - f(\lambda_k)}{\zeta - \lambda_k}\Big)^{\theta_k} \cdot \prod_{l=1}^{m+1} \Big(\frac{\zeta - u_l}{f(\zeta) - f(u_l)}\Big)$$

is in $\mathcal{G}_0^+$. To do this, it suffices to show that

$$\int_{r_0<|\zeta|<1} |R'(\zeta)|^2 \, dA(\zeta) < +\infty \tag{2.24}$$

for some $r_0 \in (0,1)$. But we have

$$\begin{aligned} R'(\zeta) = R(\zeta)\Big(\sum_{k=1}^{n} \theta_k \Big(\frac{f'(\zeta)}{f(\zeta) - f(\lambda_k)} - \frac{1}{\zeta - \lambda_k}\Big) \\ - \sum_{l=1}^{m+1} \Big(\frac{f'(\zeta)}{f(\zeta) - f(u_l)} - \frac{1}{\zeta - u_l}\Big)\Big). \end{aligned} \tag{2.25}$$

In the case where $\theta < m+1$, we have

$$|R(\zeta)| \asymp \frac{1}{|f(\zeta)|^{m+1-\theta}}$$

near $\mathbb{T}$ (we suppose without loss of generality that f does not assume zero values near $\mathbb{T}$) and hence

$$\int_{r_0<|\zeta|<1} |R'(\zeta)|^2 \, dA(\zeta) \asymp \int_{r_0<|\zeta|<1} \frac{|f'(\zeta)|^2}{|f(\zeta)|^{2+2(m+1-\theta)}} \, dA(\zeta),$$

which is finite by geometric reasons. In the case where $\theta = m+1$, we have

$$\sum_{k=1}^{n} \theta_k \frac{f'(\zeta)}{f(\zeta) - f(\lambda_k)} - \sum_{l=1}^{m+1} \frac{f'(\zeta)}{f(\zeta) - f(u_l)} = O\Big(\Big|\frac{f'(\zeta)}{f(\zeta)^2}\Big|\Big)$$

near $\mathbb{T}$ due to the cancellation of first order terms as $f(\zeta) \to \infty$. Hence

$$|R'(\zeta)| = O\Big(\Big|\frac{f'(\zeta)}{f(\zeta)^2}\Big|\Big),$$

which gives again a finite integral (2.24). •

Corollary 2.2. *Let f be a univalent function defined on $\mathbb{D}$. If $\theta_1 + \dots + \theta_n \leqslant 1$, then for any function P of the form* (2.20) *we have*

$$[P \circ f, P \circ f] \leqslant 0.$$

Moreover, equality occurs for full mappings f (i.e., such that $\mathbb{C} \setminus f(\mathbb{D})$ is an area-null set).

§3. Fundamental decompositions

In this section, we study certain fundamental decompositions of the spaces $\mathcal{G}_\theta$ with indefinite Dirichlet form into orthogonal positive and negative subspaces.

Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a finite sequence of distinct points of $\mathbb{D}$, and $\Theta = \{\theta_1, \dots, \theta_n\}$ a sequence of numbers $\theta_k \in (0,1)$. Let the function $\omega_{\Lambda,\Theta}$ be defined by (2.9) and (2.10). An easy argument which is left to the reader shows that multiplication by $\omega_{\Lambda,\Theta}$ is an invertible transformation from $\mathcal{G}_\eta$ onto $\mathcal{G}_\nu$, where

$$\nu = \eta + \theta_1 + \dots + \theta_n.$$

In the case where both η and ν are not integers, this multiplication operator is continuous and invertible as an operator between Kreĭn spaces.

Definition 3.1. Let $\theta = \theta_1 + \dots + \theta_n$. A subspace of $\mathcal{G}_\theta$ consisting of functions f of the form

$$f = \omega_{\Lambda,\Theta} g \quad \text{with} \quad g \in \mathcal{G}_0^+$$

will be denoted by $\mathcal{G}_{\Lambda,\Theta}^+$.

Applying (2.12), we see that

$$\begin{aligned}[\omega_{\Lambda,\Theta} g, \omega_{\Lambda,\Theta} g] &= \int_{\mathbb{T}} \Big(\sum_{k=1}^{n} \theta_k \frac{1 - |\lambda_k|^2}{|1 - \bar{\lambda}_k \zeta|^2}\Big) |g(\zeta)|^2 \, dm(\zeta) + [g,g] \\ &= \int_{\mathbb{T}} \Big(\sum_{k=1}^{n} \theta_k \frac{1 - |\lambda_k|^2}{|1 - \bar{\lambda}_k \zeta|^2}\Big) |g(\zeta)|^2 \, dm(\zeta) + \sum_{k \geqslant 1} k |\hat{g}(k)|^2 \qquad (3.1)\end{aligned}$$

for any $g \in \mathcal{G}_0^+$, which shows that $\mathcal{G}_{\Lambda,\Theta}^+$ is a positive subspace of $\mathcal{G}_\theta$. Moreover, it is also uniformly positive as a subspace of a Kreĭn space in the case where $\theta \notin \mathbb{Z}$, which means that

$$[f, f] \geqslant c\|f\|_{J,\theta}^2$$

for any $f \in \mathcal{G}_{\Lambda,\Theta}^+$. Later on in this section, we shall prove that $\mathcal{G}_{\Lambda,\Theta}^+$ is a maximal positive subspace (which means that it is not contained in any bigger positive subspace) in the case where $\theta_1 + \cdots + \theta_n < 1$. The Dirichlet space $\mathcal{G}_0^+$ (regarded as a space of functions analytic in $\mathbb{D}$) supplied with the norm

$$\|g\|_{\mathcal{D}_{\Lambda,\Theta}}^2 := \int_{\mathbb{T}} \left(\sum_{k=1}^n \theta_k \frac{1 - |\lambda_k|^2}{|1 - \bar{\lambda}_k \zeta|^2} \right) |g(\zeta)|^2 \, dm(\zeta) + \sum_{k \geqslant 1} k |\hat{g}(k)|^2 = [\omega_{\Lambda,\Theta} g, \omega_{\Lambda,\Theta} g] \tag{3.2}$$

will be denoted by $\mathcal{D}_{\Lambda,\Theta}$. Let $k_{\Lambda,\Theta}^+$ denote the reproducing kernel for the space $\mathcal{D}_{\Lambda,\Theta}$. In other words, for each $\mu \in \mathbb{D}$, the function $k_{\Lambda,\Theta}^+(\cdot, \mu) \in \mathcal{D}_{\Lambda,\Theta}$ is uniquely determined by the identity

$$g(\mu) = [\omega_{\Lambda,\Theta} g, \omega_{\Lambda,\Theta} k_{\Lambda,\Theta}^+(\cdot, \mu)], \quad g \in \mathcal{G}_0^+.$$

The following lemma gives an integral representation of the norm in the space $\mathcal{D}_{\Lambda,\Theta}$.

Lemma 3.1. *For any $g \in \mathcal{D}_{\Lambda,\Theta}$ we have*

$$\|g\|_{\mathcal{D}_{\Lambda,\Theta}}^2 = \int_{\mathbb{D}} \left| \left(\sum_{k=1}^n \theta_k \frac{1 - |\lambda_k|^2}{(1 - \bar{\lambda}_k z)^2} \omega_\Lambda^k(z) \right) g(z) + \omega_\Lambda(z) g'(z) \right|^2 |\omega_{\Lambda,\Theta - \mathbf{1}}(z)|^2 \, dA(z). \tag{3.3}$$

Here, ω_Λ is defined by (2.11),

$$\omega_\Lambda^k(z) = \frac{\omega_\Lambda(z)}{\omega_{\lambda_k}(z)},$$

and $\Theta - \mathbf{1}$ means the sequence $(\theta_1 - 1, \ldots, \theta_n - 1)$.

Proof. Let $\mathbb{D}_{\Lambda,\varepsilon}$be the unit disk with deleted disks of small radius $\varepsilon > 0$ around points λ_k, $k = 1, \ldots, n$. Then

$$\begin{aligned} \|g\|_{\Lambda,\Theta}^2 &= [\omega_{\Lambda,\Theta} g, \omega_{\Lambda,\Theta} g] = \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{\partial \mathbb{D}_{\Lambda,\varepsilon}} \omega_{\Lambda,\Theta}(z) g(z) \, d\left(\overline{\omega_{\Lambda,\Theta}(z) g(z)} \right) \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{D}_{\Lambda,\varepsilon}} \left| \partial_z \left(\omega_{\Lambda,\Theta}(z) g(z) \right) \right|^2 dA(z) \\ &= \int_{\mathbb{D}} \left| \left(\sum_{k=1}^n \theta_k \frac{1 - |\lambda_k|^2}{(z - \lambda_k)(1 - \bar{\lambda}_k z)} \right) \omega_{\Lambda,\Theta}(z) g(z) + \omega_{\Lambda,\Theta}(z) g'(z) \right|^2 dA(z), \end{aligned}$$

which gives (3.3). •

Our next goal is a characterization and the study of the orthogonal companion to $\mathcal{G}^+_{\Lambda,\Theta}$ with respect to the Dirichlet form. Let $\theta = \theta_1 + \cdots + \theta_n$.

Definition 3.2. A subspace of $\mathcal{G}_{-\theta}$ consisting of functions H satisfying

$$[H^\sharp, g] = 0$$

for any $g \in \mathcal{G}^+_{\Lambda,\Theta}$ will be denoted by $\mathcal{G}^-_{\Lambda,\Theta}$.

In other words, $(\mathcal{G}^-_{\Lambda,\Theta})^\sharp$ is the orthogonal companion to $\mathcal{G}^+_{\Lambda,\Theta}$.

Let $\mathcal{B}^+_0$ denote the usual Bergman space of single-valued functions x analytic in $\mathbb{D}$ and satisfying

$$\|x\|^2_{\mathcal{B}^+_0} = \int_{\mathbb{D}} |x(z)|^2 \, dA(z) = \sum_{n \geqslant 0} \frac{|\hat{x}(n)|^2}{n+1} < +\infty.$$

We shall need also a two-sided character-automorphic Bergman space $\mathcal{B}_\theta$ consisting of distributions h defined on the universal covering of $\mathbb{T}$, having Fourier series

$$h(u) \sim \sum_{k \in \mathbb{Z}} h_k u^{k+\theta},$$

and satisfying

$$\sum_{k \in \mathbb{Z}} \frac{|h_k|^2}{|k|+1} < +\infty.$$

Obviously, $z\partial_z f \in \mathcal{B}_\theta$ if and only if $f \in \mathcal{G}_\theta$. Multiplication by $\omega_{\Lambda,\Theta}$ is an invertible transformation from $\mathcal{B}_\eta$ onto $\mathcal{B}_\nu$, where $\nu = \eta + \theta_1 + \cdots + \theta_n$ (the proof of this fact is left to the reader).

Lemma 3.2. *A function $H \in \mathcal{G}_{-\theta}$ belongs to $\mathcal{G}^-_{\Lambda,\Theta}$ if and only if*

$$u\partial_u H(u) = u{\omega_{\Lambda,\Theta}}^{-1}(u) x(u) \tag{3.4}$$

on (*the universal covering of*) $\mathbb{T}$, *where x is a function in $\mathcal{B}^+_0$.*

Proof. By definition, $H \in \mathcal{G}^-_{\Lambda,\Theta}$ is equivalent to

$$[H^\sharp, \omega_{\Lambda,\Theta} p] = 0$$

for any analytic polynomial p, which, by (2.7) and (2.16), can be rewritten as

$$0 = \int_{\mathbb{T}} z\partial_z H^\sharp(z)\overline{\omega_{\Lambda,\Theta}(z)p(z)}\,dm(z)$$
$$= -\int_{\mathbb{T}} (u\partial_u H)^\sharp\,(u)\overline{\omega_{\Lambda,\Theta}(u)p(u)}\,dm(u)$$
$$= -\overline{\int_{\mathbb{T}} \omega_{\Lambda,\Theta}(u)u\partial_u H(u)p(u)\,dm(u)}.$$

The assertion is obvious now. •

A natural question in connection with this lemma is what functions x can appear in (3.4). The answer depends on whether $\theta = \theta_1 + \cdots + \theta_n$ is an integer or not.

If θ is an integer, then ${\omega_{\Lambda,\Theta}}^{-1}x \in \mathcal{B}_0$ for $x \in \mathcal{B}_0^+$ and the existence of $H \in \mathcal{G}_0$ satisfying (3.4) is equivalent to the property

$$\int_{\mathbb{T}} u{\omega_{\Lambda,\Theta}}^{-1}(u)x(u)\,dm(u) = 0$$

or

$$\int_\gamma \frac{x(u)}{\omega_{\Lambda,\Theta}(u)}\,du = 0, \tag{3.5}$$

where γ is an arbitrary simple contour in $\mathbb{D}$ enclosing the points $\lambda_1, \dots, \lambda_n$. In other words, the function ${\omega_{\Lambda,\Theta}}^{-1}(u)x(u)$ which is analytic and single-valued in some annulus $\{r < |u| < 1\}$, where $|\lambda_k| < r$, $k = 1, \dots, n$, should have a single-valued primitive function in this annulus.

In the case where θ is not an integer, formula (2.8) shows that any function $x \in \mathcal{B}_0^+$ can appear in (3.4).

The next question is how to express the Dirichlet form $[H, H]$ in terms of the function x if H satisfies (3.4). The difficulty here is that H satisfying (3.4) need not be a boundary value of a $(\Lambda, -\theta)$-character-automorphic function and hence one cannot apply Green's formula to the domain $\mathbb{D} \setminus \{\lambda_1, \dots, \lambda_n\}$ as we did in the proof of Lemma 3.1.

First, we consider the case where θ is not an integer. So, we assume that $H \in \mathcal{G}^-_{\Lambda,\Theta}$ and

$$\partial_u H(u) = h(u) = \frac{x(u)}{\omega_{\Lambda,\Theta}(u)} \tag{3.6}$$

on (the universal covering of) $\mathbb{T}$, where $x \in \mathcal{B}_0^+$. Obviously, the right-hand side is a $(\Lambda, -\theta)$-character-automorphic function in $\mathbb{D} \setminus \{\lambda_1, \dots, \lambda_n\}$. We choose some simple smooth paths $\gamma_1, \dots, \gamma_{n-1}$ such that

- γ_k connects λ_n with λ_k;
- γ_i intersects γ_j only at the point λ_n, and

- one can choose $\varphi_k = \arg\gamma_k'$ at the point λ_n so that

$$\varphi^* < \varphi_1 < \varphi_2 < \cdots < \varphi_{n-1} < \varphi^* + 2\pi,$$

where φ^* is some value (see Figure 1). A path γ in $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$ is then constructed as follows. A starting point λ_n^0 is chosen on the path γ_1 infinitesimally close to λ_n. Then one goes from $\lambda_n^0$to λ_1 along γ_1 obtaining a path γ_1^-; then around λ_1 along an infinitesimally small circle in positive direction ; then back to λ_n along γ_1 obtaining a path γ_1^+; then to the path γ_2 along an arc of an infinitesimally small circle centered at λ_n in positive direction; then along γ_2 to λ_2 obtaining a path γ_2^-, around λ_2 and back to λ_n along γ_2 obtaining γ_2^+, and so on, returning finally to λ_n^0 along an arc of an infinitesimally small circle centered at λ_n in positive direction (see Figure 1). We fix also some point λ_n^* on the universal covering of $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$ whose projection to $\mathbb{D}$ is λ_n^0. The lifting of γ to the universal covering of $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$ will be denoted by γ^*. Let also $\Omega \subset \mathbb{D}$ be some smooth simply connected domain containing all paths γ_k except the endpoints λ_j, $j = 1, \ldots, n$. Let h_0 be a single-valued branch of the function h defined on Ω and satisfying $h_0(\lambda_n^0) = h(\lambda_n^*)$. We put

$$I_k := \int_{\gamma_k^-} h_0(u)\, du, \quad k = 1, \ldots, n-1. \tag{3.7}$$

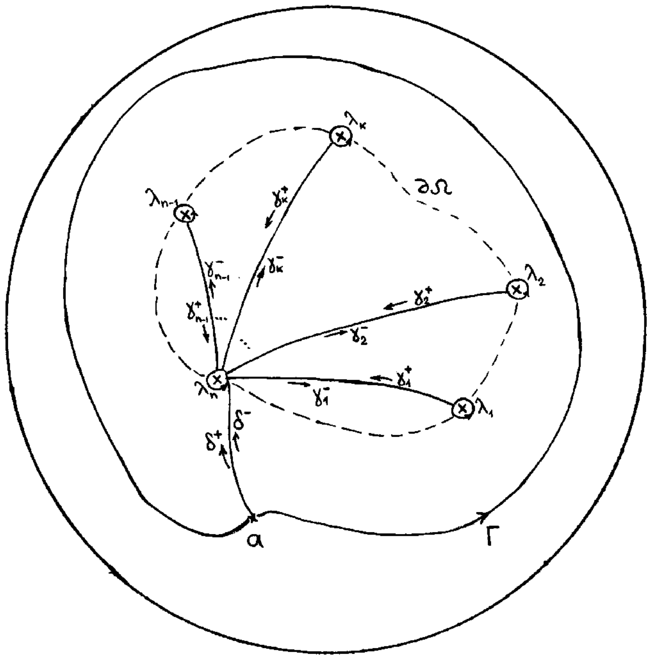
The path γ constructed above is homotopically equivalent to $\mathbb{T}$ in $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$. If a simply-connected domain $\Omega \subset \mathbb{D}$ containing the points $\lambda_1, \ldots, \lambda_n$ on its boundary is already given, then the existence of the paths $\gamma_1, \ldots, \gamma_{n-1}$ lying in Ω and satisfying the above requirements is equivalent to the property that the sequence $\lambda_1, \ldots, \lambda_n$ follows the positive orientation of $\partial\Omega$.

Lemma 3.3. *There exists a unique function H defined and multivalued in $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$, satisfying* (3.6) *and* $(\Gamma, -\theta)$*-character-automorphic for any simple positively oriented path Γ in $\mathbb{D}$ enclosing the points $\{\lambda_1, \ldots, \lambda_n\}$. Moreover, this function H satisfies*

$$H(\lambda_n^*) = \left(e^{-2\pi i\theta} - 1\right)^{-1} \cdot \sum_{k=1}^{n} (1 - e^{-2\pi i\theta_k}) e^{-2\pi i(\theta_1 + \cdots + \theta_{k-1})} I_k. \tag{3.8}$$

Proof. Let H be some function satisfying (3.6), and let $H(\lambda_n^*) = C$. We show first that this constant C (and hence the function H also) is uniquely determined by the requirement that H be $(\gamma, -\theta)$-character-automorphic. First, a process of analytic continuation of the function h along the path γ^* shows that

$$h(u) = h_0(u) \text{ on } \gamma_1^-; \quad h(u) = e^{-2\pi i\theta_1} h_0(u) \text{ on } \gamma_1^+;$$
$$h(u) = e^{-2\pi i\theta_1} h_0(u) \text{ on } \gamma_2^-; \quad h(u) = e^{-2\pi i(\theta_1+\theta_2)} h_0(u) \text{ on } \gamma_2^+,$$



The path γ.

and so on. Hence

$$\begin{aligned}
\int_{\gamma^*} h(u)\,du &= \sum_{k=1}^{n-1}\Big(\int_{\gamma_k^-}+\int_{\gamma_k^+}\Big)h(u)\,du \\
&= \sum_{k=1}^{n-1}\Big(\int_{\gamma_k^-} e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}h_0(u)\,du - \int_{\gamma_k^-} e^{-2\pi i(\theta_1+\dots\theta_k)}h_0(u)\,du\Big) \\
&= \sum_{k=1}^{n-1}\int_{\gamma_k^-}\big(1-e^{-2\pi i\theta_k}\big)e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}h_0(u)\,du \\
&= \sum_{k=1}^{n-1}\big(1-e^{-2\pi i\theta_k}\big)e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}I_k.
\end{aligned}$$

The requirement that H be $(\gamma, -\theta)$-character-automophic implies that

$$H(\lambda_n^*) + \int_{\gamma^*} h(u)\,du = e^{-2\pi i\theta} H(\lambda_n^*),$$

which gives

$$C = (e^{-2\pi i\theta} - 1)^{-1} \sum_{k=1}^{n-1} \left(1 - e^{-2\pi i\theta_k}\right) e^{2\pi i(\theta_1+\cdots+\theta_{k-1})} I_k \tag{3.9}$$

and proves (3.8).

It remains to show that if H is the primitive function to h satisfying $H(\lambda_n^*) = C$, where C is given by (3.9), then H is $(\Gamma, -\theta)$-character-automorphic for any simple positively oriented path Γ enclosing the points $\{\lambda_1, \ldots, \lambda_n\}$ (i.e., homotopically equivalent to $\mathbb{T}$ in $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$). Let Γ^* denote the lifting of such a path to the universal covering of $\mathbb{D} \setminus \{\lambda_1, \ldots, \lambda_n\}$ having the initial point a^- and the endpoint a^+. Suppose also δ^- is an arbitrary path on the universal covering connecting a^- and λ_n^*, and $(\delta^+)^{-1}$ is the image of δ^- under the canonical isomorphism of the universal covering induced by the homotopy class of paths τ equivalent to $\mathbb{T}$, so that, in particular, the endpoint of δ^+ is a^+ (see Figure 1). Since γ and Γ are both homotopically equivalent to $\mathbb{T}$, we have

$$\int_{\Gamma^*} h(u)\,du = \int_{\delta^-} h(u)\,du + \int_{\gamma^*} h(u)\,du + \int_{\delta^+} h(u)\,du,$$

which implies

$$\begin{aligned}
H(a^+) &= H(a^-) + \int_{\delta^-} h(u)\,du + (e^{-2\pi i\theta} - 1)H(\lambda_n^*) - e^{-2\pi i\theta} \int_{\delta^-} h(u)\,du \\
&= H(a^-) + (e^{-2\pi i\theta} - 1)\Big(H(\lambda_n^*) - \int_{\delta^-} h(u)\,du\Big) \\
&= H(a^-) + (e^{-2\pi i\theta} - 1)H(a^-) = e^{-2\pi i\theta} H(a^-).
\end{aligned}$$

This proves that H is $(\Gamma, -\theta)$-character-automorphic. •

Now, we apply Green's formula to the domain $\mathbb{D} \setminus \operatorname{supp}\gamma$, and we get

$$\begin{aligned}
[H, H] &= \int_{\mathbb{T}} \partial_u H(u) \cdot \overline{H(u)} \frac{du}{2\pi i} \\
&= \int_{\gamma} h(u)\overline{H(u)}\, \frac{du}{2\pi i} + \int_{\mathbb{D}\setminus \operatorname{supp}\gamma} |\partial_u H(u)|^2\, dA(u) \\
&= \int_{\gamma} h(u)\overline{H(u)} \frac{du}{2\pi i} + \int_{\mathbb{D}} |x(u)|^2\, \frac{dA(u)}{|\omega_{\Lambda,\Theta}(u)|^2}.
\end{aligned}$$

The first term in this formula can be expressed via the integrals I_k. First, we have

$$H(u^-) = C + \int_{\lambda_n^*}^{u^-} h(v)\,dv = C + \int_{\lambda_n}^{u^-} h_0(v)\,dv \quad \text{on the path} \quad \gamma_1^-; \tag{3.10}$$

here $C = H(\lambda_n^*)$ is given by (3.9) and the last integral is taken along γ_1. Since $h(v) = e^{-2\pi i\theta_1} h_0(v)$ along γ_1^+, we obtain

$$\begin{aligned} H(u^+) &= C + I_1 + \int_{\lambda_1}^{u^+} e^{-2\pi i\theta_1} h_0(v)\,dv \\ &= C + (1 - e^{-2\pi i\theta_1}) I_1 + e^{-2\pi i\theta_1} \int_{\lambda_n}^{u^+} h_0(v)\,dv \end{aligned} \tag{3.11}$$

on the path γ_1^+, the last integral being taken along γ_1. Continuing similarly, we find

$$H(u^-) = C + \sum_{l=1}^{k-1} (1 - e^{-2\pi i\theta_l}) e^{-2\pi i(\theta_1+\cdots+\theta_{l-1})} I_l + e^{-2\pi i(\theta_1+\cdots+\theta_{k-1})} \int_{\lambda_n}^{u^-} h_0(v)\,dv \tag{3.12}$$

on the path γ_k^- and

$$H(u^+) = C + \sum_{l=1}^{k} (1 - e^{-2\pi i\theta_l}) e^{-2\pi i(\theta_1+\cdots+\theta_{l-1})} I_l + e^{-2\pi i(\theta_1+\cdots+\theta_k)} \int_{\lambda_n}^{u^+} h_0(v)\,dv \tag{3.13}$$

on the path γ_k^+, both integrals in these formulas being taken along γ_k. In particular,

$$H(u^+) - H(u^-) = e^{-2\pi i(\theta_1+\cdots+\theta_{k-1})} (1 - e^{-2\pi i\theta_k}) \int_{u}^{\lambda_k} h_0(v)\,dv \tag{3.14}$$

if u^-, u^+ denote some point $u \in \operatorname{supp} \gamma_k$ interpreted as lying on γ_k^-, γ_k^+, respectively.

Now, we have

$$
\begin{aligned}
\left(\int_{\gamma_k^-}+\int_{\gamma_k^+}\right)h(u)\overline{H(u)}\,\frac{du}{2\pi i} &= \int_{\lambda_n}^{\lambda_k}\Big(h(u^-)\overline{H(u^-)}-h(u^+)\overline{H(u^+)}\Big)\,\frac{du}{2\pi i}\\
&= \int_{\lambda_n}^{\lambda_k}\bigg(h(u^-)\overline{H(u^-)}-h(u^+)\overline{H(u^-)}\\
&\qquad - h(u^+)e^{2\pi i(\theta_1+\cdots+\theta_{k-1})}(1-e^{2\pi i\theta_k})\overline{\int_u^{\lambda_k}h_0(v)\,dv}\bigg)\,\frac{du}{2\pi i}\\
&= \int_{\gamma_k^-}\bigg((1-e^{-2\pi i\theta_k})h(u^-)\overline{H(u^-)}\\
&\qquad - h(u^-)e^{2\pi i(\theta_1+\cdots+\theta_{k-1})}(e^{-2\pi i\theta_k}-1)\overline{\int_u^{\lambda_k}h_0(v)\,dv}\bigg)\,\frac{du}{2\pi i}. \qquad (3.15)
\end{aligned}
$$

Using the fact that

$$
\begin{aligned}
&H(u^-)+e^{-2\pi i(\theta_1+\ldots\theta_{k-1})}\int_u^{\lambda_k}h_0(v)\,dv\\
&\quad = C+\sum_{l=1}^{k-1}(1-e^{-2\pi i\theta_l})e^{-2\pi i(\theta_1+\cdots+\theta_{l-1})}I_l+e^{-2\pi i(\theta_1+\cdots+\theta_{k-1})}I_k,
\end{aligned}
$$

we see that the last expression in (3.15) reduces to

$$
\begin{aligned}
&(1-e^{-2\pi i\theta_k})\int_{\gamma_k^-}h(u^-)\\
&\qquad\times\Big(\overline{C}+\sum_{l=1}^{k-1}(1-e^{2\pi i\theta_l})e^{2\pi i(\theta_1+\cdots+\theta_{l-1})}\overline{I_l}+e^{2\pi i(\theta_1+\cdots+\theta_{k-1})}\overline{I_k}\Big)\,\frac{du}{2\pi i}\\
&\quad=\frac{1}{2\pi i}(1-e^{-2\pi i\theta_k})e^{-2\pi i(\theta_1+\cdots+\theta_{k-1})}I_k\\
&\qquad\times\Big(\overline{C}+\sum_{l=1}^{k-1}(1-e^{2\pi i\theta_l})e^{2\pi i(\theta_1+\cdots+\theta_{l-1})}\overline{I_l}+e^{2\pi i(\theta_1+\cdots+\theta_{k-1})}\overline{I_k}\Big). \qquad (3.16)
\end{aligned}
$$

Using the identity

$$
\begin{aligned}
C + \sum_{l=1}^{k-1} (1 - e^{-2\pi i\theta_l}) e^{-2\pi i(\theta_1+\dots+\theta_{l-1})} I_l + e^{-2\pi i(\theta_1+\dots+\theta_{k-1})} I_k \\
= \sum_{l=1}^{k-1} e^{-2\pi i\theta} \frac{(1 - e^{-2\pi i\theta_l}) e^{-2\pi i(\theta_1+\dots+\theta_{l-1})}}{e^{-2\pi i\theta} - 1} I_l \\
+ \frac{e^{-2\pi i\theta} - e^{-2\pi i\theta_k}}{e^{-2\pi i\theta} - 1} \cdot e^{-2\pi i(\theta_1+\dots+\theta_{k-1})} I_k \\
+ \sum_{l=k+1}^{n-1} \frac{(1 - e^{-2\pi i\theta_l}) e^{-2\pi i(\theta_1+\dots+\theta_{l-1})}}{e^{-2\pi i\theta} - 1} I_l,
\end{aligned}
$$

we find

$$
\begin{aligned}
&\left(\int_{\gamma_k^-} + \int_{\gamma_k^+} \right) h(u)\overline{H(u)} \, \frac{du}{2\pi i} \\
&= \frac{1}{2\pi i} \sum_{l=1}^{k-1} (1 - e^{-2\pi i\theta_k}) \overline{(1 - e^{-2\pi i\theta_l})} e^{-2\pi i(\theta_1+\dots+\theta_{k-1})} e^{2\pi i(\theta_1+\dots+\theta_{l-1})} \\
&\times \frac{e^{2\pi i\theta}}{e^{2\pi i\theta} - 1} I_k \overline{I_l} + \frac{1}{2\pi i} \frac{(1 - e^{-2\pi i\theta_k})(e^{2\pi i\theta} - e^{2\pi i\theta_k})}{e^{2\pi i\theta} - 1} |I_k|^2 \\
&+ \frac{1}{2\pi i} \sum_{l=k+1}^{n-1} (1 - e^{-2\pi i\theta_k}) \overline{(1 - e^{-2\pi i\theta_l})} e^{-2\pi i(\theta_1+\dots+\theta_{k-1})} e^{2\pi i(\theta_1+\dots+\theta_{l-1})} \\
&\times \frac{1}{e^{2\pi i\theta} - 1} I_k \overline{I_l}.
\end{aligned}
$$

If we denote

$$
J_k := (1 - e^{-2\pi i\theta_k}) e^{-2\pi i(\theta_1+\dots+\theta_{k-1})} I_k, \tag{3.17}
$$

then the last expression reduces to

$$
\frac{1}{4\pi \sin \pi\theta} \left(-\sum_{l=1}^{k-1} e^{\pi i\theta} J_k \overline{J_l} + \frac{\sin \pi(\theta - \theta_k)}{\sin \pi\theta_k} |J_k|^2 - \sum_{l=k+1}^{n-1} e^{-\pi i\theta} J_k \overline{J_l} \right).
$$

This implies the identity

$$
\int_\gamma h(u)\overline{H(u)} \, \frac{du}{2\pi i} = \frac{1}{4\pi \sin \pi\theta} \mathbf{Q}_\Theta(J_1, \dots, J_{n-1}),
$$

where the quadratic form $\mathbf{Q}_\Theta$ is given by the formula

$$\mathbf{Q}_\Theta(J_1,\dots,J_{n-1}) := \sum_{k=1}^{n-1} \frac{\sin\pi(\theta-\theta_k)}{\sin\pi\theta_k}|J_k|^2 - 2\operatorname{Re}\Bigg(\sum_{1\leqslant l\leqslant k-1} e^{\pi i\theta} J_k \overline{J_l}\Bigg). \quad (3.18)$$

Finally, we arrive at the following statement.

Theorem 3.1. *Assume that $\theta=\theta_1+\dots+\theta_n$ is not an integer. Then $H\in\mathcal{G}^-_{\Lambda,\Theta}$ if and only if there exists a function $x\in\mathcal{B}^+_0$ such that*

$$\partial_u H(u) = h(u) = \frac{x(u)}{\omega_{\Lambda,\Theta}(u)}.$$

Moreover, let $\Omega\subset\mathbb{D}$ be a smooth simply connected domain such that the sequence $\{\lambda_1,\dots,\lambda_n\}$ lies on the boundary $\partial\Omega$ and follows the positive orientation of $\partial\Omega$. Let h_0 be some single-valued branch of h defined in Ω, and let

$$J_k = (1-e^{-2\pi i\theta_k})e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}\int_{\lambda_n}^{\lambda_k} h_0(u)\,du, \quad k=1,\dots,n-1 \quad (3.19)$$

(*the integral is taken along some path lying in* Ω). *Then*

$$[H,H] = \int_{\mathbb{D}} |x(u)|^2 \frac{dA(u)}{|\omega_{\Lambda,\Theta}(u)|^2} + \frac{1}{4\pi\sin\pi\theta}\mathbf{Q}_\Theta(J_1,\dots,J_{n-1}), \quad (3.20)$$

where the quadratic form $\mathbf{Q}_\Theta$ is defined by (3.18).

Remark. If a is an arbitrary point in Ω, then we can choose formally $\lambda_{n+1}=a$, $\theta_{n+1}=0$ and then apply Theorem 3.1 to the sequences $\Lambda'=\{\lambda_1,\dots,\lambda_n,a\}$ and $\Theta'=\{\theta_1,\dots,\theta_n,0\}$. Then we obtain

$$[H,H] = \int_{\mathbb{D}} |x(u)^2| \frac{dA(u)}{|\omega_{\Lambda,\Theta}(u)|^2} + \frac{1}{4\pi\sin\pi\theta}\mathbf{Q}_{\Theta'}(J'_1,\dots,J'_n),$$

where

$$J'_k = (1-e^{-2\pi i\theta_k})e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}\int_a^{\lambda_k} h_0(u)\,du, \quad k=1,\dots,n.$$

We turn now to the case where $\theta=\theta_1+\dots+\theta_n$ is an integer. We keep the same notation as before (except that for C, which can be an arbitrary constant). The function H is not uniquely determined by h now. Choosing an arbitrary constant C and then taking H to be a primitive function to h satisfying $H(\lambda_n^*)=C$, we obtain a function which is $(\Gamma,0)$-character-automorphic for any simple path Γ enclosing points $\{\lambda_1,\dots,\lambda_n\}$ because

$$\int_\Gamma h(u)\,du = 0 \quad (3.21)$$

(see the discussion after Lemma 3.2). This identity can be rewritten also as

$$\sum_{k=1}^{n-1}(1-e^{-2\pi i\theta_k})e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}I_k=0. \tag{3.22}$$

Formulas (3.10)–(3.16) remain valid without changes, and we obtain

$$\begin{aligned}\int_\gamma h(u)\overline{H(u)}\,\frac{du}{2\pi i}&=\frac{1}{2\pi i}\left(\sum_{k=1}^{n-1}(1-e^{-2\pi i\theta_k})e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}I_k\right)\cdot\overline{C}\\&+\frac{1}{2\pi i}\sum_{k=1}^{n-1}(1-e^{-2\pi i\theta_k})e^{-2\pi i(\theta_1+\dots+\theta_{k-1})}I_k\\&\times\sum_{l=1}^{k-1}(1-e^{2\pi i\theta_l})e^{2\pi i(\theta_1+\dots+\theta_{l-1})}\overline{I_l}\\&+\sum_{k=1}^{n-1}\frac{1-e^{-2\pi i\theta_k}}{2\pi i}|I_k|^2.\end{aligned} \tag{3.23}$$

As before, we define J_k by (3.17), and we observe that (3.22) reduces to

$$J_1+\dots+J_{n-1}=0. \tag{3.24}$$

The expression (3.23) then becomes

$$\sum_{k=1}^{n-1}\frac{|J_k|^2}{2\pi i(1-e^{2\pi i\theta_k})}+\frac{1}{2\pi i}\sum_{1\leqslant l<k\leqslant n-1}J_k\overline{J_l}.$$

Subtracting the expression

$$0=\frac{1}{4\pi i}|J_1+\dots+J_{n-1}|^2=\frac{1}{4\pi i}\sum_{1\leqslant l,k\leqslant n-1}J_k\overline{J_l}$$

from this formula, we finally get

$$\int_\gamma h(u)\overline{H(u)}\,\frac{du}{2\pi i}=\frac{1}{4\pi}\tilde{\mathbf{Q}}_\Theta(J_1,\dots,J_{n-1}),$$

where

$$\widetilde{\mathbf{Q}}_\Theta(J_1,\dots,J_{n-1}):=\sum_{k=1}^{n-1}\cot\pi\theta_k|J_k|^2-2\operatorname{Re}\left(i\sum_{1\leqslant l<k\leqslant n-1}J_k\overline{J_l}\right). \tag{3.25}$$

Therefore, we have proved the following statement.

Theorem 3.2. *Assume that $\theta = \theta_1 + \cdots + \theta_n$ is an integer. Then $H \in \mathcal{G}^-_{\Lambda,\Theta}$ if and only if there exists a function $x \in \mathcal{B}^+_0$ such that*

$$\partial_u H(u) = h(u) = \frac{x(u)}{\omega_{\Lambda,\Theta}(u)}, \quad u \in \mathbb{D},$$

and the integral of h along any simple path enclosing points $\{\lambda_1, \dots, \lambda_n\}$ vanishes.

Moreover, let Ω, h_0, and J_k be as in Theorem 3.1. Then $J_1 + \cdots + J_{n-1} = 0$ and

$$[H, H] = \int_{\mathbb{D}} |x(u)|^2 \frac{dA(u)}{|\omega_{\Lambda,\Theta}(u)|^2} + \widetilde{\mathbf{Q}}_\Theta(J_1, \dots, J_{n-1}), \tag{3.26}$$

where the quadratic form $\widetilde{\mathbf{Q}}_\Theta$ is defined by (3.25).

The quadratic form $\mathbf{Q}_\Theta$ is positive definite in the case where $\theta_1 + \cdots + \theta_n = \theta \leqslant 1$. This follows from the next lemma by the Sylvester criterion.

Lemma 3.4. *Let $\theta_1, \dots, \theta_m$ and θ be real numbers such that θ_k, $k = 1, \dots, m$ are not integers. Let $A = A(\theta_1, \dots, \theta_m; \theta) = (A_{kj})_{1 \leqslant k,j \leqslant m}$ denote the matrix with entries*

$$A_{kj} = \begin{cases} \frac{\sin \pi(\theta - \theta_k)}{\sin \pi\theta_k} & \text{if } k = j; \\ -e^{i\pi\theta} & \text{if } k < j; \\ -e^{-i\pi\theta} & \text{if } k > j. \end{cases}$$

Then

$$\det\left[A(\theta_1, \dots, \theta_m; \theta)\right] = \frac{\left(\sin \pi\theta\right)^{m-1} \cdot \sin \pi(\theta - \theta_1 - \cdots - \theta_m)}{\sin \pi\theta_1 \cdot \ldots \cdot \sin \pi\theta_m}. \tag{3.27}$$

Proof. Without loss of generality we may assume that all numbers $\theta - (\theta_1 + \cdots + \theta_k)$, $k = 1, \dots, m-1$ are not integers. Replacing each row $A_{k\cdot}$, $k = 2, \dots, n$ of the matrix A by the row

$$A_{k\cdot} + \frac{\sin \pi\theta_1}{\sin \pi(\theta - \theta_1)} e^{-i\pi\theta} A_{1\cdot},$$

we obtain the matrix

$$\begin{pmatrix} \frac{\sin \pi(\theta - \theta_1)}{\sin \pi\theta_1} & -e^{i\pi\theta} & \dots & -e^{i\pi\theta} \\ 0 & B_{11} & \dots & B_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{m-1,1} & \dots & B_{m-1,m-1} \end{pmatrix},$$

where the new matrix B is given by

$$B = \frac{\sin \pi\theta}{\sin \pi(\theta - \theta_1)} A(\theta_2, \dots, \theta_m; \theta - \theta_1).$$

Hence

$$\det\big[A(\theta_1,\dots,\theta_m;\theta)\big] = \frac{\sin\pi\theta}{\sin\pi\theta_1}\det\big[A(\theta_2,\dots,\theta_m;\theta-\theta_1)\big].$$

Iterating this relation, we arrive at (3.27). •

Corollary 3.1. *If* $\theta_1+\dots+\theta_n=\theta\leqslant 1$, *then the quadratic form* $\mathbf{Q}_\Theta$ *given by* (3.18) *is positive definite.*

Proposition 3.1. *If* θ_k, $k=1,\dots,n$ *are positive and* $\theta_1+\dots+\theta_n=\theta=1$, *then the quadratic form* $\widetilde{\mathbf{Q}}_\Theta(J_1,\dots,J_{n-1})$ *given by* (3.25) *is positive definite on the hyperplane*

$$J_1+\dots+J_{n-1}=0. \tag{3.28}$$

Proof. Let $r\Theta$ be the sequence $\{r\theta_1,\dots,r\theta_n\}$, where $r\in(0,1]$. By the preceding discussion, the quadratic form $\mathbf{Q}_{r\Theta}$ is positive definite for any $r\in(0,1]$. Moreover, $\mathbf{Q}_\Theta(J_1,\dots,J_{n-1})=0$ on the hyperplane (3.28). Hence

$$-\frac{\partial}{\partial r}\Big|_{r=1}\big[\mathbf{Q}_{r\Theta}(J_1,\dots,J_{n-1})\big]\geqslant 0$$

on the hyperplane (3.28). But the last expression is equal to $\pi\widetilde{\mathbf{Q}}(J_1,\dots,J_{n-1})$, which gives the desired result. •

Therefore, we have proved the following.

Theorem 3.3. *If* $\theta_1+\dots+\theta_n=\theta\leqslant 1$, *then*

$$[H,H]\geqslant 0$$

for any $H\in\mathcal{G}^-_{\Lambda,\Theta}$.

Corollary 3.2. (a) *If* $\theta_1+\dots+\theta_n=\theta<1$, *then* $\mathcal{G}^+_{\Lambda,\Theta}$ *is a maximal positive subspace* (*with respect to the Dirichlet form*) *in* $\mathcal{G}_\theta$.

(b) *If* $\theta_1+\dots+\theta_n=\theta=1$, *then the subspace* $\{C\}+\mathcal{G}^+_{\Lambda,\Theta}$ *is a maximal positive subspace in* $\mathcal{G}_1(=\mathcal{G}_0)$. *Here* $\{C\}$ *is the subspace of constant functions.*

Proof. (a) If θ is not an integer, then $\mathcal{G}_\theta$ is a Kreĭn space and, clearly, $\mathcal{G}^+_{\Lambda,\Theta}$ is a closed positive subspace there. Since its orthogonal companion $(\mathcal{G}^+_{\Lambda,\Theta})^\perp=(\mathcal{G}^-_{\Lambda,\Theta})^\sharp$ is a negative subspace, Lemma V.4.5 of [3] implies the claim.

(b) In the case where $\theta=1$, the subspace $\{C\}$ of constant functions is an isotropic subspace in $\mathcal{G}_0$ and the factorspace $\mathcal{G}_0/\{C\}$ is a Kreĭn space. As before, $\{C\}+\mathcal{G}^+_{\Lambda,\Theta}$ is a maximal positive subspace in $\mathcal{G}_0/\{C\}$ and hence it is also a maximal positive subspace in $\mathcal{G}_0$. •

The results on the maximal uniformly positive subspaces of Kreĭn spaces presented in [3, Ch. V], imply now that subspaces $\mathcal{G}^+_{\Lambda,\Theta}$ and $(\mathcal{G}^+_{\Lambda,\Theta})^\perp = (\mathcal{G}^-_{\Lambda,\Theta})^\sharp$ form a fundamental decomposition of $\mathcal{G}_\theta$ into a sum of orthogonal positive and negative subspaces in the case where $\theta_1 + \cdots + \theta_n = \theta \leqslant 1$ (if $\theta = 1$, we obtain first a decomposition of $\mathcal{G}_0/\{C\}$, and then we realize that the constant subspace {C} is included in $(\mathcal{G}^-_{\Lambda,\Theta})^\sharp$). Therefore, we have proved the following theorem.

Theorem 3.4. *Let* $\theta_1 \cdots + \theta_n = \theta \leqslant 1$. *Then* $\mathcal{G}_\theta$ *is the direct orthogonal sum*

$$\mathcal{G}_\theta = \mathcal{G}^+_{\Lambda,\Theta} \dot{+} (\mathcal{G}^-_{\Lambda,\Theta})^\sharp \tag{3.29}$$

of the positive subspace $\mathcal{G}^+_{\Lambda,\Theta}$ *and the negative subspace* $(\mathcal{G}^-_{\Lambda,\Theta})^\sharp$.

Theorem 3.3 shows that in the case where $\theta_1 + \cdots + \theta_n \leqslant 1$, the right hand side of (3.20) or of (3.26) is a certain Hilbert space norm of a function $x \in \mathcal{B}^+_0$. This norm will be denoted by $\|\cdot\|_{\mathcal{B}_{\Lambda,\Theta}}$ and the corresponding scalar product by $\langle\cdot,\cdot\rangle_{\Lambda,\Theta}$. In the case where $\theta_1 + \cdots + \theta_n < 1$, the space $\mathcal{B}^+_0$ supplied with the norm $\|\cdot\|_{\Lambda,\Theta}$ will be denoted by $\mathcal{B}_{\Lambda,\Theta}$. In the case where $\theta_1 + \cdots + \theta_n = 1$, the notation $\mathcal{B}_{\Lambda,\Theta}$ stands for the space of functions x such that

$$\partial_u H(u) = \frac{x(u)}{\omega_{\Lambda,\Theta}(u)} \tag{3.30}$$

for some $H \in \mathcal{G}^-_{\Lambda,\Theta}$; this space is supplied with the norm $\|\cdot\|_{\mathcal{B}_{\Lambda,\Theta}}$. Let also $k^-_{\Lambda,\Theta}$ denote the reproducing kernel for the space $\mathcal{B}_{\Lambda,\Theta}$. In other words, $k^-_{\Lambda,\Theta}(\cdot,\mu)$ is a function analytic in $\mathbb{D}$ such that there exists a function $H_\mu \in \mathcal{G}^-_{\Lambda,\Theta}$ with

$$\partial_u H_\mu(u) = \frac{k^-_{\Lambda,\Theta}(u,\mu)}{\omega_{\Lambda,\Theta}(u)},$$

and for any function x satisfying (3.30) we have

$$x(\mu) = [H, H_\mu] = \langle x, k^-_{\Lambda,\Theta}(\cdot,\mu)\rangle_{\Lambda,\Theta}.$$

The kernel $k^-_{\Lambda,\Theta}$ should be thought of as a Bergman-type kernel, because the norm $\|\cdot\|_{\mathcal{B}_{\Lambda,\Theta}}$ is a Bergman-type norm, while the kernel $k^+_{\Lambda,\Theta}$ introduced earlier is a Dirichlet-type kernel because $\|\cdot\|_{\Lambda,\Theta}$ is a Dirichlet-type norm. Later on in this section, we find a relationship between the kernels $k^-_{\Lambda,\Theta}$ and $k^+_{\Lambda,\Theta}$. Both these kernel functions appear later in §5 in connection with area theorems.

Our next goal is to find an explicit form of decompositions (3.29) for certain functions in $\mathcal{G}_\theta$.

Definition 3.3. For each $a \in \mathbb{D}$, we define

$$f_a(\zeta) = f_{\Lambda,\Theta,a}(\zeta) := \frac{1}{\zeta - a} - \sum_{\nu=1}^{n} \frac{\theta_\nu \bar{\lambda}_\nu}{1 - a\bar{\lambda}_\nu} k^+_{\Lambda,\Theta}(\zeta, \lambda_\nu) \tag{3.31}$$

as an element of $\mathcal{G}_0$.

Remark. In what follows in this section, we assume that Λ and Θ are fixed and do not indicate the dependence of f_a on Λ and Θ explicitly.

Lemma 3.5. *For each $a \in \mathbb{D}$, the function $\omega_{\Lambda,\Theta} f_a$ is orthogonal to $\mathcal{G}^+_{\Lambda,\Theta}$.*

Proof. Let g be an arbitrary function in $\mathcal{G}^+_0$. Then

$$\begin{aligned}
&\big[\omega_{\Lambda,\Theta} f_a, \omega_{\Lambda,\Theta} g\big] \\
&= \Big[\frac{\omega_{\Lambda,\Theta}(\zeta)}{\zeta - a}, \omega_{\Lambda,\Theta}(\zeta) g(\zeta)\Big] - \sum_{\nu=1}^{n} \frac{\theta_\nu \bar{\lambda}_\nu}{1 - a\bar{\lambda}_\nu} \big[\omega_{\Lambda,\Theta} k^+_{\Lambda,\Theta}(\cdot, \lambda_\nu), \omega_{\Lambda,\Theta} g\big] \\
&= \int_{\mathbb{T}} \Big(\sum_{\nu=1}^{n} \theta_\nu \frac{1 - |\lambda_\nu|^2}{|1 - \bar{\lambda}_\nu z|^2}\Big) \frac{\overline{g(z)}}{z - a}\, dm(z) + \Big[\frac{1}{\zeta - a}, g(\zeta)\Big] - \sum_{\nu=1}^{n} \frac{\theta_\nu \bar{\lambda}_\nu}{1 - a\bar{\lambda}_\nu} \overline{g(\lambda_\nu)} \\
&= \int_{\mathbb{T}} \Big(\sum_{\nu=1}^{n} \theta_\nu \frac{1 - |\lambda_\nu|^2}{|1 - \bar{\lambda}_\nu z|^2}\Big) \overline{\Big(\frac{z g(z)}{1 - \bar{a} z}\Big)}\, dm(z) - \sum_{\nu=1}^{n} \frac{\theta_\nu \bar{\lambda}_\nu}{1 - a\bar{\lambda}_\nu} \overline{g(\lambda_\nu)} = 0. \quad \bullet
\end{aligned}$$

Corollary 3.3. *The function* $\dfrac{\omega_{\Lambda,\Theta}(\zeta)}{\zeta - a}$ *decomposes with respect to* (3.29) *as*

$$\frac{\omega_{\Lambda,\Theta}(\zeta)}{\zeta - a} = \omega_{\Lambda,\Theta}(\zeta) \Big[\sum_{\nu=1}^{n} \frac{\theta_\nu \bar{\lambda}_\nu}{1 - a\bar{\lambda}_\nu} k^+_{\Lambda,\Theta}(\zeta, \lambda_\nu)\Big] + \omega_{\Lambda,\Theta}(\zeta) f_a(\zeta). \tag{3.32}$$

Lemma 3.6. *Let $H \in \mathcal{G}^-_{\Lambda,\Theta}$, and let $x \in \mathcal{B}_{\Lambda,\Theta}$ be defined by* (3.30). *Then*

$$-\big[H^\sharp, \omega_{\Lambda,\Theta} f_a\big] = \overline{x(a)}. \tag{3.33}$$

for any $a \in \mathbb{D}$.

Proof. We have

$$\begin{aligned}
-\,[H^\sharp, \omega_{\Lambda,\Theta} f_a] &= -\int_{\mathbb{T}} z \partial_z H^\sharp(z) \cdot \overline{\omega_{\Lambda,\Theta}(z) f_a(z)}\, dm(z) \\
&= \int_{\mathbb{T}} \bar{z} \omega_{\Lambda,\Theta}(z) \overline{x(z)}\, \overline{\omega_{\Lambda,\Theta}(z)} \Big(\frac{1}{\bar{z} - \bar{a}} - \sum_{\nu=1}^{n} \frac{\bar{\theta}_\nu \lambda_\nu}{1 - \bar{a}\lambda_\nu} \overline{k^+_{\Lambda,\Theta}(z, \lambda_\nu)}\Big)\, dm(z) \\
&= \int_{\mathbb{T}} \frac{\bar{z}\overline{x(z)}}{\bar{z} - \bar{a}}\, dm(z) = \overline{x(a)}. \quad \bullet
\end{aligned}$$

As a corollary, we obtain the following fact.

Proposition 3.2. *If* $\theta_1 + \cdots + \theta_n \leqslant 1$, *then*

$$\partial_u (\omega_{\Lambda,\Theta} f_a)^\sharp (u) = \frac{k^-_{\Lambda,\Theta}(u,a)}{\omega_{\Lambda,\Theta}(u)} \tag{3.34}$$

for any $a \in \mathbb{D}$.

Proof. Since $(\omega_{\Lambda,\Theta} f_a)^\sharp \in \mathcal{G}^-_{\Lambda,\Theta}$, we have

$$\partial_u (\omega_{\Lambda,\Theta} f_a)^\sharp (u) = \frac{x_a(u)}{\omega_{\Lambda,\Theta}(u)}$$

for some function $x_a \in \mathcal{B}_{\Lambda,\Theta}$. But then for any function $x \in \mathcal{B}_{\Lambda,\Theta}$ satisfying (3.30) we obtain

$$\langle x, x_a \rangle_{\Lambda,\Theta} = \big[H, (\omega_{\Lambda,\Theta} f_a)^\sharp\big] = -\big[\omega_{\Lambda,\Theta} f_a, H^\sharp\big] = x(a),$$

which shows that $x_a = k^-_{\Lambda,\Theta}(\cdot, a)$. •

Finally, we get another corollary.

Corollary 3.4. *If* $\theta_1 + \cdots + \theta_n \leqslant 1$, *then*

$$-\big[\omega_{\Lambda,\Theta} f_a, \omega_{\Lambda,\Theta} f_b\big] = k^-_{\Lambda,\Theta}(a,b) \tag{3.35}$$

for any $a, b \in \mathbb{D}$.

Proof. We have

$$\begin{aligned} -\big[\omega_{\Lambda,\Theta} f_a, \omega_{\Lambda,\Theta} f_b\big] &= \big[(\omega_{\Lambda,\Theta} f_b)^\sharp, (\omega_{\Lambda,\Theta} f_a)^\sharp\big] \\ &= \langle k^-_{\Lambda,\Theta}(\cdot, b), k^-_{\Lambda,\Theta}(\cdot, a)\rangle_{\Lambda,\Theta} = k^-_{\Lambda,\Theta}(a,b). \quad \bullet \end{aligned}$$

The last inner product can be calculated explicitly in terms of the kernel $k^+_{\Lambda,\Theta}$. We formulate this as a separate statement.

Proposition 3.3. *We have*

$$\begin{aligned} k^-_{\Lambda,\Theta}(a,b) &= -\big[\omega_{\Lambda,\Theta} f_a, \omega_{\Lambda,\Theta} f_b\big] \\ &= \frac{1}{(1-\bar{b}a)^2} - \sum_{\nu=1}^{n} \theta_k \frac{1 - |\lambda_\nu|^2 \bar{b} a}{(1-\bar{b}\lambda_\nu)(1-a\bar{\lambda}_\nu)(1-\bar{b}a)} \\ &\quad + \sum_{1 \leqslant \nu, l \leqslant n} \frac{\theta_\nu \theta_l \bar{\lambda}_\nu \lambda_l}{(1-a\bar{\lambda}_\nu)(1-\bar{b}\lambda_l)} \, k^+_{\Lambda,\Theta}(\lambda_l, \lambda_\nu). \end{aligned} \tag{3.36}$$

Proof. Since all functions $k^+_{\Lambda,\Theta}(\cdot,\lambda_\nu)$ are orthogonal to f_b, we have

$$\begin{aligned}
-\left[\omega_{\Lambda,\Theta} f_a, \omega_{\Lambda,\Theta} f_b\right] &= -\Big[\frac{\omega_{\Lambda,\Theta}(\zeta)}{\zeta - a}, \omega_{\Lambda,\Theta} f_b\Big] \\
&= -\int_{\mathbb{T}} z\partial_z\Big(\frac{\omega_{\Lambda,\Theta}(z)}{z-a}\Big)\cdot\overline{\omega_{\Lambda,\Theta}(z) f_b(z)}\,dm(z) \\
&= -\int_{\mathbb{T}}\Big(-\frac{z}{(z-a)^2} + \sum_{\nu=1}^n \theta_\nu \frac{z(1-|\lambda_\nu|^2)}{(z-\lambda_\nu)(1-\bar\lambda_\nu z)(z-a)}\Big)\overline{f_b(z)}\,dm(z).
\end{aligned}$$

Using the partial fraction expansion

$$\frac{z(1-|\lambda_\nu|^2)}{(z-\lambda_\nu)(1-\bar\lambda_\nu z)(z-a)} = \frac{\frac{\lambda_\nu}{\lambda_\nu - a}}{z-\lambda_\nu} + \frac{\frac{a(1-|\lambda_\nu|^2)}{(a-\lambda_\nu)(1-\bar\lambda_\nu a)}}{z-a} + \frac{\frac{\bar\lambda_\nu}{1-\bar\lambda_\nu a}}{1-\bar\lambda_\nu z}$$

and calculating the integrals

$$\begin{aligned}
\int_{\mathbb{T}} \frac{\overline{f_b(z)}}{z-\mu}\,dm(z) &= \frac{1}{1-\bar b\mu}; \\
\int_{\mathbb{T}} \frac{z\overline{f_b(z)}}{(z-a)^2}\,dm(z) &= \frac{1}{(1-\bar b a)^2}; \\
\int_{\mathbb{T}} \frac{\overline{f_b(z)}}{1-\bar\lambda z}\,dm(z) &= -\sum_{l=1}^n \frac{\theta_l\lambda_l}{1-\bar b\lambda_l} k^+_{\Lambda,\Theta}(\lambda_l,\lambda),
\end{aligned}$$

we arrive at (3.36). •

§4. Domination of kernels

In this section we develop an abstract language of domination relation for kernel functions. It allows us to formulate area theorems of different type in an economical form avoiding too long inequalities.

Let $\mathcal{X}$, $\mathcal{Y}$ be some abstract sets, and let functions $L(x,y)$, $K_1(x,x')$, and $K_2(y,y')$ be such that

- $K_1(x,x')$ is defined on $\mathcal{X}\times\mathcal{X}$ and positive definite;
- $K_2(y,y')$ is defined on $\mathcal{Y}\times\mathcal{Y}$ and positive definite;
- $L(x,y)$ is defined on $\mathcal{X}\times\mathcal{Y}$.

Definition 4.1. In the above context, we shall say that the kernel $L(x,y)$ is dominated by the pair $\{K_1, K_2\}$ and write

$$L(x,y) \ll \big\{K_1(x,x'); K_2(y,y')\big\} \tag{4.1}$$

if for any choice of $x_1, \dots, x_m \in \mathcal{X}$, $y_1, \dots, y_n \in \mathcal{Y}$, $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \mathbb{C}$ we have the following inequality:

$$\Big|\sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_i \bar{\beta}_j L(x_i, y_j)\Big|^2 \leqslant \Big(\sum_{i=1}^{m}\sum_{i'=1}^{m} \alpha_i \bar{\alpha}_{i'} K_1(x_i, x_{i'})\Big)\Big(\sum_{j=1}^{n}\sum_{j'=1}^{n} \beta_j \bar{\beta}_{j'} K_2(y_j, y_{j'})\Big). \tag{4.2}$$

An example of a domination relation is given by the Lebedev inequalities (see [5, p. 125])

$$\Big|\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \mu_j \log\Big(\frac{g(z_i)-g(\zeta_j)}{z_i-\zeta_j}\Big)\Big|^2 \leqslant \Big\{-\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \bar{\lambda}_j \log\Big(1-\frac{1}{z_i \bar{z}_j}\Big)\Big\} \cdot \Big\{-\sum_{i=1}^{N}\sum_{j=1}^{N} \mu_i \bar{\mu}_j \log\Big(1-\frac{1}{\zeta_i \bar{\zeta}_j}\Big)\Big\} \tag{4.3}$$

valid for the univalent functions g in the class Σ (i.e., for the functions g univalent in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and obeying the normalization $g(z) = z + O(1)$ at infinity). This family of inequalities is in fact the domination $L \ll \{K_1; K_2\}$, where

$$L(z,\zeta) = \log\Big(\frac{g(z)-g(\zeta)}{z-\zeta}\Big);$$
$$K_1(z,z') = -\log\Big(1-\frac{1}{z\bar{z}'}\Big);$$
$$K_2(\zeta,\zeta') = -\log\Big(1-\frac{1}{\bar{\zeta}\zeta'}\Big);$$

and all variables are in $\mathbb{C} \setminus \overline{\mathbb{D}}$. It is well known that, by the symmetry of L and the relation $K_1(z, z') = K_2(z', z)$, the inequalities (4.3) are equivalent to the formally weaker Goluzin inequalities

$$\Big|\sum_{1\leqslant i,j\leqslant N} \lambda_i \lambda_j \log\Big(\frac{g(z_i)-g(z_j)}{z_i-z_j}\Big)\Big| \leqslant \sum_{1\leqslant i,j\leqslant N} -\lambda_i \bar{\lambda}_j \log\Big(1-\frac{1}{z_i \bar{z}_j}\Big) \tag{4.4}$$

(see, e.g., the discussion in [10, Section 3.6]).

Another example of domination inequalities (4.2) appears in the paper [2] (see inequality (4.14a) of that paper). It reads as

$$l(\zeta,\eta) \ll \big\{K(\zeta,\zeta'); K(\eta',\eta) - \Gamma(\eta',\eta)\big\}$$

where

$$l(\zeta,\eta) = \frac{1}{\pi(\zeta-\eta)^2} + \frac{2}{\pi}\frac{\partial^2 G_\Omega(\zeta,\eta)}{\partial\zeta\partial\eta};$$
$$K(\eta,\eta') = -\frac{2}{\pi}\frac{\partial^2 G_\Omega(\eta,\eta')}{\partial\eta\partial\bar\eta'};$$
$$\Gamma(\eta,\eta') = \frac{1}{\pi}\int_{\mathbb{C}\setminus\Omega}\frac{dA(z)}{(z-\eta)^2\overline{(z-\eta')^2}};$$

$G_\Omega(\cdot,\cdot)$ is the Green function for the Laplacian in a bounded smooth domain $\Omega\subset\mathbb{C}$, and all variables are in Ω.

A requirement formally weaker than but equivalent to (4.2) is that for any choice of x_i, y_j, α_j, and β_j we have the inequality

$$\Big|\sum_{i,j}\alpha_i\bar\beta_j L(x_i,y_j)\Big| \leqslant \frac{1}{2}\Big(\sum_{i,i'}\alpha_i\bar\alpha_{i'}K_1(x_i,x_{i'}) + \sum_{j,j'}\beta_j\bar\beta_{j'}K_2(y_j,y_{j'})\Big). \quad (4.5)$$

Indeed, replacing in (4.5) α_i by $t\alpha_i$ and β_j by β_j/t with $t>0$ and then minimizing the right-hand side as a function of t, we obtain (4.2) (and the converse implication is trivial).

As a corollary, we obtain the following property: if $L \ll \{K_1; K_2\}$ and $\tilde L \ll \{\tilde K_1; \tilde K_2\}$, then $L+\tilde L \ll \{K_1+\tilde K_1; K_2+\tilde K_2\}$.

The following theorem gives several equivalent conditions for the domination relation for kernel functions. For a positive definite kernel $K(x,x')$, $x,x'\in\mathcal{X}$, the standard notation $\mathcal{H}(K)$ stands for the Hilbert space of functions on $\mathcal{X}$ whose reproducing kernel is K.

Theorem 4.1. *Let kernels $L(x,y)$, $K_1(x,x')$, and $K_2(y,y')$ be as above. Then the following conditions are equivalent*:

(1) $L(x,y) \ll \big\{K_1(x,x'); K_2(y,y')\big\}$;

(2) *There is a contractive operator $T : \mathcal{H}(K_2)\mapsto\mathcal{H}(K_1)$ such that*

$$L(x,y) = (TK_2(\cdot,y), K_1(\cdot,x))_{\mathcal{H}(K_1)} \quad (4.6)$$

for any $x\in\mathcal{X}$, $y\in Y$;

(3a) *$L(\cdot,y)\in\mathcal{H}(K_1)$ for any $y\in\mathcal{Y}$ and, moreover,*

$$\Big\|\sum_{j=1}^{n}\bar\beta_j L(\cdot,y_j)\Big\|^2_{\mathcal{H}(K_1)} \leqslant \sum_{1\leqslant j,j'\leqslant n}\beta_j\bar\beta_{j'}K_2(y_j,y_{j'}) \quad (4.7)$$

for any choice of $y_1,\dots,y_n\in\mathcal{Y}$, $\beta_1,\dots,\beta_n\in\mathbb{C}$;

(3b) *$\overline{L(x,\cdot)}\in\mathcal{H}(K_2)$ for any $x\in\mathcal{X}$ and, moreover,*

$$\Big\| \sum_{i=1}^{m} \bar{\alpha}_i \overline{L(x_i, \cdot)} \Big\|^2_{\mathcal{H}(K_2)} \leqslant \sum_{1 \leqslant i,i' \leqslant m} \alpha_i \bar{\alpha}_{i'} K_1(x_i, x_{i'}). \tag{4.8}$$

for any choice of $x_1, \dots, x_m \in \mathcal{X}$*,* $\alpha_1, \dots, \alpha_m \in \mathbb{C}$.

Proof. We start with the implication (1) $\Rightarrow$ (2). First, we prove that $L(\cdot, y) \in \mathcal{H}(K_1)$ for any $y \in \mathcal{Y}$. Let y be fixed. From (4.2) it follows that

$$\Big| \sum_{i=1}^{n} \alpha_i L(x_i, y) \Big| \leqslant K_2^{1/2}(y, y) \left(\sum_{i,i'} \alpha_i \bar{\alpha}_{i'} K_1(x_i, x_{i'}) \right)^{1/2}$$
$$= K_2^{1/2}(y, y) \Big\| \sum_{i'} \bar{\alpha}_{i'} K_1(\cdot, x_{i'}) \Big\|_{\mathcal{H}(K_1)}.$$

Therefore, the linear mapping

$$\lambda_y \, : \, \sum_i \bar{\alpha}_i K_1(\cdot, x_i) \mapsto \sum_i \bar{\alpha}_i \overline{L(x_i, y)}$$

extends to a bounded linear functional on $\mathcal{H}(K_1)$ and hence it is determined by a scalar product with some function $l_y \in \mathcal{H}(K_1)$. Since

$$\overline{l_y(x)} = (K_1(\cdot, x), l_y)_{\mathcal{H}(K_1)} = \lambda_y(K_1(\cdot, x)) = \overline{L(x, y)},$$

we obtain $L(\cdot, y) = l_y \in \mathcal{H}(K_1)$.

Now, we define an operator $T \, : \, \mathcal{H}(K_2) \mapsto \mathcal{H}(K_1)$ first on finite linear combinations of the functions $K_2(\cdot, y)$ as follows:

$$T\Big(\sum_j \bar{\beta}_j K_2(\cdot, y_j) \Big) := \sum_j \bar{\beta}_j L(\cdot, y_j).$$

Then we have the inequality

$$\Big| \Big(T\Big(\sum_j \bar{\beta}_j K_2(\cdot, y_j) \Big), \sum_i \bar{\alpha}_i K_1(\cdot, x_i) \Big)_{\mathcal{H}(K_1)} \Big|$$
$$= \Big| \sum_{i,j} \alpha_i \bar{\beta}_j L(x_i, y_j) \Big| \leqslant \Big(\sum_{i,i'} \alpha_i \bar{\alpha}_{i'} K_1(x_i, x_{i'}) \Big)^{1/2} \cdot \Big(\sum_{j,j'} \beta_j \bar{\beta}_{j'} K_2(y_j, y_{j'}) \Big)^{1/2}$$
$$= \Big\| \sum_i \bar{\alpha}_i K_1(\cdot, x_i) \Big\| \cdot \Big\| \sum_j \bar{\beta}_j K_2(\cdot, y_j) \Big\|$$

showing that T can be extended to a contractive operator from $\mathcal{H}(K_2)$ to $\mathcal{H}(K_1)$.

To prove the implication (2) $\Rightarrow$(3a), we observe that the assumption

$$L(\cdot, y) = T K_2(\cdot, y)$$

equivalent to (4.6) implies

$$\Big\| \sum_{j=1}^{n} \bar\beta_j L(\cdot, y_j) \Big\|^2_{\mathcal{H}(K_1)} \leqslant \|T\|^2 \Big\| \sum_j \bar\beta_j K_2(\cdot, y_j) \Big\|^2_{\mathcal{H}(K_2)}$$
$$= \|T\|^2 \sum_{j,j'} \beta_j \bar\beta_{j'} K_2(y_j, y_{j'}).$$

Finally, (3a) implies (1) because

$$\Big| \sum_{i,j} \alpha_i \bar\beta_j L(x_i, y_j) \Big|^2 = \Big| \Big(\sum_j \bar\beta_j L(\cdot, y_j),\ \sum_i \bar\alpha_i K_1(\cdot, x_i) \Big)_{\mathcal{H}(K_1)} \Big|^2$$
$$\leqslant \Big\| \sum_j \bar\beta_j L(\cdot, y_j) \Big\|^2_{\mathcal{H}(K_1)} \cdot \Big\| \sum_i \bar\alpha_i K_1(\cdot, x_i) \Big\|^2_{\mathcal{H}(K_1)}$$
$$\leqslant \Big(\sum_{i=1}^{m} \sum_{i'=1}^{m} \alpha_i \bar\alpha_{i'} K_1(x_i, x_{i'}) \Big) \Big(\sum_{j=1}^{n} \sum_{j'=1}^{n} \beta_j \bar\beta_{j'} K_2(y_j, y_{j'}) \Big).$$

Obvious modifications needed to prove implications (2) $\Rightarrow$ ($3b$) $\Rightarrow$ (1) are left to the reader. •

We saw above that domination relations can be added. The next proposition shows that they can be multiplied as well.

Proposition 4.1. *Assume that we have the dominations*

$$L(x, y) \ll \big\{ K_1(x, x'); K_2(y, y') \big\} \text{ and } \tilde L(x, y) \ll \big\{ \tilde K_1(x, x'); \tilde K_2(y, y') \big\},$$

where $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$.

Then

$$L(x, y) \tilde L(x, y) \ll \big\{ K_1(x, x') \tilde K_1(x, x'); K_2(y, y') \tilde K_2(y, y') \big\}.$$

Proof. The proof is completely similar to that of the clssical Schur theorem on elementwise product of positive definite matrices.

By Theorem 4.1, there exist contractions $T : \mathcal{H}(K_2) \mapsto \mathcal{H}(K_1)$ and $\tilde T : \mathcal{H}(\tilde K_2) \mapsto \mathcal{H}(\tilde K_1)$ such that

$$L(x, y) = (T K_2(\cdot, y), K_1(\cdot, x))_{\mathcal{H}(K_1)} \text{ and } \tilde L(\tilde x, \tilde y) = \Big(\tilde T \tilde K_2(\cdot, \tilde y), \tilde K_1(\cdot, \tilde x) \Big)_{\mathcal{H}(\tilde K_1)}.$$

Let $T \otimes \tilde T : \mathcal{H}(K_2) \otimes \mathcal{H}(\tilde K_2) \mapsto \mathcal{H}(K_1) \otimes \mathcal{H}(\tilde K_1)$ denote the tensor product of T and $\tilde T$. Then we have

$$L(x, y) \tilde L(\tilde x, \tilde y) = \Big(T \otimes \tilde T \left(K_2(\cdot, y) \otimes \tilde K_2(\cdot, \tilde y) \right), K_1(\cdot, x) \otimes \tilde K_1(\cdot, \tilde x) \Big)_{\mathcal{H}(K_1) \otimes \mathcal{H}(\tilde K_1)},$$

which implies the domination

$$L(x,y)\tilde{L}(\tilde{x},\tilde{y}) \ll \big\{K_1(x,x')\tilde{K}_1(\tilde{x},\tilde{x}'); K_2(y,y')\tilde{K}_2(\tilde{y},\tilde{y}')\big\}.$$

Restricting it to the diagonal $x=\tilde{x}$, $y=\tilde{y}$, $x'=\tilde{x}'$, $y'=\tilde{y}'$, we obtain the desired conclusion. •

Combining addition and multiplication of dominations, we obtain, in particular

Corollary 4.1. *The domination $L(x,y) \ll \{K_1(x,x'); K_2(y,y')\}$ implies that*

$$\exp\big(cL(x,y)\big) \ll \big\{\exp\big(cK_1(x,x')\big); \exp\big(cK_2(y,y')\big)\big\}$$

for any $c>0$.

In the case of the domination (4.3), the last corollary is a well-known result on exponentiation of Golusin inequalities (see, e.g., [5, Chapter 5]).

The next proposition shows how to differentiate domination relations.

Proposition 4.2. *Assume that the domination occurs*

$$L(z,y) \ll \big\{K_1(z,z'); K_2(y,y')\big\}; \quad z,z' \in \Omega,\ y,y' \in \mathcal{Y},$$

where Ω is a domain in $\mathbb{C}$. Assume also that K_1 is differentiable with respect to the variables z and $\bar{z}'$. Then $L(z,y)$ is differentiable with respect to the variable z for any $y \in \mathcal{Y}$, and

$$\frac{\partial L}{\partial z}(z,y) \ll \left\{\frac{\partial^2 K_1(z,z')}{\partial z \partial \bar{z}'}; K_2(y,y')\right\}. \tag{4.9}$$

Remark. Of course, a similar result is true for differentiation with respect to the variable y.

Proof. The property that K_1 is differentiable with respect to z and $\bar{z}'$ implies that all functions in $\mathcal{H}(K_1)$ are differentiable in Ω. Moreover, the mapping $\bar{z} \mapsto K_1(\cdot,z)$ is differentiable as a $\mathcal{H}(K_1)$-valued function defined on $\overline{\Omega}$, and for any $f \in \mathcal{H}(K_1)$ we have

$$\partial_z f(z) = (f, \partial_{\bar{z}} K_1(\cdot,z))_{\mathcal{H}(K_1)}\,.$$

Since $L(\cdot,y) \in \mathcal{H}(K_1)$ by Theorem 4.1, we deduce that $L(z,y)$ is differentiable with respect to z and, moreover,

$$\partial_z L(z,y) = (L(\cdot,y), \partial_{\bar{z}} K_1(\cdot,z))_{\mathcal{H}(K_1)}\,.$$

By Theorem 4.1, there exists a contraction $T : \mathcal{H}(K_2) \mapsto \mathcal{H}(K_1)$ such that $L(\cdot,y) = TK_2(\cdot,y)$. We obtain therefore

$$\partial_z L(z,y) = (TK_2(\cdot,y), \partial_{\bar{z}} K_1(\cdot,z))_{\mathcal{H}(K_1)}\,.$$

It remains to apply the same arguments as in the proof of the implication (2) ⇒(3a) of Theorem 4.1 together with identity

$$\big(\partial_{\bar z'}K_1(\cdot,z'),\partial_{\bar z}K_1(\cdot,z)\big)_{\mathcal{H}(K_1)}=\frac{\partial^2K_1(z,z')}{\partial z\partial\bar z'}. \quad \bullet$$

Remark. The differentiation operator $\dfrac{\partial}{\partial z}$ can be replaced by more general differential (or integral) operators.

Definition 4.2. Assume that the domination $L(x,y)\ll\{K_1(x,x');K_2(y,y')\}$ occurs. We shall say that it is *exact* if the operator $T:\mathcal{H}(K_2)\mapsto\mathcal{H}(K_1)$ such that $L(\cdot,y)=TK_2(\cdot,y)$ is a unitary operator.

The domination (4.3) given by Lebedev inequalities is exact if and only if g is a full mapping, i.e., it maps $\mathbb{C}\setminus\mathbb{D}$ onto a complement of an area-null set. This is because the operator that maps $K_2(\cdot,\zeta)$ to $L(\cdot,\zeta)$ in this context is (up to an appropriate normalization on Taylor coefficients) the classical Grunsky operator, which is unitary if and only if g is a full mapping.

The following proposition is almost obvious.

Proposition 4.3. *The following conditions are equivalent*:

(1) *The domination* $L(x,y)\ll\{K_1(x,x');K_2(y,y')\}$ *is exact.*

(2) *The functions* $L(\cdot,y)$, $y\in\mathcal{Y}$, *form a complete system of functions in* $\mathcal{H}(K_1)$ *and for any choice of* $y_1,\dots,y_n\in\mathcal{Y}$, $\beta_1,\dots,\beta_n\in\mathbb{C}$ *we have*

$$\Big\|\sum_{j=1}^n\bar\beta_jL(\cdot,y_j)\Big\|^2_{\mathcal{H}(K_1)}=\sum_{1\leqslant j,j'\leqslant n}\beta_j\bar\beta_{j'}K_2(y_j,y_{j'}). \tag{4.10}$$

(3) *The functions* $\overline{L(x,\cdot)}$, $x\in\mathcal{X}$, *form a complete system of functions in* $\mathcal{H}(K_2)$ *and for any choice of* $x_1,\dots,x_m\in\mathcal{X}$, $\alpha_1,\dots,\alpha_m\in\mathbb{C}$ *we have*

$$\Big\|\sum_{i=1}^m\bar\alpha_i\overline{L(x_i,\cdot)}\Big\|^2_{\mathcal{H}(K_2)}=\sum_{1\leqslant i,i'\leqslant n}\alpha_i\bar\alpha_{i'}K_1(x_i,x_{i'}). \tag{4.11}$$

(4) *The two identities* (4.10) *and* (4.11) *are true for any choice of parameters* x_i, y_j, α_i, *and* β_j.

Our last proposition of technical nature can be proved by the same arguments as Proposition 4.2. For $\theta>0$, the differential operator D^θ is defined as follows:

$$D^\theta\,:\,g(z)\mapsto z^{1-\theta}\partial_z(z^\theta g(z))=\theta g(z)+z\partial_zg(z). \tag{4.12}$$

Clearly, D^θ is invertible as an operator on functions analytic in $\mathbb{D}$, and the inverse operator is

$$h(z) \mapsto \int_0^1 t^{\theta-1} h(tz)\,dt.$$

Proposition 4.4. *Let $L(z,y)$, $K_1(z,z')$, and $K_2(y,y')$ be as in Proposition 4.2. Then the domination*

$$L(z,y) \ll \{K_1(z,z'); K_2(y,y')\}$$

is exact if and only if the differentiated domination

$$D_z^\theta L(z,y) \ll \{D_z^\theta D_{\bar z'}^\theta K_1(z,z'); K_2(y,y')\}$$

is exact.

§5. Area theorems

In this section, we obtain two branching point area theorems (Theorems 5.1 and 5.2 below), which are the main results of this paper. Both theorems are formulated as certain domination relations for kernel functions. Using equivalent conditions in Theorem 4.1, we can easily reformulate them in terms of integral inequalities.

Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a fixed sequence of distinct points in $\mathbb{D}$, and let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a fixed sequence of positive numbers such that $\theta = \theta_1 + \dots + \theta_n \leqslant 1$. We recall (see §3) that $k^+_{\Lambda,\Theta}(\cdot,\cdot)$ and $k^-_{\Lambda,\Theta}(\cdot,\cdot)$ are reproducing kernels relative to the norms defined by (3.2) and (3.20) (or (3.26) if $\theta_1+\dots+\theta_n = 1$), respectively. The functions $f_{\Lambda,\Theta,a}$ are defined by (3.31).

Theorem 5.1. *Let $\psi : \mathbb{D} \mapsto \mathbb{D}$ be a conformal self-map of the unit disk. Then we have the domination*

$$l_{\psi,\Lambda,\Theta}(\zeta,u) \ll \{k^+_{\Lambda,\Theta}(\zeta,\zeta'); \overline{k^-_{\Lambda,\Theta}(u,u') - k^-_{\psi,\Lambda,\Theta}(u,u')}\}, \tag{5.1}$$

where

$$k^-_{\psi,\Lambda,\Theta}(u,u') = \frac{\omega_{\Lambda,\Theta}(u)}{\omega_{\psi(\Lambda),\Theta}(\psi(u))} \overline{\Big(\frac{\omega_{\Lambda,\Theta}(u')}{\omega_{\psi(\Lambda),\Theta}(\psi(u'))}\Big)} \psi'(u)\overline{\psi'(u')} k^-_{\psi(\Lambda),\Theta}(\psi(u),\psi(u')) \tag{5.2}$$

and

$$l_{\psi,\Lambda,\Theta}(\zeta,u) = \frac{\omega_{\psi(\Lambda),\Theta}(\psi(\zeta))}{\omega_{\Lambda,\Theta}(\zeta)} \frac{\omega_{\Lambda,\Theta}(u)}{\omega_{\psi(\Lambda),\Theta}(\psi(u))} \psi'(u) f_{\psi(\Lambda),\Theta,\psi(u)}(\psi(\zeta)) - f_{\Lambda,\Theta,u}(\zeta) \tag{5.3}$$

(*and all variables are in $\mathbb{D}$*).

Proof. We pick some points $u_1, \ldots, u_N \in \mathbb{D}$ and some $c_1, \ldots, c_N \in \mathbb{C}$ and consider the function

$$g(\zeta) = \omega_{\psi(\Lambda),\Theta}(\zeta) \sum_{k=1}^{N} c_k \frac{\omega_{\Lambda,\Theta}(u_k)}{\omega_{\psi(\Lambda),\Theta}(\psi(u_k))} \psi'(u_k) f_{\psi(\Lambda),\Theta,\psi(u_k)}(\zeta).$$

By Corollary 3.4, we have

$$-[g,g] = \sum_{k,l} c_k \bar{c}_l k^{-}_{\psi,\Lambda,\Theta}(u_k, u_l). \tag{5.4}$$

Proposition 2.2 implies now that

$$[g \circ \psi, g \circ \psi] \leqslant [g,g]. \tag{5.5}$$

On the other hand, $g \circ \psi$ admits the following explicit decomposition into a sum of positive and negative terms related to (3.29):

$$g \circ \psi = G_+ + G_-,$$

where

$$G_+(\cdot) = \omega_{\Lambda,\Theta}(\cdot) \sum_{k=1}^{N} c_k l_{\psi,\Lambda,\Theta,}(\cdot, u_k)$$

and

$$G_-(\cdot) = \omega_{\Lambda,\Theta}(\cdot) \sum_{k=1}^{N} c_k f_{\Lambda,\Theta,u_k}(\cdot),$$

so that

$$[g \circ \psi, g \circ \psi] = [G_+, G_+] + [G_-, G_-].$$

By Corollary 3.4,

$$-[G_-, G_-] = \sum_{k,l} c_k \bar{c}_l k^{-}_{\Lambda,\Theta}(u_k, u_l). \tag{5.6}$$

Combining (5.4), (5.5), and (5.6), we obtain

$$\begin{aligned} &\Big[\omega_{\Lambda,\Theta}(\cdot) \sum_{k=1}^{N} c_k l_{\psi,\Lambda,\Theta}(\cdot, u_k), \omega_{\Lambda,\Theta}(\cdot) \sum_{k=1}^{N} c_k l_{\psi,\Lambda,\Theta}(\cdot, u_k)\Big] \\ &\qquad \leqslant \sum_{k,l} c_k \bar{c}_l \left(k^{-}_{\Lambda,\Theta}(u_k, u_l) - k^{-}_{\psi,\Lambda,\Theta}(u_k, u_l)\right). \end{aligned} \tag{5.7}$$

By the definition of the kernel $k^{+}_{\Lambda,\Theta}$, this inequality is equivalent to the domination (5.1) by Theorem 4.1. •

Theorem 5.2. *Let φ be a function univalent in $\mathbb{D}$. Then we have the domination*

$$L_{\varphi,\Lambda,\Theta}(\zeta,u) \ll \big\{k^+_{\Lambda,\Theta}(\zeta,\zeta');\overline{k^-_{\Lambda,\Theta}(u,u')}\big\}, \tag{5.8}$$

where

$$L_{\varphi,\Lambda,\Theta}(\zeta,u) = \Big[\prod_{k=1}^{n}\Big(\frac{\varphi(\zeta)-\varphi(\lambda_k)}{\zeta-\lambda_k}\cdot\frac{1-\bar\lambda_k\zeta}{1-\bar\lambda_k u}\cdot\frac{u-\lambda_k}{\varphi(u)-\varphi(\lambda_k)}\Big)^{\theta_k}\Big] \times \frac{\varphi'(u)}{\varphi(\zeta)-\varphi(u)} - f_{\Lambda,\Theta,u}(\zeta), \tag{5.9}$$

and all variables are in $\mathbb{D}$.

Proof. We argue as in the proof of Theorem 5.1. Suppose $u_1,\dots,u_N\in\mathbb{D}$ and $c_1,\dots,c_N\in\mathbb{C}$. Consider the function

$$P(z) = \Big[\prod_{k=1}^{n}(z-\varphi(\lambda_k))^{\theta_k}\Big] \times \sum_{\nu=1}^{N}\Big[\prod_{k=1}^{n}\Big(\frac{u_\nu-\lambda_k}{(1-\bar\lambda_k u_\nu)(\varphi(u_\nu)-\varphi(\lambda_k))}\Big)^{\theta_k}\Big]\cdot\frac{c_\nu\varphi'(u_\nu)}{z-\varphi(u_\nu)}.$$

By Corollary 2.2,

$$[P\circ\varphi, P\circ\varphi]\leqslant 0$$

with equality in the case where φ is a full mapping. On the other hand, in accordance with the decomposition (3.29), $P\circ\varphi$ can be written explicitly as $P\circ\varphi = G_+ + G_-$, where

$$G_+(\zeta) = \omega_{\Lambda,\Theta}(\zeta)\sum_{\nu=1}^{N}c_\nu L_{\varphi,\Lambda,\Theta}(\zeta,u_\nu)$$

and

$$G_-(\zeta) = \omega_{\Lambda,\Theta}(\zeta)\sum_{\nu=1}^{N}c_\nu f_{\Lambda,\Theta,u_\nu}(\zeta),$$

so that

$$[P\circ\varphi, P\circ\varphi] = [G_+,G_+] + [G_-,G_-].$$

By Corollary 3.4,

$$-[G_-,G_-] = \sum_{k,l}c_k\bar c_l k^-_{\Lambda,\Theta}(u_k,u_l),$$

which implies

$$\Big[\omega_{\Lambda,\Theta}(\cdot)\sum_{\nu=1}^{N}c_\nu L_{\varphi,\Lambda,\Theta}(\cdot,u_\nu),\omega_{\Lambda,\Theta}(\cdot)\sum_{\nu=1}^{N}c_\nu L_{\varphi,\Lambda,\Theta}(\cdot,u_\nu)\Big]\leqslant\sum_{\nu,l}c_\nu\bar{c}_l k^-_{\Lambda,\Theta}(u_\nu,u_l) \tag{5.10}$$

(with equality if φ is a full mapping). As in the proof of Theorem 5.1, this inequality is equivalent to the domination (5.8). •

Our next aim is to prove that the domination (5.8) is exact in the case where φ is a full mapping. Without loss of generality, we assume in the remaining part of this section that $\varphi(0)=0$. First, we analyse the special case where the sequence Λ reduces to a single point at the origin:

$$\Lambda=\Lambda_0=\{0\};\quad \Theta_0=\{\theta\}. \tag{5.11}$$

We consider the case of $\theta\in(0,1)$ only; slight modifications needed in the case where $\theta=1$ are left to the reader. Our analysis partially repeats that performed in [9] and it is presented here for the sake of completeness.

By Lemma 3.1, the norm in the space $\mathcal{D}_{\Lambda_0,\Theta_0}$ is given by the formula

$$\|g\|^2_{\mathcal{D}_{\Lambda_0,\Theta_0}}=\int_{\mathbb{D}}\left|\theta g(z)+zg'(z)\right|^2|z|^{2\theta-2}\,dA(z)=\sum_{k\geqslant 0}(k+\theta)|\hat{g}(k)|^2. \tag{5.12}$$

By (3.20), the norm in $\mathcal{B}_{\Lambda_0,\Theta_0}$ is

$$\|x\|^2_{\mathcal{B}_{\Lambda_0,\Theta_0}}=\int_{\mathbb{D}}\frac{|x(u)|^2}{|u|^{2\theta}}\,dA(u)=\sum_{k\geqslant 0}\frac{|\hat{x}(k)|^2}{k+1-\theta}. \tag{5.13}$$

In particular, we see that

$$\|g\|^2_{\mathcal{D}_{\Lambda_0,\Theta_0}}=\|\theta g(z)+zg'(z)\|^2_{\mathcal{B}_{\Lambda_0,\mathbf{1}-\Theta_0}}=\|D^\theta g\|^2_{\mathcal{B}_{\Lambda_0,\mathbf{1}-\Theta_0}},$$

where $\mathbf{1}-\Theta_0=\{1-\theta\}$, and the differential operator D^θ is defined by (4.12). The explicit formulas (5.12) and (5.13) show that

$$k^+_{\Lambda_0,\Theta_0}(\zeta,\zeta')=\sum_{k\geqslant 0}\frac{(\zeta\bar{\zeta}')^k}{k+\theta}$$

and

$$k^-_{\Lambda_0,\Theta_0}(u,u')=\sum_{k\geqslant 0}(k+1-\theta)(u\bar{u}')^k=D^{1-\theta}_u D^{1-\theta}_{\bar{u}'}k^+_{\Lambda_0,\mathbf{1}-\Theta_0}(u,u'). \tag{5.14}$$

Finally, the kernel $L_{\varphi,\Lambda_0,\Theta_0}$ (defined by (5.9)) has the form

$$L_{\varphi,\Lambda_0,\Theta_0}(\zeta,u)=\Big(\frac{\varphi(\zeta)}{\zeta}\,\frac{u}{\varphi(u)}\Big)^\theta\frac{\varphi'(u)}{\varphi(\zeta)-\varphi(u)}-\frac{1}{\zeta-u}.$$

Proposition 5.1. *If φ is a full mapping, then the domination*

$$L_{\varphi,\Lambda_0,\Theta_0}(\zeta,u) \ll \{k^+_{\Lambda_0,\Theta_0}(\zeta,\zeta'); \overline{k^-_{\Lambda_0,\Theta_0}(u,u')}\} \tag{5.15}$$

is exact.

Proof. The proof of Theorem 5.2 shows that

$$\Big\| \sum_k c_k L_{\varphi,\Lambda_0,\Theta_0}(\cdot,u_k)\Big\|^2_{\mathcal{D}_{\Lambda_0,\mathbf{1}-\Theta_0}} = \sum_{k,k'} c_k\bar{c}_{k'} k^-_{\Lambda_0,\Theta_0}(u_k,u_{k'})$$

or, equivalently,

$$\Big\| \sum_k c_k D^\theta L_{\varphi,\Lambda_0,\Theta_0}(\cdot,u_k)\Big\|_{\mathcal{B}_{\Lambda_0,\mathbf{1}-\Theta_0}} = \sum_{k,k'} c_k\bar{c}_{k'} k^-_{\Lambda_0,\Theta_0}(u_k,u_{k'}). \tag{5.16}$$

But an explicit calculation yields

$$D^\theta_\zeta L_{\varphi,\Lambda_0,\Theta_0}(\zeta,u) =: M_{\varphi,\theta}(\zeta,u)$$
$$= -\Big(\frac{\zeta}{\varphi(\zeta)}\Big)^{1-\theta}\Big(\frac{u}{\varphi(u)}\Big)^{\theta}\frac{(1-\theta)\varphi(\zeta)+\theta\varphi(u)}{(\varphi(\zeta)-\varphi(u))^2}\varphi'(\zeta)\varphi'(u) + \frac{(1-\theta)\zeta+\theta u}{(\zeta-u)^2}.$$

Rewriting (5.16) as

$$\Big\| \sum_k c_k M_{\varphi,\theta}(\cdot,u_k)\Big\|^2_{\mathcal{B}_{\Lambda_0,\mathbf{1}-\Theta_0}} = \sum_{k,k'} c_k\bar{c}_{k'} k^-_{\Lambda_0,\Theta_0}(u_k,u_{k'}) \tag{5.17}$$

and using the symmetry property

$$M_{\varphi,\theta}(\zeta,u) = M_{\varphi,1-\theta}(u,\zeta),$$

we obtain also

$$\Big\| \sum_k d_k M_{\varphi,\theta}(\zeta_k,\cdot)\Big\|^2_{\mathcal{B}_{\Lambda_0,\Theta_0}} = \sum_{k,k'} d_k\bar{d}'_k k^-_{\Lambda_0,\mathbf{1}-\Theta_0}(\zeta_k,\zeta'_k). \tag{5.18}$$

By Proposition 4.3, formulas (5.17) and (5.18) imply that we have an exact domination

$$M_{\varphi,\theta}(\zeta,u) \ll \{k^-_{\Lambda_0,\mathbf{1}-\Theta_0}(\zeta,\zeta'), \overline{k^-_{\Lambda_0,\Theta_0}(u,u')}\}.$$

Since

$$k^-_{\Lambda_0,\mathbf{1}-\Theta_0}(\zeta,\zeta') = D^\theta_\zeta D^\theta_{\bar{\zeta}'} k^+_{\Lambda_0,\Theta_0}(\zeta,\zeta')$$

by (5.14), we see that the domination (5.15) is exact by Proposition 4.4. •

We recall that the space $\mathcal{G}_\theta$ consists of $(\mathbb{T},\theta)$-character-automorphic functions

$$g(z) = \sum_{k\in\mathbb{Z}} \hat{g}(k) z^{k+\theta}$$

such that

$$\|g\|_{J,\theta}^2 = \sum_{k\in\mathbb{Z}} |k+\theta|\,|\hat{g}(k)|^2 < +\infty.$$

The closure of a subset of $\mathcal{G}_\theta$ means the closure with respect to the Hilbert norm $\|\cdot\|_{J,\theta}$. Let $P^+_{\Lambda_0,\Theta_0}$ (respectively, $P^-_{\Lambda_0,\Theta_0}$) denote the canonical projection to the positive (respectively, negative) subspace of the decomposition

$$\mathcal{G}_\theta = \mathcal{G}^+_{\Lambda_0,\Theta_0} \dotplus \left(\mathcal{G}^-_{\Lambda_0,\Theta_0}\right)^\sharp. \tag{5.19}$$

For each $u \in \mathbb{D}$, let $g_{\varphi,\Lambda_0,\Theta_0,u} \in \mathcal{G}_\theta$ be defined by

$$g_{\varphi,\Lambda_0,\Theta_0,u}(\zeta) := (\varphi(\zeta))^\theta \left(\frac{u}{\varphi(u)}\right)^\theta \frac{\varphi'(u)}{\varphi(\zeta)-\varphi(u)}.$$

Clearly, $g_{\varphi,\Lambda_0,\Theta_0,u}$ decomposes with respect to (5.19) as follows:

$$g_{\varphi,\Lambda_0,\Theta_0,u} = g^+_{\varphi,\Lambda_0,\Theta_0,u} + g^-_{\varphi,\Lambda_0,\Theta_0,u},$$

where

$$g^+_{\varphi,\Lambda_0,\Theta_0,u}(\zeta) = \zeta^\theta L_{\varphi,\Lambda_0,\Theta_0}(\zeta,u)$$

and

$$g^-_{\varphi,\Lambda_0,\Theta_0,u}(\zeta) = \frac{\zeta^\theta}{\zeta-u} = \zeta^\theta f_{\Lambda_0,\Theta_0,u}(\zeta).$$

If a function g is a sum

$$g = \sum_k c_k g_{\varphi,\Lambda_0,\Theta_0,u_k}, \tag{5.20}$$

and it decomposes with respect to (5.19) as

$$g = P^+_{\Lambda_0,\Theta_0} g + P^-_{\Lambda_0,\Theta_0} g = g_+ + g_-,$$

then by Corollary 2.2 we have

$$[g_+,g_+] = -[g_-,g_-] = \frac{1}{2}\|g\|^2_{J,\theta}. \tag{5.21}$$

Let $N_{\varphi,\Lambda_0,\Theta_0}$ denote the closure in $\mathcal{G}_\theta$ of the set of all functions g given by (5.20). Clearly, $N_{\varphi,\Lambda_0,\Theta_0}$ is a closed isotropic subspace of $\mathcal{G}_\theta$ and, in particular, it is a positive subspace. The property that the linear combinations of functions

$$\zeta^\theta f_{\Lambda_0,\Theta_0,u}(\zeta)$$

are dense in $\left(\mathcal{G}^-_{\Lambda_0,\Theta_0}\right)^\sharp$ (which is true by Lemma 3.6) together with (5.21) implies that

$$P^-_{\Lambda_0,\Theta_0} N_{\varphi,\Lambda_0,\Theta_0} = \left(\mathcal{G}^-_{\Lambda_0,\Theta_0}\right)^\sharp$$

(in other words, for any $g_- \in \left(\mathcal{G}^-_{\Lambda_0,\Theta_0}\right)^\sharp$, there exists $g \in N_{\varphi,\Lambda_0,\Theta_0}$ with $g_- = P^-_{\Lambda_0,\Theta_0} g$). Similarly, the property that the functions

$$g^+_{\varphi,\Lambda_0,\Theta_0,u}(\zeta) = \zeta^\theta L_{\varphi,\Lambda_0,\Theta_0}(\zeta,u)$$

form a complete system in $\mathcal{G}^+_{\Lambda_0,\Theta_0}$ (which is true because the domination (5.15) is exact) implies that

$$P^+_{\Lambda_0,\Theta_0} N_{\varphi,\Lambda_0,\Theta_0} = \mathcal{G}^+_{\Lambda_0,\Theta_0}.$$

Lemma 5.1. *$N_{\varphi,\Lambda_0,\Theta_0}$ is a maximal positive subspace of $\mathcal{G}_\theta$.*

Proof. By Lemma V.4.5 of [3], it suffices to show that the orthogonal companion to $N_{\varphi,\Lambda_0,\Theta_0}$ is negative. Let $h \in \mathcal{G}_\theta$ be orthogonal to $N_{\varphi,\Lambda_0,\Theta_0}$, and let

$$h = P^+_{\Lambda_0,\Theta_0} h + P^-_{\Lambda_0,\Theta_0} h = h_+ + h_-.$$

By the above remarks, there exists $g \in N_{\varphi,\Lambda_0,\Theta_0}$ with $g_+ = P^+_{\Lambda_0,\Theta_0} h = h_+$. Then we have

$$0 = [g,h] = [g_+,g_+] + [g_-,h_-].$$

Since

$$[g_+,g_+]^2 = |[g_-,h_-]|^2 \leqslant [g_-,g_-]\cdot[h_-,h_-] = -[g_+,g_+]\cdot[h_-,h_-],$$

we obtain

$$-[h_-,h_-] \geqslant [g_+,g_+] = [h_+,h_+],$$

showing that

$$[h,h] \leqslant 0. \quad \bullet$$

Remark. The same arguments show also that $N_{\varphi,\Lambda_0,\Theta_0}$ is a maximal negative subspace of $\mathcal{G}_\theta$. It is easy to see that the orthogonal companion to $N_{\varphi,\Lambda_0,\Theta_0}$ coincides with $N_{\varphi,\Lambda_0,\Theta_0}$ itself.

Now, we turn to the case of general $\Lambda = \{\lambda_1,\dots,\lambda_n\}$ and $\Theta = \{\theta_1,\dots,\theta_n\}$. As before, we assume that $\theta = \theta_1 + \cdots + \theta_n \leqslant 1$. The symbols $P^+_{\Lambda,\Theta}$ and $P^-_{\Lambda,\Theta}$ stand now for canonical projections to the positive and the negative subspace of the decomposition (3.29). The functions $g_{\varphi,\Lambda,\Theta,u}$ are defined now by

$$g_{\varphi,\Lambda,\Theta,u}(\zeta) := \left[\prod_{k=1}^n (\varphi(\zeta)-\varphi(\lambda_k))^{\theta_k}\right] \times \left[\prod_{k=1}^n \left(\frac{u-\lambda_k}{(1-\bar\lambda_k u)\,(\varphi(u)-\varphi(\lambda_k))}\right)^{\theta_k}\right]\cdot\frac{\varphi'(u)}{\varphi(\zeta)-\varphi(u)},$$

so that

$$P^+_{\Lambda,\Theta} g_{\varphi,\Lambda,\Theta,u}(\zeta) = \omega_{\Lambda,\Theta}(\zeta) L_{\varphi,\Lambda,\Theta}(\zeta,u) \tag{5.22}$$

and

$$P^{-}_{\Lambda,\Theta} g_{\varphi,\Lambda,\Theta,u}(\zeta) = \omega_{\Lambda,\Theta}(\zeta) f_{\Lambda,\Theta,u}(\zeta).$$

Let $N_{\varphi,\Lambda,\Theta}$ be the closure in $\mathcal{G}_\theta$ of the linear combinations of functions $g_{\varphi,\Lambda,\Theta,u}$.

Lemma 5.2. $N_{\varphi,\Lambda_0,\Theta_0} = N_{\varphi,\Lambda,\Theta}$.

Proof. Let u be some fixed point of $\mathbb{D}$, and let Ω be some subdomain of $\mathbb{D}$ such that $\{u, \lambda_1, \dots, \lambda_n\} \in \Omega$ and $\overline{\Omega} \subset \mathbb{D}$. We define

$$F(v) = \prod_{k=1}^{n} \Big(\frac{v - \varphi(\lambda_k)}{v}\Big)^{\theta_k} \cdot \frac{1}{v - \varphi(u)}$$

choosing some branch of it regular and single-valued in $\bar{\mathbb{C}} \setminus \varphi(\overline{\Omega})$. Let γ be some simple positively oriented contour in $\mathbb{D}$ such that $\overline{\Omega} \subset \operatorname{int}\gamma$. Then we have

$$-\frac{1}{2\pi i} \int_{\varphi(\gamma)} \frac{F(v)}{v - \varphi(\zeta)}\, dv = \Bigg[\prod_{k=1}^{n} \Bigg(\frac{\varphi(\zeta) - \varphi(\lambda_k)}{\varphi(\zeta)}\Bigg)^{\theta_k}\Bigg] \cdot \frac{1}{\varphi(\zeta) - \varphi(u)}$$

if ζ is near $\mathbb{T}$, which implies

$$\begin{aligned} &-\frac{1}{2\pi i} \int_{\gamma} F(\varphi(\eta))(\varphi(\zeta))^{\theta} \frac{\varphi'(\eta)}{\varphi(\eta) - \varphi(\zeta)}\, d\eta \\ &= \Bigg[\prod_{k=1}^{n} (\varphi(\zeta) - \varphi(\lambda_k))^{\theta_k}\Bigg] \cdot \frac{1}{\varphi(\zeta) - \varphi(u)} \end{aligned}$$

for ζ near $\mathbb{T}$. Therefore, $N_{\varphi,\Lambda,\Theta} \subset N_{\varphi,\Lambda_0,\Theta_0}$. The reverse inclusion is proved similarly. •

Finally, we are ready to prove the following statement.

Theorem 5.3. *If φ is a full mapping, then the domination* (5.8) *is exact.*

Proof. It suffices to show that the functions $L_{\varphi,\Lambda,\Theta}(\cdot, u)$ form a complete system in $\mathcal{D}_{\Lambda,\Theta}$ or, equivalently, that the functions $\omega_{\Lambda,\Theta}(\cdot) L_{\varphi,\Lambda,\Theta}(\cdot, u)$ form a complete system in $\mathcal{G}^{+}_{\Lambda,\Theta}$. Let $h \in \mathcal{G}^{+}_{\Lambda,\Theta}$ be some function orthogonal to all functions $\omega_{\Lambda,\Theta}(\cdot) L_{\varphi,\Lambda,\Theta}(\cdot, u)$. Then by (5.22) h is orthogonal to all functions $g_{\varphi,\Lambda,\Theta,u}$ and hence h lies in the orthogonal companion to the subspace $N_{\varphi,\Lambda,\Theta}$. Since $N_{\varphi,\Lambda,\Theta} = N_{\varphi,\Lambda_0,\Theta_0}$ is a maximal positive subspace of $\mathcal{G}_\theta$, we see that

$$[h, h] \leqslant 0;$$

consequently $h = 0$. •

In the context of Theorem 5.1, the domination (5.1) is not exact in the case where ψ is a full self-map of $\mathbb{D}$. Indeed, if p is an analytic polynomial, then

$$\Big(l_{\psi,\Lambda,\Theta}(\cdot,u),\frac{\omega_{\psi(\Lambda),\Theta}(\psi(\cdot))}{\omega_{\Lambda,\Theta}(\cdot)}p(\psi(\cdot))\Big)_{\mathcal{D}_{\Lambda,\Theta}}$$
$$=\Big[\omega_{\psi(\Lambda),\Theta}(\psi(\cdot))\frac{\omega_{\Lambda,\Theta}(u)}{\omega_{\psi(\Lambda),\Theta}(\psi(u))}\psi'(u)\cdot f_{\psi(\Lambda),\Theta,\psi(u)}(\psi(\cdot))\,,\omega_{\psi(\Lambda),\Theta}(\psi(\cdot))p(\psi(\cdot))\Big]$$
$$=\Big[\omega_{\psi(\Lambda),\Theta}(\cdot)\frac{\omega_{\Lambda,\Theta}(u)}{\omega_{\psi(\Lambda),\Theta}(\psi(u))}\psi'(u)\cdot f_{\psi(\Lambda),\Theta,\psi(u)}(\cdot)\,,\omega_{\psi(\Lambda),\Theta}(\cdot)p(\cdot)\Big]=0,$$

showing that the functions $l_{\psi,\Lambda,\Theta}(\cdot,u)$ cannot form a complete system in $\mathcal{D}_{\Lambda,\Theta}$. Apparently, the exact domination in this context is

$$l_{\psi,\Lambda,\Theta}(\zeta,u)\ll\big\{k^+_{\Lambda,\Theta}(\zeta,\zeta')-k^+_{\psi,\Lambda,\Theta}(\zeta,\zeta')\,;\,\overline{k^-_{\Lambda,\Theta}(u,u')-k^-_{\psi,\Lambda,\Theta}(u,u')}\big\},$$

where

$$k^+_{\psi,\Lambda,\Theta}(\zeta,\zeta')=\frac{\omega_{\psi(\Lambda),\Theta}(\psi(\zeta))}{\omega_{\Lambda,\Theta}(\zeta)}\overline{\Big(\frac{\omega_{\psi(\Lambda),\Theta}(\psi(\zeta'))}{\omega_{\Lambda,\Theta}(\zeta')}\Big)}k^+_{\Lambda,\Theta}(\psi(\zeta,\psi(\zeta'))$$

and $k^-_{\psi,\Lambda,\Theta}(u,u')$ is defined by (5.2).

§6. Special cases

In this section we consider special cases of one and two branching points and obtain explicit integral inequalities as special cases of the main Theorem 5.2.

One branching point. First, let Λ consist of a single point $\lambda\in\mathbb{D}$, and let $\theta\in(0,1)$. Then the norms $\|\cdot\|_{\mathcal{D}_{\Lambda,\Theta}}$ and $\|\cdot\|_{\mathcal{B}_{\Lambda,\Theta}}$ are as follows:

$$\|g\|^2_{\mathcal{D}_{\Lambda,\Theta}}=\int_{\mathbb{D}}\Big|\theta\frac{1-|\lambda|^2}{(1-\bar\lambda z)^2}g(z)+\omega_\lambda(z)g'(z)\Big|^2\frac{dA(z)}{|\omega_\lambda(z)|^{2-2\theta}}$$

and

$$\|x\|^2_{\mathcal{B}_{\Lambda,\Theta}}=\int_{\mathbb{D}}|x(u)|^2\frac{dA(u)}{|\omega_\lambda(u)|^{2\theta}}.\tag{6.1}$$

Lemma 6.1. *Suppose $\Lambda=\{\lambda\}$ and $\Theta=\{\theta\}$. Then*

$$k^+_{\Lambda,\Theta}(z,z')=\sum_{n\geqslant0}\frac{\big(\omega_\lambda(z)\overline{\omega_\lambda(z')}\big)^n}{n+\theta}\tag{6.2}$$

and

$$k^-_{\Lambda,\Theta}(u,u') = \frac{(1-|\lambda|^2)^2}{(1-\bar\lambda u)^2(1-\lambda\bar u')^2}\sum_{n\geqslant 0}(n+1-\theta)\big(\omega_\lambda(u)\overline{\omega_\lambda(u')}\big)^n$$
$$= \frac{1}{(1-u\bar u')^2} - \frac{\theta(1-|\lambda|^2)}{(1-\bar\lambda u)(1-\lambda\bar u')(1-u\bar u')}. \tag{6.3}$$

Proof. In the special case of $\lambda = 0$, formula (6.2) was obtained in §5. On the other hand, the identity

$$[\omega^\theta_\lambda f, \omega^\theta_\lambda g] = [z^\theta(f\circ\omega_{-\lambda}), z^\theta(g\circ\omega_{-\lambda})]$$

shows that

$$k^+_{\{\lambda\},\{\theta\}}(z,z') = k^+_{\{0\},\{\theta\}}(\omega_\lambda(z),\omega_\lambda(z')),$$

which gives (6.2).

Similarly, formula (6.3) in the special case $\lambda = 0$ was obtained in §5 (see formula (5.14)). The identity

$$(f,g)_{\mathcal{B}_{\{\lambda\},\{\theta\}}} = \big((f\circ\omega_{-\lambda})\omega'_{-\lambda}\,,\,(g\circ\omega_{-\lambda})\omega'_{-\lambda}\big)_{\mathcal{B}_{\{0\},\{\theta\}}}$$

implies that

$$k^-_{\{\lambda\},\{\theta\}}(u,u') = \omega'_\lambda(u)\overline{\omega'_\lambda(u')}k^-_{\{0\},\{\theta\}}(\omega_\lambda(u),\omega_\lambda(u')),$$

which gives (6.3). •

Combining Theorems 5.2 and 4.1, we obtain the following statement.

Corollary 6.1. *Let φ be univalent in $\mathbb{D}$, and let $\zeta_1,\dots,\zeta_n\in\mathbb{D}$ and $\alpha_1,\dots,\alpha_n\in\mathbb{C}$. Then*

$$\int_{\mathbb{D}}\Big|\sum_{\nu=1}^N \alpha_\nu L_\varphi(\zeta_\nu,u)\Big|^2\frac{dA(u)}{|\omega_\lambda(u)|^{2\theta}} \leqslant \sum_{1\leqslant\nu,\nu'\leqslant N}\alpha_\nu\overline{\alpha}_{\nu'}k^+_{\Lambda,\Theta}(\zeta_\nu,\zeta_{\nu'}), \tag{6.4}$$

where $k^+_{\Lambda,\Theta}$ is given by (6.2) *and*

$$L_\varphi(\zeta,u) = \Big(\frac{\varphi(\zeta)-\varphi(\lambda)}{\zeta-\lambda}\frac{u-\lambda}{\varphi(u)-\varphi(\lambda)}\frac{1-\bar\lambda\zeta}{1-\bar\lambda u}\Big)^\theta\frac{\varphi'(u)}{\varphi(\zeta)-\varphi(u)} - \frac{1}{\zeta-u} + \frac{\bar\lambda}{1-\bar\lambda u}.$$

Moreover, equality occurs in (6.4) *if φ is a full mapping.*

In the special case where $N = 1$ and $\zeta_1 = \lambda$, inequality (6.4) reduces to

$$\int_{\mathbb{D}}\Big|\Big(\varphi'(\lambda)\frac{u-\lambda}{\varphi(u)-\varphi(\lambda)}\frac{1-|\lambda|^2}{1-\bar\lambda u}\Big)^\theta\frac{\varphi'(u)}{\varphi(\lambda)-\varphi(u)} - \frac{1}{\lambda-u} + \frac{\bar\lambda}{1-\bar\lambda u}\Big|^2$$
$$\times\frac{dA(u)}{|\omega_\lambda(u)|^{2\theta}} \leqslant \frac{1}{\theta}. \tag{6.5}$$

This integral inequality can be obtained also directly from Prawitz' inequality (1.5) by appropriate Möbius transformations, see [7]. In that paper, inequality (6.5) (written in a slightly different form) was a starting point for deriving estimates of the integral means of derivatives of univalent functions.

Two branching points. We turn now to the calculations in the case of two branching points. In the remaining part of this section, we assume that $\Lambda = \{0, \lambda_0\}$ and $\Theta = \{\theta_1, \theta_2\}$, with $\theta_1, \theta_2 > 0$; $\theta_1 + \theta_2 \leqslant 1$. The case of general $\Lambda = \{\lambda_1, \lambda_2\}$ can be reduced to $\Lambda = \{0, \lambda_0\}$ by an appropriate Möbius shift of variables.

We start with explicit formulas for the norms $\|\cdot\|_{\mathcal{D}_{\Lambda,\Theta}}$ and $\|\cdot\|_{\mathcal{B}_{\Lambda,\Theta}}$. We have

$$\|g\|^2_{\mathcal{D}_{\Lambda,\Theta}} = \int_{\mathbb{D}} \Big|\Big(\theta_1\omega_{\lambda_0}(z) + \theta_2 z\frac{1-|\lambda_0|^2}{(1-\bar\lambda_0 z)^2}\Big)g(z) + z\omega_{\lambda_0}(z)g'(z)\Big|^2 \times \frac{dA(z)}{|z|^{2-2\theta_1}|\omega_{\lambda_0}(z)|^{2-2\theta_2}}$$

and

$$\|x\|^2_{\mathcal{B}_{\Lambda,\Theta}} = \int_{\mathbb{D}} |x(u)|^2 \frac{dA(u)}{|u|^{2\theta_1}|\omega_{\lambda_0}(u)|^{2\theta_2}} + \frac{\sin\pi\theta_1\cdot\sin\pi\theta_2}{\pi\sin(\pi(\theta_1+\theta_2))}\bigg|\int_0^{\lambda_0}\frac{x(u)}{u^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(u)}\,du\bigg|^2 \tag{6.6}$$

in the case where $\theta_1+\theta_2<1$, and

$$\|x\|^2_{\mathcal{B}_{\Lambda,\Theta}} = \int_{\mathbb{D}} |x(u)|^2 \frac{dA(u)}{|u|^{2\theta_1}|\omega_{\lambda_0}(u)|^{2\theta_2}} \tag{6.7}$$

in the case where $\theta_1+\theta_2 = 1$, and in the latter case $\mathcal{B}_{\Lambda,\Theta}$ consists of the functions $x \in \mathcal{B}_0^+$ that satisfy

$$\int_0^{\lambda_0}\frac{x(u)}{u^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(u)}\,du = 0.$$

The next lemma shows that the kernel $k^+_{\Lambda,\Theta}(z,\mu)$ is a solution of a certain first order differential equation.

Lemma 6.2. *We have*

$$z\partial_z\Big(\omega_{\Lambda,\Theta}(z)k^+_{\Lambda,\Theta}(z,\mu)\Big) = \omega_{\Lambda,\Theta}(z)\Big(\frac{1}{1-\bar\mu z} + \frac{\theta_2\lambda_0}{z-\lambda_0}k^+_{\Lambda,\Theta}(\lambda_0,\mu)\Big). \tag{6.8}$$

Proof. For any polynomial p, we can write

$$\overline{p(\mu)} = \Big[\omega_{\Lambda,\Theta}(\cdot)k^+_{\Lambda,\Theta}(\cdot,\mu),\, \omega_{\Lambda,\Theta}(\cdot)p(\cdot)\Big]$$
$$= \int_{\mathbb{T}} z\partial_z\left(\omega_{\Lambda,\Theta}(z)k^+_{\Lambda,\Theta}(z,\mu)\right)\overline{\omega_{\Lambda,\Theta}(z)p(z)}\,dm(z), \tag{6.9}$$

which shows that

$$\overline{\omega_{\Lambda,\Theta}(z)}z\partial_z\left(\omega_{\Lambda,\Theta}(z)k^+_{\Lambda,\Theta}(z,\mu)\right) = \frac{1}{1-\bar{\mu}z} + \overline{zr(z)}, \quad z\in\mathbb{T}, \tag{6.10}$$

where $r(\cdot)$ is some function from $\mathcal{B}_0^+$. On the other hand, for $z\in\mathbb{D}$ we have

$$\frac{1}{\omega_{\Lambda,\Theta}(z)}z\partial_z\left(\omega_{\Lambda,\Theta}(z)k^+_{\Lambda,\Theta}(z,\mu)\right)$$
$$= \Big(\theta_1 + \theta_2 z\frac{1-|\lambda_0|^2}{(z-\lambda_0)(1-\bar{\lambda}_0 z)}\Big)k^+_{\Lambda,\Theta}(z,\mu) + z\partial_z k^+_{\Lambda,\Theta}(z,\mu)$$
$$= \theta_2\frac{\lambda_0 k^+_{\Lambda,\Theta}(\lambda_0,\mu)}{z-\lambda_0} + h(z), \tag{6.11}$$

where $h\in\mathcal{B}_0^+$. Comparing (6.10) and (6.11), we obtain

$$\overline{zr(z)} = \theta_2\frac{\lambda_0 k^+_{\Lambda,\Theta}(\lambda_0,\mu)}{z-\lambda_0}, \quad z\in\mathbb{T},$$

which proves (6.8). •

Remark. Similar arguments show that in the case of general $\Lambda=\{\lambda_1,\dots,\lambda_n\}$ and $\Theta=\{\theta_1,\dots,\theta_n\}$ we have

$$z\partial_z\left(\omega_{\Lambda,\Theta}(z)k^+_{\Lambda,\Theta}(z,\mu)\right) = \omega_{\Lambda,\Theta}(z)\Big(\frac{1}{1-\bar{\mu}z} + \sum_{k=1}^{n}\frac{\theta_k\lambda_k}{z-\lambda_k}k^+_{\Lambda,\Theta}(\lambda_k,\mu)\Big).$$

We return to the special case of $\Lambda=\{0,\lambda_0\}$, $\Theta=\{\theta_1,\theta_2\}$. The kernels $k^+_{\Lambda,\Theta}(\cdot,0)$ and $k^+_{\Lambda,\Theta}(\cdot,\lambda_0)$ can be described by alternative formulas. First, we have the following statement.

Lemma 6.3.

$$\theta_1 k^+_{\Lambda,\Theta}(\mu,0) + \theta_2 k^+_{\Lambda,\Theta}(\mu,\lambda_0) = 1, \quad \mu\in\mathbb{D}. \tag{6.12}$$

Proof. We have

$$1 = \left[z^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(z)\cdot 1,\; z^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(z)k^+_{\Lambda,\Theta}(z,\mu)\right]$$
$$= \int_{\mathbb{T}} z\partial_z\left(z^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(z)\right)\overline{z^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(z)k^+_{\Lambda,\Theta}(z,\mu)}\,dm(z).$$

Since

$$z\partial_z\left(z^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(z)\right) = z^{\theta_1}{\omega_{\lambda_0}}^{\theta_2}(z)\left(\theta_1 + \theta_2\frac{1-|\lambda_0|^2}{|1-\bar\lambda_0 z|^2}\right), \quad z\in\mathbb{T},$$

we obtain

$$1 = \int_{\mathbb{T}}\left(\theta_1 + \theta_2\frac{1-|\lambda_0|^2}{|1-\bar\lambda_0 z|^2}\right)\overline{k^+_{\Lambda,\Theta}(z,\mu)}\,dm(z) = \theta_1 k^+_{\Lambda,\Theta}(0,\mu) + \theta_2 k^+_{\Lambda,\Theta}(\lambda_0,\mu)$$

which implies (6.12). •

Our next goal is to describe the kernel $k^+_{\Lambda,\Theta}(\cdot,0)$. We introduce the function

$$X_0(z) = z + \bar\lambda_0\theta_1 z\omega_\lambda(z)k^+_{\Lambda,\Theta}(z,0), \tag{6.13}$$

so that

$$k^+_{\Lambda,\Theta}(z,0) = \frac{X_0(z)-z}{\bar\lambda_0\theta_1 z\omega_{\lambda_0}(z)}. \tag{6.14}$$

Lemma 6.4. *The function $z^{\theta_1-1}\omega_\lambda(z)^{\theta_2-1}X_0(z)$ is orthogonal* (*with respect to the Dirichlet form*) *to $\mathcal{G}^+_{\Lambda,\Theta}$.*

Proof. For any polynomial p, we can write

$$\begin{aligned}
&\left[z^{\theta_1-1}\omega_\lambda(z)^{\theta_2-1}X_0(z), z^{\theta_1}\omega_\lambda^{\theta_2}(z)p(z)\right]\\
&\quad= \left[z^{\theta_1}\omega_\lambda(z)^{\theta_2-1}, z^{\theta_1}\omega_\lambda(z)^{\theta_2-1}\cdot\omega_\lambda(z)p(z)\right]\\
&\qquad+ \bar\lambda_0\theta_1\left[z^{\theta_1}\omega_\lambda^{\theta_2}(z)k^+_{\Lambda,\Theta}(z,0), z^{\theta_1}\omega_\lambda^{\theta_2}(z)p(z)\right]\\
&\quad= \int_{\mathbb{T}}\left(\theta_1 + (\theta_2-1)\frac{1-|\lambda_0|^2}{|1-\bar\lambda_0\zeta|^2}\right)\overline{\omega_\lambda(\zeta)p(\zeta)}\,dm(\zeta) + \bar\lambda_0\theta_1\overline{p(0)}\\
&\quad= -\bar\lambda_0\theta_1\overline{p(0)} + \bar\lambda_0\theta_1\overline{p(0)} = 0. \quad \bullet
\end{aligned}$$

For any $a\in\mathbb{R}$, let $(a)_k$ denote the Pochhammer symbol

$$(a)_k := a(a+1)\cdot\ \dots\ \cdot(a+k-1).$$

Let also $\mathcal{S}$ denote the shift operator:

$$(\mathcal{S}f)(z) := zf(z),$$

and $\mathcal{L}$ denote the left inverse to $\mathcal{S}$, i.e., the backward shift operator:

$$(\mathcal{L}g)(z) := \frac{g(z)-g(0)}{z}$$

(both operators are defined on functions analytic in $\mathbb{D}$). Then, for any h analytic in $\overline{\mathbb{D}}$, we have

$$\int_{\mathbb{T}} h(\zeta)f(\zeta)\overline{g(\zeta)}\,dm(\zeta) = \int_{\mathbb{T}} f(\zeta)\overline{\left[h_*(\mathcal{L})g\right](\zeta)}\,dm(\zeta) \tag{6.15}$$

(in all situations where both sides of this identity have sense), where

$$h_*(z) = \overline{h(\bar z)}.$$

Proposition 6.1. *The function* X_0 *defined by* (6.13) *admits the following representation*:

$$\begin{aligned} X_0(z) &= a_0 z (I - \lambda_0 \mathcal{L})^{1-\theta_2} \Big[\sum_{k \geqslant 0} \frac{(\theta_1)_k}{(\theta_1+\theta_2)_k} \bar\lambda_0^k z^k \Big] \\ &= a_0 z \sum_{k\geqslant 0} \Big\{ \sum_{l \geqslant 0} \frac{(\theta_2 - 1)_l (\theta_1)_{k+l}}{l!(\theta_1+\theta_2)_{k+l}} |\lambda_0|^{2l} \Big\} \bar\lambda_0^k z^k, \end{aligned} \tag{6.16}$$

where the constant a_0 *is chosen so that* $X_0(\lambda_0) = \lambda_0$.

Proof. Lemma 6.4 implies that

$$[z^{\theta_1}(z-\lambda_0)^{\theta_2} p(z), z^{\theta_1 - 1}\omega_{\lambda_0}(z)^{\theta_2 - 1} X_0(z)] = 0$$

for any polynomial p. This identity can be written as

$$\begin{aligned} 0 &= \int_{\mathbb{T}} \big[(\theta_1(z-\lambda_0) + \theta_2 z)\, p(z) + z(z-\lambda_0)\partial_z p(z) \big] \\ &\quad \times z^{\theta_1 - 1}(z-\lambda_0)^{\theta_2 - 1} \cdot \overline{z^{\theta_1-1}\omega_{\lambda_0}(z)^{\theta_2-1} X_0(z)}\, z dm(z) \\ &= \int_{\mathbb{T}} \big[((\theta_1+\theta_2)z - \theta_1\lambda_0)\, p(z) + (z-\lambda_0) z \partial_z p(z) \big] (1 - \bar\lambda_0 z)^{\theta_2 - 1} z \overline{X_0(z)}\, dm(z) \\ &= \int_{\mathbb{T}} p(z) \cdot \overline{\big[(\theta_1+\theta_2)\mathcal{L} - \theta_1 \bar\lambda_0 + z\partial_z(\mathcal{L} - \bar\lambda_0 I)\big]\, (I - \lambda_0\mathcal{L})^{\theta_2 - 1}\mathcal{L} X_0(z)}\, dm(z), \end{aligned}$$

which gives

$$\big[(\theta_1+\theta_2)\mathcal{L} - \theta_1\bar\lambda_0 + z\partial_z(\mathcal{L} - \bar\lambda_0 I)\big](I-\lambda_0\mathcal{L})^{\theta_2-1}\mathcal{L}X_0 = 0. \tag{6.17}$$

Let $Y_0 = (I - \lambda_0\mathcal{L})^{\theta_2 - 1}\mathcal{L}X_0$. Since $X_0(0) = 0$, we have

$$X_0(z) = z\big[(I - \lambda_0 \mathcal{L})^{1-\theta_2} Y_0\big](z). \tag{6.18}$$

If

$$Y_0(z) = \sum_{k\geqslant 0} a_k z^k,$$

then (6.17) is equivalent to

$$\big[(\theta_1+\theta_2)\mathcal{L} - \theta_1\bar\lambda_0 + z\partial_z(\mathcal{L} - \bar\lambda_0 I)\big] Y_0 = 0$$

or

$$(\theta_1 + \theta_2 + k) a_{k+1} - (\theta_1 + k)\bar\lambda_0 a_k = 0,$$

which gives

$$a_k = \frac{(\theta_1)_k \bar{\lambda}_0^k}{(\theta_1 + \theta_2)_k} a_0$$

and implies (6.16). The normalization $X_0(\lambda_0) = \lambda_0$ follows from the definition (6.13). •

The right-hand side of (6.16) admits an integral representation. Namely, we observe that

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

and hence by standard properties of Euler integrals we have

$$\frac{(\theta_1)_k}{(\theta_1+\theta_2)_k} = \frac{\Gamma(\theta_1+k)\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_1+\theta_2+k)} = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^1 t^{\theta_1-1+k}(1-t)^{\theta_2-1}\, dt.$$

Therefore,

$$\sum_{k\geqslant 0} \frac{(\theta_1)_k}{(\theta_1+\theta_2)_k} (\bar{\lambda}_0 z)^k = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^1 \frac{t^{\theta_1-1}(1-t)^{\theta_2-1}}{1-\bar{\lambda}_0 z t}\, dt.$$

Since

$$\mathcal{L}\Big[\frac{1}{1-\bar{\lambda}_0 z t}\Big] = \frac{\bar{\lambda}_0 t}{1-\bar{\lambda}_0 z t},$$

we obtain

$$X_0(z) = a_0 z \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^1 \frac{(1-|\lambda_0|^2 t)^{1-\theta_2} t^{\theta_1-1}(1-t)^{\theta_2-1}}{1-\bar{\lambda}_0 z t}\, dt.$$

Let E_{θ_1,θ_2} denote the integral (depending also on λ_0)

$$E_{\theta_1,\theta_2}(z) := \int_0^1 \frac{(1-|\lambda_0|^2 t)^{1-\theta_2} t^{\theta_1-1}(1-t)^{\theta_2-1}}{1-\bar{\lambda}_0 z t}\, dt. \tag{6.19}$$

Then we obtain finally

$$X_0(z) = z E^{-1}_{\theta_1,\theta_2}(\lambda_0) E_{\theta_1,\theta_2}(z)$$

and

$$k^+_{\Lambda,\Theta}(z,0) = \frac{E^{-1}_{\theta_1,\theta_2}(\lambda_0)E_{\theta_1,\theta_2}(z) - 1}{\bar{\lambda}_0 \theta_1 \omega_{\lambda_0}(z)} \tag{6.20}$$

Combining this formula with (6.12), we can find also an integral representation of $k^+_{\Lambda,\Theta}(z,\lambda_0)$:

$$k^+_{\Lambda,\Theta}(z,\lambda_0) = \frac{\dfrac{1-|\lambda_0|^2}{1-\bar{\lambda}_0 z} - E^{-1}_{\theta_1,\theta_2}(\lambda_0)E_{\theta_1,\theta_2}(z)}{\bar{\lambda}_0 \theta_2 \omega_{\lambda_0}(z)} \tag{6.21}$$

Finally, we write down an important special case of the domination (5.8). Namely, for any φ univalent in $\mathbb{D}$ we have

$$\|L_{\varphi,\Lambda,\Theta}(\lambda_0,\cdot)\|^2_{\mathcal{B}_{\Lambda,\Theta}} \leqslant k^+_{\Lambda,\Theta}(\lambda_0,\lambda_0). \tag{6.22}$$

Here, explicit formulas for the norm $\|\cdot\|_{\mathcal{B}_{\Lambda,\Theta}}$ are given by (6.6) and (6.7); the kernel $L_{\varphi,\Lambda,\Theta}(\lambda_0,u)$ is

$$\begin{aligned} L_{\varphi,\Lambda,\Theta}(\lambda_0,u) &= \Big(\frac{u}{\varphi(u)-\varphi(0)}\frac{\varphi(\lambda_0)-\varphi(0)}{\lambda_0}\Big)^{\theta_1} \\ &\times \Big(\frac{u-\lambda_0}{\varphi(u)-\varphi(\lambda_0)}\varphi'(\lambda_0)\frac{1-|\lambda_0|^2}{1-\bar\lambda_0 u}\Big)^{\theta_2}\frac{\varphi'(u)}{\varphi(\lambda_0)-\varphi(u)} \\ &- \frac{1}{\lambda_0-u}+\frac{\theta_2\bar\lambda_0}{1-\bar\lambda_0 u}k^+_{\Lambda,\Theta}(\lambda_0,\lambda_0). \end{aligned}$$

The constant $k^+_{\Lambda,\Theta}(\lambda_0,\lambda_0)$ admits an easier representation than that given by the limit as $z\to\lambda_0$ in (6.21). Namely, a Möbius shift of variables shows that for any $\lambda_1,\lambda_2\in\mathbb{D}$ and a Möbius transformation ω of $\mathbb{D}$ we have

$$k^+_{\{\omega(\lambda_1),\omega(\lambda_2)\},\{\theta_1,\theta_2\}}(z,u) = k^+_{\{\lambda_1,\lambda_2\},\{\theta_1,\theta_2\}}(\omega(z),\omega(u)).$$

In particular, applying this formula with

$$\omega(z) = \frac{\lambda_0-z}{1-\bar\lambda_0 z},$$

we obtain

$$k^+_{\{0,\lambda_0\},\{\theta_1,\theta_2\}}(\lambda_0,\lambda_0) = k^+_{\{\lambda_0,0\},\{\theta_1,\theta_2\}}(0,0) = k^+_{\{0,\lambda_0\},\{\theta_2,\theta_1\}}(0,0).$$

From (6.20) it follows that

$$\begin{aligned} &k^+_{\{0,\lambda_0\},\{\theta_2,\theta_1\}}(0,0) \\ &\quad= \frac{1}{\theta_2|\lambda_0|^2}\bigg\{1-\bigg(\int_0^1(1-|\lambda_0|^2t)^{-\theta_1}t^{\theta_2-1}(1-t)^{\theta_1-1}dt\bigg)^{-1} \\ &\qquad\qquad\times\int_0^1(1-|\lambda_0|^2t)^{1-\theta_1}t^{\theta_2-1}(1-t)^{\theta_1-1}dt\bigg\}, \end{aligned}$$

which gives

$$\begin{aligned} k^+_{\Lambda,\Theta}(\lambda_0,\lambda_0) &= \frac{1}{\theta}\bigg(\int_0^1(1-|\lambda_0|^2t)^{-\theta_1}t^{\theta_2-1}(1-t)^{\theta_1-1}dt\bigg)^{-1} \\ &\quad\times\int_0^1(1-|\lambda_0|^2t)^{-\theta_1}t^{\theta_2}(1-t)^{\theta_1-1}dt \end{aligned} \tag{6.23}$$

(for $\Lambda=\{0,\lambda_0\}$, $\Theta=\{\theta_1,\theta_2\}$).

Derivation of Goluzin's inequality. In this subsection, we derive the pointwise estimate (1.8) from a special case of the integral inequality (6.22).

In the special case of $\Lambda = \{0, \lambda_0\}$, $\Theta = \{\frac{1}{2}, \frac{1}{2}\}$, the estimate (6.22) takes the form

$$\int_{\mathbb{D}} |L_{\varphi,\Lambda,\Theta}(\lambda_0, u)|^2 \frac{dA(u)}{|u||\omega_{\lambda_0}(u)|} \leqslant k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0), \tag{6.24}$$

where

$$L_{\varphi,\Lambda,\Theta}(\lambda_0, u) = \Big(\frac{u}{\varphi(u)}\frac{\varphi(\lambda_0)}{\lambda_0}\Big)^{1/2}\Big(\frac{u-\lambda_0}{\varphi(u)-\varphi(\lambda_0)}\varphi'(\lambda_0)\frac{1-|\lambda_0|^2}{1-\bar\lambda_0 u}\Big)^{1/2} \times \frac{\varphi'(u)}{\varphi(\lambda_0)-\varphi(u)} - \frac{1}{\lambda_0 - u} + \frac{\theta_2\bar\lambda_0}{1-\bar\lambda_0 u}k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0). \tag{6.25}$$

in the case where $\varphi(0) = 0$. Since the left-hand side of (6.24) is the norm of $L_{\varphi,\Lambda,\Theta}(\lambda_0, \cdot)$ in $\mathcal{B}_{\Lambda,\Theta}$, we can conclude that

$$|L_{\varphi,\Lambda,\Theta}(\lambda_0, \lambda_0)| \leqslant \Big[k^-_{\Lambda,\Theta}(\lambda_0, \lambda_0)k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0)\Big]^{1/2}. \tag{6.26}$$

An explicit calculation shows that

$$L_{\varphi,\Lambda,\Theta}(\lambda_0, \lambda_0) = -\frac{1}{4}\Big(\frac{\varphi''(\lambda_0)}{\varphi'(\lambda_0)} - 2\frac{\varphi'(\lambda_0)}{\varphi(\lambda_0)} + \frac{2}{\lambda_0} + \frac{2\bar\lambda_0}{1-|\lambda_0|^2} - \frac{2\bar\lambda_0}{1-|\lambda_0|^2}k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0)\Big).$$

By Proposition 3.3, we have

$$k^-_{\Lambda,\Theta}(\lambda_0, \lambda_0) = \frac{1}{4}|\lambda_0|^2\frac{k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0)}{(1-|\lambda_0|^2)^2}.$$

Finally, formula (6.23) shows that

$$k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0) = 2|\lambda_0|^{-2}\Big(1 - \Big(\int_0^1 \frac{dt}{\sqrt{t(1-t)(1-|\lambda_0|^2 t)}}\Big)^{-1} \cdot \int_0^1 \sqrt{\frac{1-|\lambda_0|^2 t}{t(1-t)}}\,dt\Big).$$

If $E(z)$ and $K(z)$ denote standard complete elliptic integrals

$$E(z) = \int_0^1 \sqrt{\frac{1-zt^2}{1-t^2}}\,dt \quad \text{and} \quad K(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}},$$

we obtain

$$k^+_{\Lambda,\Theta}(\lambda_0, \lambda_0) = \frac{2}{|\lambda_0|^2}\Big(1 - \frac{E(|\lambda_0|^2)}{K(|\lambda_0|^2)}\Big).$$

Substituting this to previous formulas, we arrive at the following pointwise estimate for functions $\varphi \in S$:

$$\Big|\lambda_0\frac{\varphi''(\lambda_0)}{\varphi'(\lambda_0)} - 2\frac{\lambda_0\varphi'(\lambda_0)}{\varphi(\lambda_0)} + 2 + \frac{2|\lambda_0|^2-4}{1-|\lambda_0|^2} + 4\frac{E(|\lambda_0|^2)}{K(|\lambda_0|^2)}\frac{1}{1-|\lambda_0|^2}\Big| \leqslant 4\Big(1 - \frac{E(|\lambda_0|^2)}{K(|\lambda_0|^2)}\Big)$$

Originally, it appeared in [6, Chapter IV, Section 3, formula (30)] and it is equivalent to the pointwise estimate (1.8) for functions of the class Σ.

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Поступило 10 октября 2005 г.