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**ESTIMATES OF SUITABLE WEAK
SOLUTIONS TO THE NAVIER–STOKES
EQUATIONS IN CRITICAL MORREY SPACES**

ABSTRACT. We prove some estimates for suitable weak solutions to the nonstationary three-dimensional Navier–Stokes equations under assumptions that certain invariant functionals of the velocity field are bounded.

1. INTRODUCTION

Consider the nonstationary 3D Navier–Stokes equations

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla p, \quad \operatorname{div} v = 0 \quad (1.1)$$

in the unit space-time cylinder $Q = B \times]-1, 0[\subset \mathbb{R}^3 \times \mathbb{R}^1$. Here, $B(r)$ is the ball of radius r in \mathbb{R}^3 centered at the space origin $x = 0$, $Q(r) = B(r) \times]-r^2, 0[$ is a standard parabolic cylinder, $B = B(1)$, $Q = Q(1)$, v and p stand for the velocity and for the pressure, respectively.

It is known that equations (1.1) are invariant with respect to the following scaling (we call it the natural scaling)

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

In the so-called ε -regularity theory, the important role plays certain critical Morrey spaces. Their norms are generated by functionals which are invariant with respect to the natural scaling. Among such functionals, there are

$$C(r) = \frac{1}{r^2} \int_{Q(r)} |v|^3 dz, \quad A(r) = \operatorname{ess\,sup}_{-r^2 < t < 0} \frac{1}{r} \int_{B(r)} |v(x, t)|^2 dx,$$
$$E(r) = \frac{1}{r} \int_{Q(r)} |\nabla v|^2 dz, \quad H(r) = \frac{1}{r^3} \int_{Q(r)} |v|^2 dz,$$

The work was supported by the Alexander von Humboldt Foundation, by the RFFI grant 05-01-00941-a, and by the CRDF grant RU-M1-2596-ST-04.

$$D(r) = \frac{1}{r^2} \int_{Q(r)} |p|^{\frac{3}{2}} dz, \quad D_0(r) = \frac{1}{r^2} \int_{Q(r)} |p - [p]_{B(r)}|^{\frac{3}{2}} dz,$$

$$D_1(r) = \frac{1}{r^{\frac{3}{2}}} \int_{-r^2}^0 \left(\int_{B(r)} |\nabla p|^{\frac{3}{2}} dx \right)^{\frac{4}{3}} dt,$$

where $z = (x, t)$ is a point in space-time and

$$[p]_{B(r)}(t) = \frac{1}{|B(r)|} \int_{B(r)} p(x, t) dx.$$

All conditions of ε -regularity for the so-called suitable weak solutions are stated with the help of those functionals. For example, the famous Caffarelli–Kohn–Nirenberg condition, see [1], reads as follows.

Theorem 1.1. *There is a universal positive constant ε with the following property. Assume that the pair v and p is a suitable weak solution to the Navier–Stokes equations in Q . If*

$$\sup_{0 < r \leq 1} E(r) \leq \varepsilon, \quad (1.2)$$

then the space-time origin $z = 0$ is a regular point of v .

Let us recall to the reader definitions of suitable weak solutions and regular points.

Definition 1.2. *The pair v and p is called a suitable weak solution to the Navier–Stokes equations in Q if*

$$v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q), \quad p \in L_{\frac{3}{2}}(Q); \quad (1.3)$$

$$\begin{aligned} & \text{the Navier–Stokes equations hold in } Q \\ & \text{in the sense of distributions;} \end{aligned} \quad (1.4)$$

for a.a. $t \in]-1, 0[$, the local energy inequality

$$\int_B \varphi(x, t) |v(x, t)|^2 dx + 2 \int_{-1}^t \int_B \varphi |\nabla v|^2 dx dt'$$

$$\leq \int_{-1}^t \int_B \left\{ |v|^2 (\Delta \varphi + \partial_t \varphi) + v \cdot \nabla \varphi (|v|^2 + 2p) \right\} dx dt' \quad (1.5)$$

holds for any nonnegative test function $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^1)$ vanishing in a neighborhood of the parabolic boundary of Q .

Definition 1.3. The point $z = 0$ is called a regular point of v if there is a number $r \in]0, 1]$ such that v is a Hölder continuous function in $\overline{Q}(r)$.

Here, the following abbreviations are used:

$$L_{2,\infty}(Q) = L_\infty(-1, 0; L_2(B)), \quad W_2^{1,0}(Q) = L_2(-1, 0; W_2^1(B)),$$

$L_2(B)$ and $W_2^1(B)$ are the usual Lebesgue and Sobolev spaces, respectively.

Remark 1.4. Our definition of suitable weak solutions belongs to F.-H. Lin [5]. It differs from more general definition, given by Caffarelli-Kohn-Nirenberg in [1], by the very concrete choice of the space for the pressure. To our opinion, such a choice seems to be more convenient to treat.

Remark 1.5. Definition 1.3 of regular points is due Ladyzhenskaya-Seregin [4]. In the most popular definition by Caffarelli-Kohn-Nirenberg, the Hölder space is replaced with the space of essentially bounded functions.

Roughly speaking, Theorem 1.1 and other similar statements say that smallness of functionals, which are invariant with respect to the natural scaling, is a sufficient condition for regularity. Obviously, the next problem is to figure out what happens if above functionals are bounded but not small. This seems to be a subtle and completely open question. However, there is one case, where the answer is known and positive. It is the marginal case of the so-called Ladyzhenskaya-Prodi-Serrin condition. Indeed, in the Ladyzhenskaya-Prodi-Serrin condition, the key role plays the functional $\|\cdot\|_{s,l,Q}$, which is the norm of the mixed Lebesgue space $L_{s,l}(Q) = L_l(-1, 0; L_s(B))$. This norm is invariant with the respect to the natural scaling if $3/s + 2/l = 1$ and $s \geq 3$. The regular case $s > 3$ can be reduced to the smallness of the norm $\|v\|_{s,l,Q}$ with the help of the natural scaling and absolute continuity of Lebesgue's integral. So, the only case, which seems to be not reducible to the ε -regularity theory is $s = 3$ and $l = +\infty$. It should be noticed that to treat $L_{3,\infty}$ -case we had to develop a new method based on the unique continuation theory for parabolic equations, see [3].

The aim of this paper is to contribute somehow to analysis of smoothness of suitable weak solutions under additional assumptions that certain functionals invariant with respect to natural scaling are bounded. We hope that our results can be regarded as a starting point for that analysis. Let us formulate them.

Lemma 1.6. *Assume that we are given a suitable weak solution v and p in Q . Let, in addition,*

$$\sup_{0 < r \leq 1} E(r) = E_0 < +\infty. \quad (1.6)$$

Then, there is a positive constant d depending on E_0 only such that

$$A^{\frac{3}{2}}(r) + C(r) + D_0^2(r) \leq d \left(r^{\frac{1}{2}} (A^{\frac{3}{2}}(1) + D_0^2(1)) + 1 \right) \quad (1.7)$$

for all $0 < r \leq 1/4$.

Lemma 1.7. *Suppose that the pair v and p is a suitable weak solution in Q . Let*

$$\sup_{0 < r \leq 1} C(r) = C_0 < +\infty. \quad (1.8)$$

Then

$$A(r) + D_0(r) + E(r) \leq c \left(r^2 D_0(1) + C_0 + C_0^{\frac{2}{3}} \right) \quad (1.9)$$

for all $0 < r \leq 1/2$.

Here and in what follows, c is a positive universal constant.

Lemma 1.8. *Suppose that the pair v and p is a suitable weak solution in Q . Let*

$$\sup_{0 < r \leq 1} A(r) = A_0 < +\infty. \quad (1.10)$$

Then there is a positive constant ϵ depending on A_0 only such that

$$C^{\frac{3}{4}}(r) + D_0(r) + E(r) \leq \epsilon \left(r^2 (D_0(1) + E(1)) + 1 \right) \quad (1.11)$$

for all $0 < r \leq 1/2$.

Statements similar to Lemmas 1.6–1.8 are proved by Choe–Lewis in [2], see Lemma 1 there. Our proof is different and estimates are sharper.

2. PRELIMINARY INEQUALITIES

There are three basic inequalities and their modifications. The first of them is but a multiplicative inequality and has the form

$$C(r) \leq c \left[\left(\frac{\varrho}{r} \right)^3 A^{\frac{3}{4}}(\varrho) E^{\frac{3}{4}}(\varrho) + \left(\frac{r}{\varrho} \right)^3 A^{\frac{3}{2}}(\varrho) \right] \quad (2.1)$$

for all $0 < r \leq \varrho \leq 1$. The reader can find a proof of it in [4], see also [5].

The second group of inequalities is a consequence of local energy inequality (1.5)

$$A(R/2) + E(R/2) \leq c \left[C^{\frac{2}{3}}(R) + C(R) + C^{\frac{1}{3}}(R) D_0^{\frac{2}{3}}(R) \right] \quad (2.2)$$

for all $0 < R \leq 1$. It follows from (1.5) directly. Another version of the local energy inequality is demonstrated in [4]

$$\begin{aligned} A(R/2) + E(R/2) \\ \leq c \left[C^{\frac{2}{3}}(R) + C^{\frac{1}{3}}(R) D_0^{\frac{2}{3}}(R) + A^{\frac{1}{2}}(R) C^{\frac{1}{3}}(R) E^{\frac{1}{2}}(R) \right] \end{aligned} \quad (2.3)$$

for all $0 < R \leq 1$.

A kind of a decay estimate for the pressure is the third inequality. There are several versions of such decay estimate. One of the is proved in [6] and reads

$$D_0(r) \leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 C(\varrho) \right] \quad (2.4)$$

for any $0 < r \leq \varrho \leq 1$. However, in a number of cases, it is more convenient to use a slightly different versions

$$D_0(r) \leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 A^{\frac{1}{2}}(\varrho) E(\varrho) \right] \quad (2.5)$$

or

$$D_0(r) \leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^3 A^{\frac{3}{4}}(\varrho) E^{\frac{3}{4}}(\varrho) \right]. \quad (2.6)$$

Both are valid for the same r and ϱ as in (2.4).

Inequalities (2.5) and (2.6) can be proved more or less in the same way. To show the basic arguments, let us prove the first of them. To this end, we decompose the pressure

$$p = p_1 + p_2 \quad (2.7)$$

in $B(\varrho)$ so that p_1 is a unique solution to the variational identity

$$\int_{B(\varrho)} p_1 \Delta \varphi dx = - \int_{B(\varrho)} (\tau - \tau_\varrho) : \nabla^2 \varphi dx, \quad (2.8)$$

where φ is an arbitrary test function from $W_3^2(B(\varrho))$ satisfying the boundary condition $\varphi|_{\partial B(\varrho)} = 0$ and

$$\tau = (v - c_\varrho) \otimes (v - c_\varrho), \quad \tau_\varrho = [(v - c_\varrho) \otimes (v - c_\varrho)]_{B(\varrho)}, \quad c_\varrho = [v]_{B(\varrho)}.$$

Here, time t is considered as a parameter. Obviously, then,

$$\Delta p_2 = 0 \quad (2.9)$$

in $B(\varrho)$.

We can easily find the estimate of p_1

$$\int_{B(\varrho)} |p_1|^{\frac{3}{2}} dx \leq c \int_{B(\varrho)} |\tau - \tau_\varrho|^{\frac{3}{2}} dx.$$

By the Galiardo–Nirenberg inequality,

$$\int_{B(\varrho)} |p_1|^{\frac{3}{2}} dx \leq c \left(\int_{B(\varrho)} |v - c_\varrho| |\nabla v| dx \right)^{\frac{3}{2}}$$

and thus

$$\int_{B(\varrho)} |p_1|^{\frac{3}{2}} dx \leq c \left(\int_{B(\varrho)} |v - c_\varrho|^2 dx \right)^{\frac{3}{4}} \left(\int_{B(\varrho)} |\nabla v|^2 dx \right)^{\frac{3}{4}}.$$

On the other hand, we can use the Poincaré inequality

$$\int_{B(\varrho)} |v - c_\varrho|^2 dx \leq c\varrho^2 \int_{B(\varrho)} |\nabla v|^2 dx$$

and the minimality property of c_ϱ

$$\int_{B(\varrho)} |v - c_\varrho|^2 dx \leq \int_{B(\varrho)} |v|^2 dx.$$

The latter relation leads to the estimate

$$\frac{1}{\varrho^2} \int_{-\varrho^2}^0 \int_{B(\varrho)} |p_1|^{\frac{3}{2}} dz \leq cE(\varrho)A^{\frac{1}{2}}(\varrho). \quad (2.10)$$

Since p_2 is a harmonic function in $B(\varrho)$, we have for $0 < r \leq \varrho/2$

$$\begin{aligned} \sup_{x \in B(r)} |p_2(x, t) - [p_2]_{B(r)}(t)|^{\frac{3}{2}} &\leq cr^{\frac{3}{2}} \sup_{x \in B(\varrho/2)} |\nabla p_2(x, t)|^{\frac{3}{2}} \\ &\leq c \left(\frac{r}{\varrho^4} \int_{B(\varrho)} |p_2(x, t) - [p_2]_{B(\varrho)}(t)| dx \right)^{\frac{3}{2}} \\ &\leq \frac{c}{\varrho^3} \left(\frac{r}{\varrho} \right)^{\frac{3}{2}} \int_{B(\varrho)} |p_2(x, t) - [p_2]_{B(\varrho)}(t)|^{\frac{3}{2}} dx. \end{aligned} \quad (2.11)$$

Next, by (2.7) and (2.11),

$$\begin{aligned} D(r) &\leq \frac{c}{r^2} \int_{Q(r)} |p_1 - [p_1]_{B(r)}|^{\frac{3}{2}} dz + \frac{c}{r^2} \int_{Q(r)} |p_2 - [p_2]_{B(r)}|^{\frac{3}{2}} dz \\ &\leq \frac{c}{r^2} \int_{Q(r)} |p_1|^{\frac{3}{2}} dz + \frac{c}{r^2} \frac{1}{\varrho^3} \left(\frac{r}{\varrho} \right)^{\frac{3}{2}} \int_{-r^2}^0 r^3 \int_{B(\varrho)} |p_2(x, t) - [p_2]_{B(\varrho)}(t)|^{\frac{3}{2}} dx \\ &\leq c \left(\frac{\varrho}{r} \right)^2 E(\varrho) A^{\frac{1}{2}}(\varrho) + c \left(\frac{r}{\varrho} \right)^{\frac{5}{2}} \frac{1}{\varrho^2} \int_{Q(\varrho)} |p_2 - [p_2]_{B(\varrho)}|^{\frac{3}{2}} dz \\ &\leq c \left(\frac{\varrho}{r} \right)^2 E(\varrho) A^{\frac{1}{2}}(\varrho) + c \left(\frac{r}{\varrho} \right)^{\frac{5}{2}} \left[\frac{1}{\varrho^2} \int_{Q(\varrho)} |p - [p]_{B(\varrho)}|^{\frac{3}{2}} dz \right. \\ &\quad \left. + \frac{1}{\varrho^2} \int_{Q(\varrho)} |p_1 - [p_1]_{B(\varrho)}|^{\frac{3}{2}} dz \right] \leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 E(\varrho) A^{\frac{1}{2}}(\varrho) \right]. \end{aligned}$$

So, inequality (2.5) is proved.

3. PROOF OF LEMMA 1.6

So, assume that condition (1.6) holds. Then, as it follows from (2.1), (2.2), and (1.6), we have

$$C(r) \leq c \left[\left(\frac{\varrho}{r} \right)^3 A^{\frac{3}{4}}(\varrho) E_0^{\frac{3}{4}} + \left(\frac{r}{\varrho} \right)^3 A^{\frac{3}{2}}(\varrho) \right] \quad (3.1)$$

and

$$D_0(r) \leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 A^{\frac{1}{2}}(\varrho) E_0 \right]. \quad (3.2)$$

Introducing

$$\mathcal{E}(r) = A^{\frac{3}{2}}(r) + D_0^2(r),$$

we derive from local energy inequality (2.3)

$$\begin{aligned} \mathcal{E}(r) &\leq c \left[C(2r) + C^{\frac{1}{2}}(2r) D_0(2r) + A^{\frac{3}{4}}(2r) C^{\frac{1}{2}}(2r) E_0^{\frac{3}{4}} \right] + D_0^2(r) \\ &\leq c \left[C(2r) + D_0^2(2r) + A^{\frac{3}{4}}(2r) C^{\frac{1}{2}}(2r) E_0^{\frac{3}{4}} \right]. \end{aligned} \quad (3.3)$$

Now, let us assume that $0 < r \leq \varrho/2 < \varrho \leq 1$. Replacing r with $2r$ in (3.1) and (3.2), we find from (3.3)

$$\begin{aligned} \mathcal{E}(r) &\leq c \left[\left(\frac{\varrho}{r} \right)^3 A^{\frac{3}{4}}(\varrho) E_0^{\frac{3}{4}} + \left(\frac{r}{\varrho} \right)^3 A^{\frac{3}{2}}(\varrho) \right. \\ &\quad \left. + \left(\frac{r}{\varrho} \right)^5 D_0^2(\varrho) + \left(\frac{\varrho}{r} \right)^4 A(\varrho) E_0^2 \right. \\ &\quad \left. + A^{\frac{3}{4}}(2r) \left(\left(\frac{\varrho}{r} \right)^3 A^{\frac{3}{4}}(\varrho) E_0^{\frac{3}{4}} + \left(\frac{r}{\varrho} \right)^3 A^{\frac{3}{2}}(\varrho) \right)^{\frac{1}{2}} E_0^{\frac{3}{4}} \right] \\ &\leq c \left[\left(\frac{r}{\varrho} \right)^3 A^{\frac{3}{2}}(\varrho) + \left(\frac{r}{\varrho} \right)^5 D_0^2(\varrho) + \left(\frac{r}{\varrho} \right)^{\frac{3}{2}} A^{\frac{3}{4}}(\varrho) E_0^{\frac{3}{4}} A^{\frac{3}{4}}(\varrho) \left(\frac{\varrho}{r} \right)^{\frac{3}{4}} \right. \\ &\quad \left. + \left(\frac{\varrho}{r} \right)^{\frac{3}{2} + \frac{3}{4}} A^{\frac{3}{4} + \frac{3}{8}}(\varrho) E_0^{\frac{3}{4} + \frac{3}{8}} + \left(\frac{\varrho}{r} \right)^4 A(\varrho) E_0^2 + \left(\frac{\varrho}{r} \right)^3 A^{\frac{3}{4}}(\varrho) E_0^{\frac{3}{4}} \right]. \end{aligned}$$

Here, the obvious inequality $A(2r) \leq c\varrho A(\varrho)/r$ has been used. Applying Young inequality with an arbitrary positive constant δ , we show

$$\mathcal{E}(r) \leq c \left(\frac{r}{\varrho} \right)^{\frac{3}{4}} (E_0^{\frac{3}{4}} + 1) \mathcal{E}(\varrho) + c\delta \mathcal{E}(\varrho)$$

$$+c(\delta)\left[\left(\frac{\varrho}{r}\right)^6 E_0^{\frac{3}{2}} + \left(\frac{\varrho}{r}\right)^{12} E_0^6 + \left(\frac{\varrho}{r}\right)^9 E_0^{\frac{3}{2}}\right].$$

Therefore,

$$\mathcal{E}(r) \leq c\left[\left(\frac{r}{\varrho}\right)^{\frac{3}{4}}(E_0^{\frac{3}{4}} + 1) + \delta\right]\mathcal{E}(\varrho) + c(\delta)\left(\frac{\varrho}{r}\right)^{12}(E_0^6 + E_0^{\frac{3}{2}} + E_0^{\frac{3}{2}}). \quad (3.4)$$

Inequality (3.4) holds for $r \leq \varrho/2$ and can be reduced to the form

$$\mathcal{E}(\vartheta\varrho) \leq c\left[\vartheta^{\frac{3}{4}}(E_0^{\frac{3}{4}} + 1) + \delta\right]\mathcal{E}(\varrho) + c(\delta)\vartheta^{-12}(E_0^6 + E_0^{\frac{3}{2}} + E_0^{\frac{3}{2}}) \quad (3.5)$$

for any $0 < \vartheta \leq 1/2$ and for any $0 < \varrho \leq 1$.

Now, let us fix ϑ and δ in the following way

$$c\vartheta^{\frac{1}{4}}(E_0^{\frac{3}{4}} + 1) < 1/2, \quad 0 < \vartheta \leq 1/2, \quad c\delta < \vartheta^{\frac{1}{2}}/2. \quad (3.6)$$

Obviously, ϑ and δ depend on E_0 only. So, we have

$$\mathcal{E}(\vartheta\varrho) \leq \vartheta^{\frac{1}{2}}\mathcal{E}(\varrho) + G \quad (3.7)$$

for any $0 < \varrho \leq 1$, where $\vartheta = \vartheta(E_0)$ and $G = G(E_0)$.

Iterations of (3.7) give us

$$\mathcal{E}(\vartheta^k\varrho) \leq \vartheta^{\frac{k}{2}}\mathcal{E}(\varrho) + cG$$

for any natural numbers k and for any $0 < \varrho \leq 1$. Letting $\varrho = 1$, we find

$$\mathcal{E}(\vartheta^k) \leq \vartheta^{\frac{k}{2}}\mathcal{E}(1) + cG \quad (3.8)$$

for any natural numbers k . It can be easily deduced from (3.8) that

$$\mathcal{E}(r) \leq d_1(E_0)(r^{\frac{1}{2}}\mathcal{E}(1) + 1) \quad (3.9)$$

for all $0 < r \leq 1/2$. Now, for $C(r)$, we have from (3.1)

$$C(r) \leq c\left[A^{\frac{3}{4}}(2r)E_0^{\frac{3}{4}} + A^{\frac{3}{2}}(2r)\right] \leq c\left[A^{\frac{3}{2}}(2r) + E_0^{\frac{3}{2}}\right]$$

$$\leq d_2(E_0)(\mathcal{E}(2r) + 1) \leq d_3(E_0)(r^{\frac{1}{2}}\mathcal{E}(1) + 1).$$

So, Lemma 1.6 is proved.

4. PROOF OF LEMMA 1.7

According to conditions (1.8) and inequality (2.4), the following relation is valid:

$$D_0(r) \leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 C_0 \right]$$

for all $0 < r \leq \varrho \leq 1$. Letting $r = \vartheta \varrho$ with $0 < \vartheta \leq 1$, we find

$$D_0(\vartheta \varrho) \leq c \left[\vartheta^{\frac{5}{2}} D_0(\varrho) + \vartheta^{-2} C_0 \right].$$

If we choose ϑ so that $c\vartheta^{\frac{1}{2}} \leq 1$, then

$$D_0(\vartheta \varrho) \leq \vartheta^2 D_0(\varrho) + cC_0.$$

Here, c is a universal constant. After iterations, we arrive at the inequality

$$D_0(\vartheta^k \varrho) \leq \vartheta^{2k} D_0(\varrho) + cC_0$$

for any natural k . Setting $\varrho = 1$, we find

$$D_0(\vartheta^k) \leq \vartheta^{2k} D_0(1) + cC_0$$

for any natural k or

$$D_0(r) \leq cr^2 D_0(1) + cC_0 \tag{4.1}$$

for any $0 < r \leq 1$.

Next, by (2.2) and by (4.1),

$$\begin{aligned} A(R/2) + E(R/2) &\leq c \left[C_0^{\frac{2}{3}} + C_0 + C_0^{\frac{1}{3}} D_0^{\frac{2}{3}}(R) \right] \\ &\leq c \left[D_0(R) + C_0^{\frac{2}{3}} + C_0 \right] \leq c \left[R^2 D_0(1) + C_0^{\frac{2}{3}} + C_0 \right] \end{aligned} \tag{4.2}$$

for all $0 < R \leq 1$. So, we have

$$A(r) + D_0(r) + E(r) \leq c(r^2 D_0(1) + C_0 + C_0^{\frac{2}{3}})$$

for all $0 < r \leq 1/2$. Lemma 1.7 is proved.

5. PROOF OF LEMMA 1.8

Here, we are going to use inequality (2.1) in the form

$$C(r) \leq c \left[A^{\frac{3}{4}}(r) E^{\frac{3}{4}}(r) + A^{\frac{3}{2}}(r) \right] \leq c \left[A_0^{\frac{3}{4}} E^{\frac{3}{4}}(r) + A_0^{\frac{3}{2}} \right]. \quad (5.1)$$

Next, according to local energy inequality (2.2), we have

$$\begin{aligned} \mathcal{F}(r) = E(r) + D_0(r) &\leq c \left[C^{\frac{2}{3}}(2r) + C(2r) + D_0(2r) \right] + D_0(r) \\ &\leq c \left[D_0(2r) + \left(\frac{\varrho}{r} \right)^2 C(\varrho) + \left(\frac{\varrho}{r} \right)^{\frac{4}{3}} C^{\frac{2}{3}}(\varrho) \right] \end{aligned} \quad (5.2)$$

for any $0 < r \leq \varrho/2 < \varrho \leq 1$. To prove (5.2), the inequality $C(r) \leq (\varrho/r)^2 C(\varrho)$, $0 < r \leq \varrho$, has been used.

Now, by (2.4) and by (5.1), (5.2),

$$\begin{aligned} \mathcal{F}(r) &\leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 C(\varrho) + \left(\frac{\varrho}{r} \right)^{\frac{4}{3}} C^{\frac{2}{3}}(\varrho) \right] \\ &\leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \left(\frac{\varrho}{r} \right)^2 \left(A_0^{\frac{3}{4}} E^{\frac{3}{4}}(\varrho) + A_0^{\frac{3}{2}} \right) \right. \\ &\quad \left. + \left(\frac{\varrho}{r} \right)^{\frac{4}{3}} \left(A_0^{\frac{1}{2}} E^{\frac{1}{2}}(\varrho) + A_0 \right) \right]. \end{aligned}$$

Next, we would like to exploit the fact that the power of $E(\varrho)$ is less than one. To this end, the Young inequality with an arbitrary positive constant δ is applied and we find

$$\begin{aligned} \mathcal{F}(r) &\leq c \left[\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} D_0(\varrho) + \delta E(\varrho) \right] \\ &\quad + c(\delta) \left[\left(\frac{\varrho}{r} \right)^2 A_0^{\frac{3}{2}} + \left(\frac{\varrho}{r} \right)^{\frac{4}{3}} A_0 + \left(\frac{\varrho}{r} \right)^8 A_0^3 + \left(\frac{\varrho}{r} \right)^{\frac{8}{3}} A_0 \right] \\ &\leq c \left(\left(\frac{r}{\varrho} \right)^{\frac{5}{2}} + \delta \right) \mathcal{F}(\varrho) + c(\delta) \left(\frac{\varrho}{r} \right)^8 (A_0^3 + A_0^{\frac{3}{2}} + A_0) \end{aligned}$$

for any $0 < r \leq \varrho/2 < \varrho \leq 1$. Now, let $r = \vartheta \varrho$ with $0 < \vartheta \leq 1/2$ and $0 < \varrho \leq 1$. As a result, we have

$$\mathcal{F}(\vartheta \varrho) \leq c(\vartheta^{\frac{5}{2}} + \delta) \mathcal{F}(\varrho) + c(\delta) \vartheta^{-8} (A_0^3 + A_0^{\frac{3}{2}} + A_0).$$

Fix ϑ and δ so that

$$c\vartheta^{\frac{1}{2}} \leq 1/2, \quad 0 < \vartheta \leq 1/2, \quad c\delta \leq \vartheta^2.$$

Clearly, ϑ and δ are universal constants. This implies

$$\mathcal{F}(\vartheta\varrho) \leq \vartheta^2 \mathcal{F}(\varrho) + c(A_0^3 + A_0^{\frac{3}{2}} + A_0).$$

Iterating the latter relations and then letting $\varrho = 1$, we arrive at the estimate

$$\mathcal{F}(\vartheta^k) \leq \vartheta^{2k} \mathcal{F}(1) + c(A_0^3 + A_0^{\frac{3}{2}} + A_0)$$

being valid for any natural number k or

$$\mathcal{F}(r) \leq cr^2 \mathcal{F}(1) + c(A_0^3 + A_0^{\frac{3}{2}} + A_0)$$

for any $0 < r \leq 1/2$.

Next, we can derive from (5.1) that

$$C^{\frac{4}{3}}(r) \leq c(A_0 E(r) + A_0^2) \leq c(A_0 \mathcal{F}(r) + A_0^2) \leq e_1(A_0)(r^2 \mathcal{F}(1) + 1).$$

So, Lemma 1.8 is proved.

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С.-Петербургское отделение
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Поступило 25 января 2006 г.