ТРУДЫ МАТЕМАТИЧЕСКОГО ИНСТИТУТА ИМ. В.А. СТЕКЛОВА, 2005, т. 250, с. 262–271

УДК 517.97

Γ-convergence for Strongly Local Dirichlet Forms in Open Sets with Holes

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Поступило в январе 2005 г.

We consider homogenization problems with holes for strongly local Dirichlet forms in the cases of the Dirichlet and Neumann homogeneous conditions on the boundaries of the holes. In the second case, the main difficulties arise from the absence of a group structure on the underlying space and from the nonperiodic distribution of the holes. Complete proofs of the results will appear later.

1. INTRODUCTION

In this paper, we present some results on the Γ-convergence of problems with holes with respect to a Riemannian or a Poincaré-type Dirichlet form. We consider the cases of homogeneous Dirichlet and Neumann boundary conditions on the boundaries of the holes. In the second case, one of the main difficulties is that we operate on a locally compact metrizable Hausdorff space, where no structure of a group is available, so there is no notion of periodicity in our space. We divide the introduction to the paper into two parts: in the first part, we recall previous results obtained for the elliptic and subelliptic cases, and in the second part, we give the notions of Poincaré-type and Riemannian Dirichlet forms with some properties. We observe that some of the results also hold for Dirichlet forms of a more general type, and one can find the results in full generality in the references cited; for simplicity, we prefer to use a framework in which all the results hold, but in some cases, some results hold under more general assumptions.

First, we consider the case of the Dirichlet homogeneous conditions. In this case, the Euclidean elliptic case was studied first in [19, 22]; however, our reference papers for the Euclidean case are [11, 12], where a result on the Γ-convergence is given for a general distribution of the holes. The papers [11, 12] point out that the limit problems contain a term of the type μu, where μ is a positive Borel measure (and u is the unknown function). This result suggests that our problem falls under a more general class of problems, i.e., the relaxed Dirichlet problems as defined in [15, 16], where the Γ-convergence for this type of problems is studied (see also [14]). For the case of the Neumann boundary condition, the first paper in which the convergence to a limit problem is rigorously proved for a particular Euclidean elliptic problem with periodically distributed holes is [13]; more general Euclidean cases are considered in [20]. L. Tartar made the remark that all the proofs depend on the existence of suitable extension operators (in the holes); the problem in the Euclidean setting without periodicity but with the assumption of the existence of suitable extension operators (in the holes) was solved in [10]. We note that the question of the existence of extension operators with the required properties is solved in the Euclidean setting under a certain assumption on the regularity of the boundary of the holes but becomes very delicate in the subelliptic setting; therefore, it is important to investigate methods that can operate in the absence of the classical extension theory (see [20, 1, 9, 17, 26–28] for the Euclidean case and [7] for the periodic case in the Heisenberg (subelliptic) group).

Now we introduce the notions of Poincaré-type and Riemannian Dirichlet forms.

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Let X be a metrizable locally compact Hausdorff space with a positive Radon measure m such that $\operatorname{supp}[m] = X$. We assume that we are given a *strongly local regular Dirichlet form* a *of diffusion type* in the Hilbert space $L^2(X, m)$ in the sense of M. Fukushima [18]; its domain is denoted by $D[a]$. Such a form a admits the following integral representation: $a(u, v) = \int_X \alpha(u, v)(dx)$ for every $u, v \in D[a]$, where $\alpha(u, v)$ is a signed Radon measure on X that is uniquely associated with the functions u and v. Moreover, for any open subset Ω of X, the restriction of $\alpha(u, v)$ to Ω depends only on the restrictions of u and v to Ω, and $\alpha(u, v) = 0$ whenever $u = \text{const}$ in a neighborhood of the support of v. Let Ω be an open set; by $D_0[a, \Omega]$, we denote the closure of $C_0(\Omega) \cap D[a]$ in $D[a]$. By $D_{\mathrm{loc}}[a, \Omega]$, we denote the space of all m-measurable functions u and v in X that coincide m-a.e. on every compact subset of Ω with some functions of $D[a]$. The measure $\alpha(u, v)$ is defined unambiguously in Ω for all $u, v \in D_{\mathrm{loc}}[a, \Omega]$. We refer the reader to [4, 18, 23–25] for the properties of $\alpha(u, v)$ with respect to the Leibnitz, chain, and truncation rules. We assume now that the space X is endowed with a pseudodistance d and is complete with respect to d (we also assume that d defines a topology on X that is equivalent to the initial topology). We denote $B(x, r) = \{y;\ d(x, y) < r\}$; $B(r)$ will denote balls $B(x, r)$ with a fixed center x. We say that the Dirichlet form is of *Poincaré type* if the following two assumptions hold:

(H_1) there exist constants $0 < R_0 < +\infty$, $\nu > 0$, and $c_0 > 0$ such that

$$0 < c_0 \left(\frac{r}{R}\right)^{\nu} m(B(x, R)) \le m(B(x, r))$$

for every x in a given compact set and every $0 < r < R_0$;

(H_2) for every ball $B(x, r)$ such that $B(x, kr) \Subset \Omega$ and every $f \in D_{\mathrm{loc}}[\Omega]$, the following *scaled Poincaré inequality* holds:

$$\int\limits_{B(x,r)} |f - f_{x,r}|^2\, m(dx) \le c_1 r^2 \int\limits_{B(x,kr)} \alpha(f, f)(dx),$$

where c_1 and $k \ge 1$ are constants independent of x and r, and Ω is a relatively compact open set.

Remark 1.1. We observe that (H_1) is verified if a *duplication property* holds for the balls $B(x, r)$, $0 < r < R_0$, that is,

$$m(B(x, 2r)) \le c_0^\star\, m(B(x, r)),$$

where $c_0^\star$ is a positive constant independent of x and r. In this case, we have $\nu \ge \log_2 c_0^\star$.

Remark 1.2. We recall [18] that there is a Choquet capacity associated with a given Dirichlet form of Poincaré type. If d is a distance on X and $d \in D_{\mathrm{loc}}[a]$ with $\alpha(d, d) \le m$, we say that our Dirichlet form is a *Riemannian* Dirichlet form.

Remark 1.3. We observe that if

$$d_a(x, y) = \sup\{\phi(x) - \phi(y);\ \forall \phi \text{ with } \alpha(\phi, \phi) \le m\}$$

and $d_a(x, y)$ is finite and separating (i.e., if $x \ne y$, then $d_a(x, y) \ne 0$), then d_a is a distance and $d_a \in D_{\mathrm{loc}}[a]$ with $\alpha(d_a, d_a) \le m$.

Remark 1.4. If a is Riemannian, then there exist cut-off functions between balls (i.e., for any two balls $B(x, r)$ and $B(x, R)$, $r < R$, there exists a function ϕ such that $\phi = 1$ on $B(x, r)$, $\phi = 0$ outside $B(x, R)$, and $\alpha(\phi, \phi) \le \frac{1}{(R-r)^2}$).

Recall that for a Dirichlet form satisfying the above assumptions, a Green function $G_\Omega(x,y)=G_\Omega(y,x)$ with respect to a relatively compact open set Ω can be defined and that, in the case $\Omega=B(x_0,R)$, the following estimate holds:

$$G_{B(x_0,R)}(x_0,y)\approx\int\limits_{d(x_0,y)}^{R}\frac{s^2}{m(B(x,s))}\frac{ds}{s}\tag{1.1}$$

for every y such that $d(x_0,y)\le\frac{R}{2}$. Finally, we observe that our assumptions hold for the Dirichlet forms associated to wide classes of

(a) weighted uniformly elliptic operators,
(b) weighted degenerate elliptic operators generated by vector fields satisfying the Hörmander condition, and
(c) subelliptic operators

(see [3, 4, 21] for more details). Moreover, the Laplacian on post critically finite fractals defines a Poincaré-type Dirichlet form [25].

2. THE CASE OF THE HOMOGENEOUS DIRICHLET BOUNDARY CONDITION

In this section, we will consider the case of homogeneous conditions on the boundaries of the holes. Let Ω be a relatively compact open set. Consider a sequence of open sets Ω_ϵ contained in Ω; in this case, the hole is given by the set $\Omega-\Omega_\epsilon$.

Consider the problems

$$\int\limits_{\Omega_\epsilon}\alpha(u,v)(dx)=\langle f,v\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]},\qquad u\in D_0[a,\Omega_\epsilon],\qquad\text{for all } v\in D_0[a,\Omega_\epsilon],\tag{2.1_ϵ}$$

where $\alpha(u,v)$ is the energy density of a Dirichlet form of Poincaré type. Equation (2.1_ϵ) makes sense since every function $v\in D_0[a,\Omega_\epsilon]$ can be extended by 0 to a function (denoted again by v) in $D_0[a,\Omega]$.

Theorem 2.1. *Let u_ϵ be a solution of problem* (2.1_ϵ)*. The sequence u_ϵ weakly converges* (*at least after extraction of subsequences*) *in $D_0[a,\Omega]$ to u_0, which is a unique solution of the problem*

$$\int\limits_{\Omega}\alpha(u,v)(dx)+\int\limits_{\Omega}uv\,\mu(dx)=\langle f,v\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]},\qquad u\in D_0[a,\Omega],\quad\textit{for all } v\in D_0[a,\Omega],\tag{2.1_0}$$

where μ is a positive Borel measure that does not charge sets of zero capacity (*relative to a*) (*i.e., the measure of a set of zero capacity iz zero*).

We recall that the proof of Theorem 2.1 is obtained by embedding the class of problems (2.1_ϵ) into the more general class of relaxed Dirichlet problems. Consider the problems

$$\int\limits_{X}\alpha(u,v)(dx)+\int\limits_{X}uv\,\mu_\epsilon(dx)=\langle f,v\rangle_{(D[a])',D[a]},\qquad u\in D[a]\cap L^2(X,\mu_\epsilon),\quad\text{for all } v\in D[a]\cap L^2(X,\mu_\epsilon),\tag{$2.1'_\epsilon$}$$

where μ_ϵ are positive Borel measures that do not charge sets of zero capacity (relative to a). We observe that $(2.1'_\epsilon)$ reduces to (2.1_ϵ) if we choose

$$\mu_\epsilon(B) = m(B) \tag{2.2}$$

if B is a Borel set such that $B \cap \Omega_\epsilon^{\mathrm{c}}$ has zero capacity and

$$\mu_\epsilon(B) = +\infty \tag{2.3}$$

if B is a Borel set such that $B \cap \Omega_\epsilon^{\mathrm{c}}$ has positive capacity.

Definition 2.1. We say that *the sequence* μ_ϵ Γ*-converges to* μ if the sequence u_ϵ of solutions of problems $(2.1'_\epsilon)$ weakly converges in $D_0[a,\Omega]$ to u_0, which is a unique solution of the problem

$$\begin{gathered}\int\limits_X \alpha(u,v)(dx) + \int\limits_X uv\,\mu(dx) = \langle f, v\rangle_{(D[a])',D[a]}, \qquad u \in D[a] \cap L^2(X,\mu),\\ \text{for all } v \in D[a] \cap L^2(X,\mu).\end{gathered} \tag{$2.1'_0$}$$

Theorem 2.1 will be proved if we prove the following result.

Theorem 2.2. *The set* $\mathcal{M}^+$ *of positive Borel measures on* X *that do not charge sets of zero capacity is compact under* Γ*-convergence.*

For the proof of this result in the Euclidean case, we refer the reader to [14, 16], and this proof can be carried over to the general case (see [23]). We also observe that Theorem 2.2 implies that the Borel measure μ in Theorem 2.1 does not change if we enlarge Ω preserving the holes. Moreover, the following result was proved in [6].

Theorem 2.3. *The set of Borel measures defined by holes, i.e., defined as in* (2.2) *and* (2.3), *is dense for the* Γ*-convergence in* $\mathcal{M}^+$.

As in the Euclidean case [16], it is possible to characterize the points at which a measure in $\mathcal{M}^+$ represents some holes in the Riemannian case as well [2].

Definition 2.2. Let μ be in $\mathcal{M}^+$. A point x_0 is called a *regular point* for μ if any solution of the problem

$$\begin{gathered}\int\limits_\Omega \alpha(u,v)(dx) + \int\limits_\Omega uv\,\mu(dx) = \langle f, v\rangle_{(D[a,\Omega])',D[a,\Omega]}, \qquad u \in D_{\mathrm{loc}}[a,\Omega] \cap L^2(\Omega,\mu),\\ \text{for all } v \in D[a] \cap L^2(\Omega,\mu) \text{ with compact support in } \Omega,\end{gathered} \tag{2.4}$$

where Ω is any relatively compact neighborhood of x_0, assumes the value 0 with continuity at x_0.

It is possible to characterize a regular point for μ by a Wiener-type criterion using a suitable notion of capacity.

Definition 2.3. Let μ be in $\mathcal{M}^+$, E be a Borel set contained in the open set Ω, and μ_E be the restriction of μ to E. The set E is *μ-admissible* (in Ω) if there exists $w \in D[a,\Omega]$ such that $(w-1) \in L^2(X,\mu_E)$; if E is μ-admissible, we define

$$\operatorname{cap}_\mu(E,\Omega) = \min\left\{a(v,v) + \int (v-1)\mu_E(dx);\ v \in D[a,\Omega]\right\}.$$

Let

$$\delta(r,x_0)=\frac{\mathrm{cap}_\mu(B(x_0,r),B(x_0,k^\star r))}{\mathrm{cap}(B(x_0,r),B(x_0,k^\star r))}.$$

The following result was proved in [2].

Theorem 2.4. *A point x_0 is a regular point for $\mu\in\mathcal{M}^+$ if and only if x_0 is a Wiener point for μ, i.e.,*

$$\lim_{r\to0}\int\limits_r^1\delta(\rho,x_0)\frac{d\rho}{\rho}=+\infty.$$

Remark 2.1. If, in the Riemannian case, the energy density $\alpha(u,u)(dx)$ is absolutely continuous with respect to m, i.e., $\alpha(u,u)(dx)=\alpha(u,u)(x)m(dx)$, where $\alpha(u,u)(\cdot)\in L^1(X,m)$, for every $u\in D[a]$, and there exists n bounded linear operators $L_i\colon D[a]\to L^2(X,m)$, $i=1,\dots,n$, such that

$$\lambda\sum_{i=1}^n|L_iu(x)|^2\le\alpha(u,u)(x)\le\Lambda\sum_{i=1}^n|L_iu(x)|^2\qquad\text{a.e. in }X,$$

then we can define $D_0^p[a,\Omega]$, $p>1$, as the closure of $D[a]\cap C_0(\Omega)$ with respect to the norm

$$\|u\|_p=\left(\int\limits_\Omega\sum_{i=1}^n|L_iu(x)|^pm(dx)+\int\limits_\Omega|u|^pm(dx)\right)^{1/p},$$

where Ω is an open set. The space $D_0^p[a,\Omega]$ is a uniformly convex Banach space, and it can be proved that if Ω is relatively compact, the solutions of (2.1_ϵ) converge strongly (at least after extraction of subsequences) in $D_0^p[a,\Omega]$ for $1<p<2$ (see [5]).

3. THE CASE OF THE HOMOGENEOUS NEUMANN BOUNDARY CONDITION

In this section, we outline the proofs of the results obtained by N.A. Tchou and the present author.[2] We consider a denumerable covering $\mathcal{B}_\epsilon$ of the space obtained by balls of radius $(1+\delta)\epsilon$ ($\epsilon>0$ and $\delta>0$ are fixed) such that the homothetic balls of radius ϵ do not intersect. We also assume that $\mathcal{B}_\epsilon$ has a property of finite intersection that is uniform with respect to ϵ (i.e., every point of X belongs to at most Q balls in the covering, where Q does not depend on ϵ).

We denote by $B_{i,(1+\delta)\epsilon}$, $i=1,2,\dots$, the balls in $\mathcal{B}_\epsilon$ and by $B_{i,\epsilon}$, $i=1,2,\dots$, the homothetic balls of radius ϵ. Consider now a relatively compact open set Ω with boundary Γ and denote by $B_{j,\epsilon}$, $j=1,2,\dots,q$, a subfamily of the balls $B_{i,\epsilon}$ such that $B_{i,\epsilon(1+\delta)}$ is contained in Ω (the number of the balls $B_{j,\epsilon}$ is finite due to the homogeneous structure of X). In every ball $B_{j,\epsilon(1-\delta)}$, we consider a compact set $T_{j,\epsilon}$. Denote $\Omega_\epsilon=\Omega-\bigcup_jT_{j,\epsilon}$. Let θ_ϵ be the characteristic function of Ω_ϵ. Assume that θ_ϵ converges in the weak* topology of $L^\infty(\Omega,m)$ to a function θ (this property always holds at least after extraction of subsequences) with $0<\sigma<\theta\le1$. Consider a Riemannian-type form $a(u,v)=\int\alpha(u,v)(dx)$ on X with a domain $D[a]$; denote by $D_0[a,\Omega]$ the domain of the restriction of the form to Ω and by $V_\epsilon(\Omega)$ the closure of the space of the functions v in $C_0(\Omega)\cap D_{\mathrm{loc}}[a,\Omega]$ with respect to the norm

$$\|v\|_\epsilon=\left[\int\limits_{\Omega_\epsilon}\alpha(v,v)(dx)+\int\limits_{\Omega_\epsilon}|v|^2\,m(dx)\right]^{1/2}.$$

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[2] *Biroli M., Tchou N.A.* In preparation.

We assume that the following scaled Poincaré inequality holds:

$$\int\limits_{B_{j,\epsilon}-T_{j,\epsilon}} |v-\overline{v}_{j,\epsilon}|^2\, m(dx) \le C_2\epsilon^2 \int\limits_{B_{j,\epsilon(1+\delta)}-T_{j,\epsilon}} \alpha(v,v)(dx) \tag{P$_1$}$$

for every $v \in V_\epsilon(\Omega)$, where C_2 is a constant independent of j and ϵ, and $\overline{v}_{j,\epsilon}$ is the average of v for the measure $m(dx)$ on $B_{j,\epsilon}-B_{j,\epsilon(1-\delta)}$. From (P$_1$), we can derive the following coercivity inequality.

Proposition 3.1. *There exists $\epsilon_0 > 0$ such that, for $\epsilon \le \epsilon_0$, we have*

$$\int\limits_{\Omega_\epsilon} \alpha(v,v)(dx) \ge \lambda \|v\|^2_{L^2(\Omega_\epsilon)},$$

where $v \in V_\epsilon(\Omega)$ and λ is a positive constant.

Now let $\eta_{j,\epsilon}$ be a cut-off function of $B_{j,\epsilon(1-\delta)}$ with respect to $B_{j,\epsilon}$, and define a linear operator $P^\epsilon\colon V_\epsilon \to D_0[a,\Omega]$ as

$$P^\epsilon v = \Biggl(1-\sum_j \eta_{j,\epsilon}\Biggr)v + \sum_j \eta_{j,\epsilon}\overline{v}_{j,\epsilon} \quad \text{on } \Omega_\epsilon, \qquad P^\epsilon v = \sum_j \eta_{j,\epsilon}\overline{v}_{j,\epsilon} \quad \text{on } \Omega-\Omega_\epsilon.$$

Proposition 3.2. *The linear operator P^ϵ is bounded. Moreover, denoting by $\widetilde{v}_\epsilon$ the extension of v by $\overline{v}_{j,\epsilon}$ to $T_{j,\epsilon}$ (thus, $\widetilde{v}_\epsilon$ is defined on Ω), we find that $\lim_{\epsilon\to 0}(P^\epsilon v - \widetilde{v}_\epsilon) = 0$ in $L^2(\Omega,m)$. Consider now a sequence v_ϵ such that $\|v_\epsilon\|_\epsilon$ is bounded; then there exists a subsequence, denoted again by v_ϵ, such that $\lim_{\epsilon\to 0} P^\epsilon v_\epsilon = w$ weakly in $D_0[a,\Omega]$ and strongly in $L^2(\Omega,m)$ and $\lim_{\epsilon\to 0}(P^\epsilon v_\epsilon - (\widetilde{v}_\epsilon)_\epsilon) = 0$ in $L^2(\Omega,m)$ (equivalently, $\|P^\epsilon v_\epsilon - v_\epsilon\|_{L^2(\Omega_\epsilon,m)}$ converges to 0); more exactly,*

$$\|P^\epsilon v_\epsilon - (\widetilde{v}_\epsilon)_\epsilon\|^2_{L^2(\Omega)} \le C\epsilon^2 \int\limits_{\Omega_\epsilon} \alpha(v_\epsilon,v_\epsilon)(dx), \qquad \|P^\epsilon v_\epsilon - v_\epsilon\|^2_{L^2(\Omega_\epsilon)} \le C\epsilon^2 \int\limits_{\Omega_\epsilon} \alpha(v_\epsilon,v_\epsilon)(dx),$$

$(\widetilde{v}_\epsilon)_\epsilon$ converges to w in $L^2(\Omega)$, and $\|v_\epsilon - w\|_{L^2(\Omega_\epsilon,m)}$ converges to 0. Moreover, if v is a function in $D[a,\Omega]$, then $P^\epsilon v - v$ converges to 0 in $L^2(\Omega,m)$.

The proof is based on the Poincaré inequality (P$_1$) and the compact embedding of $D[a,\Omega]$ into $L^2(X,m)$ [8].

In what follows, we will use P^ϵ as an approximate extension operator.

Now consider the following problems:

$$\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon,v)(dx) = \langle f, P^\epsilon v\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]}, \qquad u_\epsilon \in V_\epsilon(\Omega), \qquad \forall v \in V_\epsilon(\Omega). \tag{3.1$_\epsilon$}$$

These problems have a unique solution u_ϵ. Choosing $v = u_\epsilon$, we obtain

$$\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) \le C,$$

where C is a constant independent of ϵ. We define a linear operator $B_\epsilon\colon (D_0[a,\Omega])' \to D_0[a,\Omega]$ as

$$B_\epsilon f = P^\epsilon u_\epsilon.$$

It is easy to prove that the operators B_ϵ are uniformly bounded and, at least after extraction of a subsequence denoted again by B_ϵ, weakly converge at every $f \in (D_0[a,\Omega])'$ to a linear operator $B_0\colon (D_0[a,\Omega])' \to D_0[a,\Omega]$ (i.e., $\lim_{\epsilon\to 0} B_\epsilon u = B_0 f$ weakly in $D_0[a,\Omega]$ for every fixed $f \in (D_0[a,\Omega])'$). From Proposition 3.2, it follows that $P^\epsilon u_\epsilon$ is bounded in $D_0[a,\Omega]$; then (taking into account that B_ϵ weakly converges to B_0) we find that

$$\lim_{\epsilon\to 0} P^\epsilon u_\epsilon = u_0$$

weakly in $D[a,\Omega]$ and strongly in $L^2(\Omega,m)$. Then the operator $B_0\colon (D_0[a,\Omega])' \to D_0[a,\Omega]$ is defined as

$$B_0 f = u_0.$$

First, we observe that for $f, g \in (D_0[a,\Omega])'$,

$$\langle g, B_0 f\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]} = \langle f, B_0 g\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]}$$

and

$$\begin{aligned}\langle f, B_0 f\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]} &= \langle f, u_0\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]} = \lim_{\epsilon\to 0}\langle f, P^\epsilon u_\epsilon\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]}\\ &= \lim_{\epsilon\to 0}\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) \ge C\limsup_{\epsilon\to 0}\|P^\epsilon u_\epsilon\|^2_{D_0[a,\Omega]}\\ &\ge C\|u_0\|^2_{D_0[a,\Omega]} \ge C\|B_0 f\|^2_{D_0[a,\Omega]}.\end{aligned} \tag{3.2}$$

Now we prove that B_0 is coercive. We have

$$\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon, v)(dx) = \int\limits_{\Omega} \alpha(A^{-1}f, P^\epsilon v)(dx),$$

where A is the operator defined by the form a. Choosing v as the restriction of $A^{-1}f$ to Ω_ϵ, we obtain

$$\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon, A^{-1}f)(dx) = \int\limits_{\Omega} \alpha(A^{-1}f, P^\epsilon A^{-1}f)(dx);$$

then

$$\frac{1}{2}\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) + \frac{1}{2}\int\limits_{\Omega_\epsilon} \alpha(A^{-1}f, A^{-1}f)(dx) \ge \int\limits_{\Omega} \alpha(A^{-1}f, P^\epsilon A^{-1}f)(dx).$$

In the limit as $\epsilon \to 0$, we obtain

$$\lim_{\epsilon\to 0}\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) \ge \int\limits_{\Omega} \alpha(A^{-1}f, A^{-1}f)(dx) = \|f\|_{(D_0[a,\Omega])'};$$

then, using the first two lines in (3.2), we arrive at

$$\langle f, B_0 f\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]} \ge \|f\|_{(D_0[a,\Omega])'}.$$

Taking into account the previous properties, we find that the operator B_0 is invertible and denote by $A_0\colon D_0[a,\Omega] \to (D_0[a,\Omega])'$ the operator B_0^{-1}. We can prove that A_0 is defined on the whole

$D_0[a,\Omega]$, is bounded and coercive from $D_0[a,\Omega]$ to $(D_0[a,\Omega])'$, and is such that

$$\lambda\langle Au,u\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} \le \langle A_0u,u\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} \le \Lambda\langle Au,u\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]};$$

so A_0 defines a bilinear form $a_0(u,v)$ on $D_0[a,\Omega]\times D_0[a,\Omega]$, which is positive and satisfies

$$\lambda a(u,u) \le a_0(u,u) \le \Lambda a(u,u)$$

for every $u\in D_0[a,\Omega]$. The above domination inequality implies that the bilinear form a_0 is regular ($D_0[a_0,\Omega]$ and $D_0[a,\Omega]$ coincide as sets, and their norms are equivalent). Finally, we observe that $u_0 = B_0 f$ is a unique solution of the problem

$$a_0(u_0,v) = \langle f,v\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} \qquad \text{for every } \ v\in D_0[a,\Omega]=D_0[a_0,\Omega].$$

Now we will prove that a_0 is a Dirichlet form. To this end, it suffices to prove that a Markov-type property holds for a_0. The proof of the Markov property essentially uses the following semicontinuity property.

Proposition 3.3. *Let $v_\epsilon\in V_\epsilon$ be a sequence such that*

$$\int\limits_{\Omega_\epsilon} \alpha(v_\epsilon,v_\epsilon)(dx) \le C$$

and $P^\epsilon v_\epsilon$ converges to v_0 as $\epsilon\to 0$ weakly in $D_0[a,\Omega]$. Then

$$\liminf_{\epsilon\to 0}\int\limits_{\Omega_\epsilon} \alpha(v_\epsilon,v_\epsilon)(dx) \ge a_0(v_0,v_0).$$

Now we assume that a is such that $\alpha(u,u)(dx)=\alpha(u,u)(x)m(dx)$, where $\alpha(u,u)(\cdot)\in L^1(\Omega,m)$, for all $u\in D_0[a,\Omega]$.

As in [24], we can also prove that a_0 is strongly local, and we denote by $\alpha_0(u,u)(dx)$ its energy density. As a consequence of the boundedness and coercivity of a_0 with respect to a, we obtain [24]

$$\lambda\alpha(u,u)(dx) \le \alpha_0(u,u)(dx) \le \Lambda\alpha(u,u)(dx);$$

then $\alpha_0(u,u)(dx)=\alpha_0(u,u)(x)m(dx)$, where $\alpha_0(u,u)(\cdot)\in L^1(\Omega,m)$, for all $u\in D_0[a,\Omega]$. Moreover, if a is a Riemannian form, a_0 is also a Riemannian form with the same domain as a with respect to a distance of the form $\Lambda^{-1}d$, where d is the distance associated with a. Thus, we have the following result.

Theorem 3.4. *Let u_ϵ be a sequence of solutions of problems (3.1_ϵ) such that $P^\epsilon u_\epsilon$ strongly converges in $L^2(\Omega,m)$ to u_0; then u_0 is a solution of the problem*

$$\int\limits_\Omega \alpha_0(u,v)(x)\,dx = \langle f,u_0\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]}, \qquad u_0\in D[a,\Omega], \qquad \forall v\in D[a,\Omega], \tag{3.1_0}$$

where $a_0(u,v)=\int_\Omega \alpha(u,v)(x)\,dx$ is a Riemannian-type Dirichlet form.

We observe that the convergence result in Theorem 3.4 holds again if, in problems (3.1_ϵ), we replace f by a sequence f_ϵ converging to f in $(D[a,\Omega])'$; then we obtain

Theorem 3.5. *Let u_ϵ be solutions of the problems*

$$\int\limits_{\Omega_\epsilon} \alpha(u_\epsilon, v)(dx) = \int\limits_{\Omega_\epsilon} fv\, m(dx), \qquad u_\epsilon \in V_\epsilon(\Omega), \qquad \forall v \in V_\epsilon(\Omega), \tag{$3.1'_\epsilon$}$$

where $f \in L^2(\Omega, m)$, and assume that $P^\epsilon u_\epsilon$ strongly converges to u_0 in $L^2(\Omega, m)$. Then u_0 is a solution of the problem

$$\int\limits_{\Omega} \alpha_0(u, v)(x)\, dx = \int\limits_{\Omega} \theta f u_0\, m(dx), \qquad u_0 \in D[a, \Omega], \qquad \forall v \in D[a, \Omega], \tag{$3.1'_0$}$$

where α_0 is as in Theorem 3.4 *and θ is the function defined at the beginning of the section.*

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