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In memoriam of M. Z. Solomyak

**A COMPARISON THEOREM FOR SUPER- AND
SUBSOLUTIONS OF $\nabla^2 u + f(u) = 0$ AND ITS
APPLICATION TO WATER WAVES WITH VORTICITY**

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A comparison theorem is proved for a pair of solutions that satisfy opposite nonlinear differential inequalities in a weak sense. The nonlinearity is of the form $f(u)$ with f belonging to the class L^p_{loc} and the solutions are assumed to have nonvanishing gradients in the domain, where the inequalities are considered. The comparison theorem is applied to the problem describing steady, periodic water waves with vorticity in the case of arbitrary free-surface profiles including overhanging ones. Bounds for these profiles as well as streamfunctions and admissible values of the total head are obtained.

§1. Introduction

In their remarkable article [7], Gidas, Ni and Nirenberg investigated various properties that solutions (in particular, positive solutions) of several nonlinear equations have in different (bounded as well as unbounded) domains in \mathbb{R}^n , $n \geq 1$. For this purpose several forms of the maximum principle were employed along with some other methods. The authors emphasized that their techniques could be applicable in physical situations other than those considered in the paper. During the decades past since the publication of [7], this prediction proved correct. The most spectacular results obtained in the paper deal with the equation

$$\nabla^2 u + f(u) = 0, \quad \nabla u = (u_{x_1}, \dots, u_{x_n}) \text{ and } u_{x_i} = \partial_i u = \partial u / \partial x_i, \quad (1)$$

in which f is a C^1 -function.

Key words: comparison theorem, nonlinear differential inequality, partial hodograph transform in n dimensions, periodic steady water waves with vorticity, streamfunction.

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The two-dimensional version of (1) describes, in particular, periodic steady water waves with vorticity in which case f is the given vorticity distribution. If the depth of water is finite, the domain is a quadrangle bounded by three straight segments — two of them that are opposite to each other are equal in view of periodicity — and a curve that is opposite to the third segment and corresponds to the smallest period of wave propagating on the free surface; of course, one can consider a strip with periodic upper boundary and horizontal bottom as the water domain. (The relevant free-boundary problem is derived from Euler’s equations; see, for example, [3].)

Instead of equation (1), the present paper deals with the inequality

$$\nabla^2 u + f(u) \leq 0 \quad \text{in a domain } \Omega \subset \mathbb{R}^n, \quad n \geq 2, \tag{2}$$

and its opposite which are understood in a weak sense. In the case of (2), this means that the integral inequality

$$\int_X \nabla u \cdot \nabla v \, dx \geq \int_X f(u)v \, dx \tag{3}$$

is valid for every nonnegative $v \in C_0^1(X)$, where X is an arbitrary subdomain of Ω , and so no smoothness and even continuity is required from f . Our aim is to prove the following comparison theorem for a pair of functions that satisfy (2) and the opposite inequality.

Theorem 1. *Let $f \in L_{loc}^p(\mathbb{R})$ with $p > n$, and let $u_1, u_2 \in C^1(\Omega)$ have nonvanishing gradients in Ω and satisfy (in the weak sense) the inequalities*

$$\nabla^2 u_1 + f(u_1) \geq 0 \quad \text{and} \quad \nabla^2 u_2 + f(u_2) \leq 0 \quad \text{in } \Omega, \tag{4}$$

respectively. If $u_1 \leq u_2$ in Ω and these functions are equal at some point $x^0 \in \Omega$, then u_1 and u_2 coincide throughout Ω .

Remark 1. If $f \in L_{loc}^p(\mathbb{R})$ and $u \in C^1(\Omega)$ is such that $\nabla u \neq 0$ throughout Ω , then $f(u(x))$ is a measurable function in Ω ; moreover, this superposition belongs to $L_{loc}^p(\Omega)$.

It should be mentioned that Theorem 1 is not true without the assumption that the gradients of u_1 and u_2 are nonvanishing. Indeed, even for the Hölder continuous f (the weaker condition $f \in L_{loc}^p(\mathbb{R})$, $p > n$, is imposed in Theorem 1), this follows from the example in [7, p. 220].

Namely, let $\Omega = \mathbb{R}^n$, and let u_1 be equal to zero identically. If $q > 2$, then u_2 equal to $(1 - |x|^2)^q$ when $|x| \leq 1$ and to zero for $|x| > 1$ belongs to $C^2(\mathbb{R}^n)$. It is straightforward to check that (1) holds true for u_2 with

$$f(u) = -2q(q - 2)u^{1-2/q} + 2q(n + 2q - 2)u^{1-1/q},$$

which is Hölder continuous with the exponent $1 - 2/q$ and such that $f(0) = 0$. Thus, all assumptions of Theorem 1 are fulfilled for u_1 and u_2 except for the condition concerning their gradients; for both functions they vanish when $|x| \geq 1$. Therefore, the conclusion of Theorem 1 is not true — these functions do not coincide.

The proof of Theorem 1 given in §2 is based on the so-called partial hodograph transform in n dimensions which allows us to use the weak Harnack type inequality obtained in [21]. In §3, we apply Theorem 1 to prove bounds for solutions of the free-boundary problem mentioned above; it describes steady, periodic water waves with vorticity.

§2. Proof of Theorem 1

First, the local version of the n -dimensional partial hodograph transform is introduced. It is used in the proof of an auxiliary lemma required for proving Theorem 1. Then a version of Hopf's lemma is discussed; the latter is applied in considerations of §3.

2.1. The partial hodograph transform in n dimensions. Being defined locally, it generalises the transform introduced by Dubreil–Jacotin [6] in her studies of water waves with vorticity in the two-dimensional case. Moreover, the transform proposed here extends that considered in [6] to the case of $n > 2$ dimensions. Therefore, it might be of interest for applications other than water waves with vorticity.

Let $\Omega \subset \mathbb{R}^n$ be a domain. If u is a function whose gradient does not vanish throughout this domain, then at any point $x^0 \in \Omega$ a coordinate system can be chosen so that $u_{x_n}(x^0) > 0$. This allows us to introduce the following transform in a neighborhood of x^0 . We put

$$q = (q_1, \dots, q_{n-1}) \quad \text{with} \quad q_k = x_k, \quad k = 1, \dots, n-1, \quad \text{and} \quad p = u(x),$$

and take these as new independent variables. Furthermore, instead of $u(x)$ satisfying, say inequality (2), we consider $h(q, p) = x_n$ as the unknown. Then we have

$$\frac{\partial h}{\partial x_k} = h_{q_k} + h_p \frac{\partial u}{\partial x_k} = 0 \quad \text{for} \quad k = 1, \dots, n-1 \quad \text{and} \quad \frac{\partial h}{\partial x_n} = h_p u_{x_n} = 1,$$

and so

$$h_{q_k} = -\frac{u_{x_k}}{u_{x_n}} \quad \text{for} \quad k = 1, \dots, n-1 \quad \text{and} \quad h_p = \frac{1}{u_{x_n}} > 0.$$

In view of the identities

$$u_{x_k} = -\frac{h_{q_k}}{h_p}, \quad k = 1, \dots, n-1, \quad \text{and} \quad u_{x_n} = \frac{1}{h_p},$$

the weak formulation (3) of inequality (2) for u takes the following form in terms of h :

$$\int_Q \left[-\nabla_q h \cdot \nabla_q w + \frac{1 + |\nabla_q h|^2}{h_p} w_p \right] dq dp \leq \int_Q f(p) w h_p dq dp. \tag{5}$$

Here Q — a neighborhood of the point (q^0, p^0) — is the image of X which is the neighborhood of x^0 that corresponds to (q^0, p^0) and $w(q, p)$ stands instead of $v(x(q, p))$. It is also taken into account that

$$dx = h_p dq dp \quad \text{and} \quad \nabla_q h = (h_{q_1}, \dots, h_{q_{n-1}}).$$

Like (3), the last inequality must be true for every nonnegative $w \in C_0^1(Q)$.

In the case of smooth h , a consequence of (5) is the differential inequality

$$(Lh)(q, p) \geq f(p) h_p(q, p), \quad \text{where} \quad Lh = \partial_{q_k} h_{q_k} - \partial_p \frac{1 + |\nabla_q h|^2}{h_p}.$$

Here and below, the Einstein summation notation is used. It should be noted that the right-hand side of this inequality depending on f is linear in h , whereas nonlinearity is present in the operator L on the left-hand side. It will be clear from what follows that this consequence of the introduced form of the hodograph transform is advantageous in some cases. Moreover, this differs from what we have in (2), where the differential operator is linear and nonlinearity is involved through the superposition operator $f(u)$.

Furthermore, let $h_1, h_2 \in C^1(Q)$, then we have

$$Lh_1 - Lh_2 = \partial_{q_k} (h_1 - h_2)_{q_k} - \partial_p \int_0^1 \frac{1 + |\nabla_q h^{(t)}|^2}{h_p^{(t)}} dt,$$

where $h^{(t)} = th_1 + (1 - t)h_2$. This difference can be written as an operator in divergent form with continuous coefficients. Indeed, let $W = h_1 - h_2$, then $Lh_1 - Lh_2 = \mathcal{L}W$, where

$$\mathcal{L}W = \partial_{q_k} W_{q_k} - 2 \partial_p \left[W_{q_k} \int_0^1 \frac{h_{q_k}^{(t)}}{h_p^{(t)}} dt \right] + \partial_p \left[W_p \int_0^1 \frac{1 + |\nabla_q h^{(t)}|^2}{h_p^{(t)2}} dt \right].$$

Moreover, the inequalities

$$\sum_{k=1}^{n-1} \left[\int_0^1 \frac{h_{q_k}^{(t)}}{h_p^{(t)}} dt \right]^2 \leq \sum_{k=1}^{n-1} \int_0^1 \frac{h_{q_k}^{(t)2}}{h_p^{(t)2}} dt < \int_0^1 \frac{1 + |\nabla_q h^{(t)}|^2}{h_p^{(t)2}} dt$$

show that \mathcal{L} is an elliptic operator.

2.2. Auxiliary lemma. Its assertion is an immediate corollary to Theorem 5.1 in [21] (see also [8, Theorem 8.18], where the coefficients were assumed to be bounded which is superfluous), and we give it for the reader's convenience. The lemma concerns the linear elliptic operator

$$\mathcal{P}(\partial) = \partial_j(a_{ij}\partial_i) + b_j\partial_j + c$$

whose coefficients are measurable functions in $\Omega \subset \mathbb{R}^n$; as usual, the ellipticity means that there exists $\lambda > 0$ such that

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \quad \text{for all } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \quad x \in \Omega.$$

Lemma 1. *Let $u \in C^1(\Omega)$ be a nonnegative function satisfying the inequality*

$$\mathcal{P}(\partial)u \leq 0 \quad \text{in } \Omega \tag{6}$$

in the weak sense; here all coefficients b_j belong to $L^q(\Omega)$ and $c \in L^{q/2}(\Omega)$ for some $q > n$. If there exists $x^0 \in \Omega$ such that $u(x^0) = 0$, then u vanishes identically in Ω .

Proof. Let $\rho > 0$ be such that $K_{3\rho}(x^0) \subset \Omega$; by $K_\sigma(x)$ we denote the open cube centred at x and having edges equal to σ and sides parallel to the coordinate axes. By [21, Theorem 5.1], the assumptions about u and \mathcal{P} yield the inequality

$$\rho^{-n/\gamma} \|u\|_{L^\gamma(K_{2\rho}(x^0))} \leq C \min_{x \in K_\rho(x^0)} u(x)$$

for u satisfying (6); here C is a positive constant and γ is an arbitrary number from $(1, n/(n-2))$. Since

$$\min_{x \in K_\rho(x^0)} u(x) = u(x^0) = 0,$$

there exists a neighborhood of x^0 where u vanishes identically. It is clear that the maximal neighborhood with this property is Ω because otherwise the same argument can be applied to any boundary point of the maximal neighborhood that is an interior point of Ω , thus leading to a contradiction. \square

Remark 2. The assertion of Lemma 1 remains valid for a nonpositive u satisfying the inequality $\mathcal{P}(\partial)u \geq 0$.

In the case when $\mathcal{P}(\partial)u = 0$ in Lemma 1, the assumptions on the coefficients b_j and c can be relaxed for $n \geq 3$ by using the local Kato class $K_n^{\text{loc}}(\Omega)$ instead of $L^q(\Omega)$; see [15].

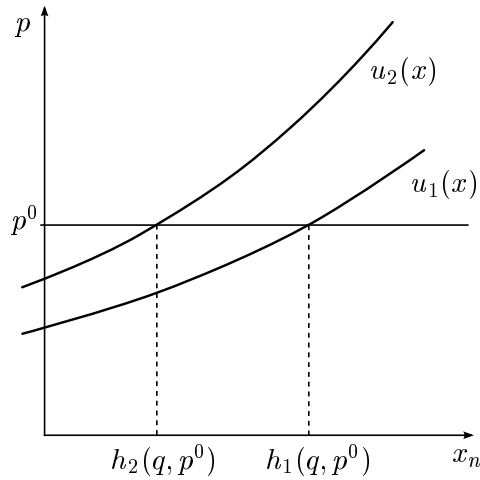


Fig. 1. A sketch of the partial hodograph transform with fixed $x_1 = q_1, \dots, x_{n-1} = q_{n-1}$. It demonstrates the relationship between inequalities for two pairs of corresponding functions. If $u_1(x) < u_2(x)$ near $(x_1, \dots, x_{n-1}, x_n^0)$, then $h_1(q, p) > h_2(q, p)$ in a neighborhood of (q, p^0) .

2.3. Proof of Theorem 1. We apply the partial hodograph transform to u_1 and u_2 in a neighborhood of x^0 . Then there exists a neighborhood Q of (q^0, p^0) — the image of x^0 — in which the inequalities

$$(-1)^j [(Lh_j)(q, p) - f(p) \partial_p h_j(q, p)] \geq 0, \quad j = 1, 2, \tag{7}$$

follow from (2); here h_1 and h_2 are the functions corresponding to u_1 and u_2 , respectively. It is clear that $h_1(q^0, p^0) = h_2(q^0, p^0)$, and Q can be taken so that $h_1 \geq h_2$ in it because $u_1 \leq u_2$ in Ω (see Fig. 1).

From (7) one deduces that $W = h_1 - h_2$ (it is nonnegative in Q) satisfies the inequality

$$(\mathcal{L}W)(q, p) - f(p)W(q, p) \leq 0 \quad \text{for } (q, p) \in Q.$$

Then Lemma 1 shows that W vanishes identically in Q , that is, $h_1 = h_2$ throughout Q , and so $u_1 = u_2$ in some neighborhood of x^0 . Thus the set, say E , where u_1 coincides with u_2 , is nonempty. It is clear that E is closed in Ω , and so if $E \neq \Omega$, then there exists $x^* \in \partial E \cap \Omega$. Applying the same considerations, we see that x^* has a neighborhood included in E , which is a contradiction.

2.4. Hopf's lemma. In §3 we will apply the following version of a well-known result whose proof we failed to find in the literature.

Lemma 2. *Let x^0 be a point on a part of $\partial\Omega$ belonging to the class $C^{1,\alpha}$, $\alpha \in (0, 1)$, and let $u \in C^1(\bar{X})$ satisfy (6) in the weak sense in X which is the intersection of a neighborhood of x^0 with Ω , whereas \mathcal{P} is such that all a_{ij} belong to $C^{0,\alpha}(X)$, $b_j \in L^q(X)$ and $c \in L^{q/2}(X)$ for some $q > n$.*

If u is nonnegative in X and $u(x^0) = 0$, then either u vanishes identically in X or $\partial_n u(x^0) < 0$, where ∂_n denotes the normal derivative on $\partial\Omega$ directed to the exterior of Ω .

Proof. The proof is essentially the same as that of Theorem 1.1 in [20], where the assumptions imposed on b_j and c (boundedness and nonnegativity for the last coefficient) are superfluous. Therefore, we restrict ourselves to some necessary remarks and begin with a couple of minor notes. First, it suffices to prove the assertion in the case when $\partial\Omega$ is flat near x_0 , to which the general case reduces by a change of variables. Second, in view of Lemma 1, one has merely to show that the inequality $\partial_n u(x^0) < 0$ is a consequence of the fact that $u > 0$ in X .

The only amendment that needs more detail concerns the proof of Lemma 3.4 in [20], in which ∇w_3 should be estimated as follows (we keep the notation used in [20]):

$$\begin{aligned} |\nabla w_3(x)| &\leq C \|\nabla v\|_{L^\infty} \int_{\Omega_\rho} \frac{|b(y)| + c(y)|x - y|}{|x - y|^{n-1}} dy \\ &\leq C \|\nabla v\|_{L^\infty} (\rho^{\alpha_1} \|b\|_{L^q} + \rho^{\alpha_2} \|c\|_{L^{q/2}}), \end{aligned}$$

where $\alpha_1 = (q - n)/(q - 1)$, $\alpha_2 = 2(q - n)/(q - 2)$. □

Remark 3. For a nonpositive u satisfying the inequality $\mathcal{P}(\partial)u \geq 0$, the assertion of Lemma 2 remains valid provided the conclusion that $\partial_n u(x^0) < 0$ is changed to $\partial_n u(x^0) > 0$.

§3. Application of Theorem 1 to water waves with vorticity

In this section, we consider the two-dimensional nonlinear problem of steady, periodic waves in an open channel occupied by an inviscid, incompressible, heavy fluid, say water. The water motion is assumed to be rotational which, according to observations, is the type of motion commonly occurring in nature. A brief characterization of results obtained earlier for this and other related problems was given in our paper [11]. Further details can be found in the survey article [19] by Strauss; see also the recent papers [5, 13] and [14]. Here, our aim is to apply Theorem 1 in order to generalise conditions guaranteeing the validity of bounds for solutions to the problem that were obtained in [12].

3.1. Statement of the problem. Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. The surface tension is neglected and the pressure is constant on the free surface. Since the water motion is assumed to be two-dimensional and rotational and in view of the water incompressibility, we seek the velocity field in the form $(\psi_y, -\psi_x)$, where the unknown function $\psi(x, y)$ is referred to as the *streamfunction* (see, for example, [16] for details of this model). It is also assumed that the vorticity distribution ω (it was a function of ψ as is explained in [16, §1]) is a prescribed function belonging to $L^p_{\text{loc}}(\mathbb{R})$ with $p > 2$. This assumption is weaker than that in our previous paper [12], where ω was assumed to be a locally Lipschitz function.

Nondimensional variables are chosen so that the constant volume rate of flow per unit span and the constant acceleration due to gravity are scaled to unity in our equations. In appropriate Cartesian coordinates (x, y) , the bottom coincides with the x -axis and gravity acts in the negative y -direction. The frame of reference is taken so that the velocity field is time-independent as well as the unknown free-surface profile. The latter is assumed to be a simple C^1 -curve, say Γ , which is Λ -periodic along the x -axis for some $\Lambda > 0$, but not necessarily representable as the graph of an x -dependent function. (It was found numerically by Vanden-Broeck [22] that there are such overhanging profiles bounding rotational flows with periodic waves, whereas Constantin, Strauss, and Varvaruca [5] recently investigated them rigorously; the relevant figures are presented in those papers.) Thus, the longitudinal section of the water domain is the strip D_Γ that lies between the x -axis and Γ , and $\psi(x, y)$ is assumed to be a Λ -periodic function of x in D_Γ .

Since the surface tension is neglected, the pair (ψ, Γ) must be found from the following free-boundary problem:

$$\psi_{xx} + \psi_{yy} + \omega(\psi) = 0, \quad (x, y) \in D_\Gamma; \quad (8)$$

$$\psi(x, 0) = 0, \quad x \in \mathbb{R}; \quad (9)$$

$$\psi(x, y) = 1, \quad (x, y) \in \Gamma; \quad (10)$$

$$|\nabla\psi(x, y)|^2 + 2y = 3r, \quad (x, y) \in \Gamma. \quad (11)$$

Here r is a constant regarded as the given problem's parameter. Notice that the boundary condition (10) allows us to write relation (11) (Bernoulli's equation) as follows:

$$[\partial_n\psi(x, y)]^2 + 2y = 3r, \quad (x, y) \in \Gamma. \quad (12)$$

Here and below ∂_n denotes the normal derivative on Γ ; the normal $n = (n_x, n_y)$ has unit length and points out of D_Γ .

In this section, we keep the notation adopted in our previous papers, and so ψ and ω stand in (8) instead of u and f , respectively, used in (1) and (2). To give the precise definition how a solution of problem (8)–(11) is understood we need the following set $\Gamma_\psi = \{(x, y) \in D_\Gamma : \nabla\psi(x, y) = 0\}$.

Definition 1. The pair (ψ, Γ) is called a solution of problem (8)–(11) with the vorticity distribution $\omega \in L^p_{\text{loc}}(\mathbb{R})$, $p > 2$, provided the following conditions are fulfilled for some $\Lambda > 0$:

- Γ is a simple C^1 -curve Λ -periodic along the x -axis;
- $\psi(x, y)$ is a Λ -periodic function of x belonging to $C^1(\overline{D_\Gamma})$;
- the boundary conditions (9)–(11) are fulfilled pointwise;
- the two-dimensional measure of Γ_ψ is equal to zero and $D_\Gamma \setminus \Gamma_\psi$ is a domain;
- for all $v \in C_0^\infty(D_\Gamma \setminus \Gamma_\psi)$ we have the identity

$$\int_{D_\Gamma} \nabla\psi \cdot \nabla v \, dx dy = \int_{D_\Gamma} \omega(\psi)v \, dx dy.$$

The last condition means that ψ is a weak solution of (8) in $D_\Gamma \setminus \Gamma_\psi$.

3.2. Auxiliary one-dimensional problems. The results presented in this section were obtained in [10] under the assumption that ω is a Lipschitz function. Since they are essential for our considerations, what follows is a digest of these results valid under the assumption that $\omega \in L^1_{\text{loc}}(\mathbb{R})$.

First, let $s > 0$, then by $U(y; s)$ we denote a strictly monotonic solution of the following Cauchy problem:

$$U'' + \omega(U) = 0, \quad y \in \mathbb{R}; \quad U(0; s) = 0, \quad U'(0; s) = s;$$

here and below $'$ stands for d/dy . It is straightforward to obtain the implicit formula

$$y = \int_0^U \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}}, \quad \Omega(\tau) = \int_0^\tau \omega(t) \, dt, \quad (13)$$

which defines U on the maximal interval of monotonicity $(y_-(s), y_+(s))$, where

$$y_\pm(s) = \int_0^{\tau_\pm(s)} \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}},$$

and the definition of $\tau_\pm(s)$ is as follows. By $\tau_+(s)$ and $\tau_-(s)$ we denote the least positive and the largest negative root, respectively, of the equation $2\Omega(\tau) = s^2$. If this equation has no positive (negative) root, we put

$$\tau_+(s) = +\infty \quad (\tau_-(s) = -\infty, \text{ respectively}).$$

Second, we consider the problem

$$u'' + \omega(u) = 0 \quad \text{on } (0, h), \quad u(0) = 0, \quad u(h) = 1, \quad (14)$$

in the class of monotonic functions. It is clear that formula (13) gives a solution of problem (14) on the interval $(0, h(s))$, where

$$h(s) = \int_0^1 \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}} \quad \text{and} \quad s > s_0 = \sqrt{2 \max_{\tau \in [0,1]} \Omega(\tau)} \geq 0. \quad (15)$$

Moreover, all monotonic solutions of problem (14) have the form (13) on the interval $(0, h)$. This remains valid for $s = s_0$ with

$$h = h_0 = \int_0^1 \frac{d\tau}{\sqrt{s_0^2 - 2\Omega(\tau)}} < \infty, \quad \text{that is, } h_0 = \lim_{s \rightarrow s_0} h(s).$$

It is clear that $h(s)$ decreases strictly monotonically from h_0 and asymptotes zero as $s \rightarrow +\infty$.

Furthermore, the pair (u, Γ) with

$$u(y) = U(y; s) \quad \text{and} \quad \Gamma = \{(x, y) : x \in \mathbb{R}, y = h(s)\}$$

is a solution of problem (8)–(11) provided s is found from the equation

$$\mathcal{R}(s) = r, \quad \text{where } \mathcal{R}(s) = [s^2 - 2\Omega(1) + 2h(s)]/3. \quad (16)$$

The last function has only one minimum, say $r_c > 0$, attained at some $s_c > s_0$. Hence if $r \in (r_c, r_0)$, where

$$r_0 = \lim_{s \rightarrow s_0+0} \mathcal{R}(s) = \frac{1}{3} [s_0^2 - 2\Omega(1) + 2h_0],$$

then equation (16) has two solutions s_+ and s_- such that $s_0 < s_+ < s_c < s_-$. By substituting s_+ and s_- in (13) and (15), one obtains the so-called *stream solutions* (u_+, H_+) and (u_-, H_-) , respectively. Indeed, these solutions satisfy Bernoulli's equation

$$[u'_\pm(H_\pm)]^2 + 2H_\pm = 3r$$

along with relations (14). It should be mentioned that s_- and the corresponding H_- exist for all values of r greater than r_c , whereas s_+ and H_+ exist only when r is less than or equal to r_0 ; in the last case $s_+ = s_0$.

3.3. Bounds for (ψ, Γ) . To express bounds for nonstream solutions of problem (8)–(11), we use solutions of problem (14) and the values r_c , H_- , and H_+ ; the last two serve as bounds for

$$\widehat{\Gamma} = \max_{(x,y) \in \Gamma} y \quad \text{and} \quad \check{\Gamma} = \min_{(x,y) \in \Gamma} y.$$

Now we formulate results generalising Theorems 1.1 and 1.2 in [12] for periodic solutions.

Theorem 2. *Let (ψ, Γ) be a nonstream solution of problem (8)–(11) in the sense of Definition 1. Then the following two assertions are true provided $\psi \leq 1$ on $\overline{D_\Gamma}$.*

1. *If $\check{\Gamma} < h_0$, then*

$$\psi(x, y) < U(y; \check{s}) \text{ in the strip } \mathbb{R} \times (0, \check{\Gamma}), \quad (17)$$

where U is defined by formula (13) and $\check{s} > s_0$ is such that $h(\check{s}) = \check{\Gamma}$. Moreover, the inequalities (A) $r \geq r_c$, (B) $H_- \leq \check{\Gamma}$ hold, and if $r \leq r_0$, then (C) $\check{\Gamma} \leq H_+$.

2. *If $h_0 \neq +\infty$ and $\check{\Gamma} = h_0$, then inequality (17) is nonstrict, whereas inequalities (A)–(C) are true.*

Theorem 3. *Let (ψ, Γ) be a nonstream solution of problem (8)–(11) in the sense of Definition 1. If $\widehat{\Gamma} < h_0$ and $\psi \geq 0$ on $\overline{D_\Gamma}$, then*

$$\psi(x, y) > U(y; \widehat{s}) \text{ in } D_\Gamma, \quad (18)$$

where U is defined by formula (13) and $\widehat{s} > s_0$ is such that $h(\widehat{s}) = \widehat{\Gamma}$. Moreover, $\widehat{\Gamma} \geq H_+$ provided $r \leq r_0$ and $\psi \leq 1$ on $\overline{D_\Gamma}$.

The following assertions demonstrate that some inequalities in Theorems 2 and 3 are strict under the assumption that the first derivatives of ψ are Hölder continuous near the points where the values $\check{\Gamma}$ and $\widehat{\Gamma}$ are attained.

Proposition 1. *Let (ψ, Γ) be a nonstream solution of problem (8)–(11) in the sense of Definition 1 and such that $\psi \leq 1$ on $\overline{D_\Gamma}$. Also, let Γ be of class $C^{1,\alpha}$, $\alpha \in (0, 1)$, near some point $(x^0, \check{\Gamma}) \in \Gamma$. If $\psi \in C^{1,\alpha}(\check{X})$, where X is the intersection of D_Γ with a sufficiently small neighborhood of $(x^0, \check{\Gamma})$, then the inequalities are strict in (A) and (B), and if $r \leq r_0$, then the inequality in (C) is also strict.*

Proposition 2. *Let (ψ, Γ) be a nonstream solution of problem (8)–(11) in the sense of Definition 1 and such that $\psi \geq 0$ on $\overline{D_\Gamma}$. Also, let Γ be of class $C^{1,\alpha}$, $\alpha \in (0, 1)$, near some point $(x^0, \widehat{\Gamma}) \in \Gamma$. If $\psi \in C^{1,\alpha}(\widehat{X})$, where X is the intersection of D_Γ with a sufficiently small neighborhood of $(x^0, \widehat{\Gamma})$, then $\widehat{\Gamma} > H_+$ provided $r \leq r_0$ and $\psi \leq 1$ on $\overline{D_\Gamma}$.*

3.4. Proof of Theorem 2. First, suppose that $\check{\Gamma} < h_0$, and so there exists $\check{s} > s_0$ such that $h(\check{s}) = \check{\Gamma}$, whereas the function $U(y; \check{s})$ solves problem (14) on $(0, \check{\Gamma})$. Moreover, formula (13) defines $U(y; \check{s})$ on the half-axis $y \geq 0$ provided $\omega(t)$ is extended by -1 for $t > 1$. This implies that $\check{s}^2 > 2 \max_{\tau \geq 0} \Omega(\tau)$, and we have

$$U'(y; \check{s}) = \sqrt{\check{s}^2 - 2\Omega(U(y; \check{s}))} > 0 \quad \text{for } y \geq 0.$$

Hence $U(y; \check{s})$ is a monotonically increasing function of y , $U(y; \check{s}) > 1$ for $y > h_0$ and $U(y; \check{s}) \rightarrow +\infty$ as $y \rightarrow +\infty$.

Putting $U_\ell(y) = U(y + \ell; \check{s})$ for $\ell \geq 0$, we see that $U_\ell(y) > 1$ on $[0, \check{\Gamma}]$ when $\ell > \check{\Gamma}$. Therefore, $U_\ell - \psi > 0$ on \overline{D}_Γ for $\ell > \check{\Gamma}$. Let us show that there is no $\ell_0 \in (0, \check{\Gamma})$ such that

$$\min_{(x,y) \in D_\Gamma} \{U_{\ell_0}(y) - \psi(x, y)\} = 0. \tag{19}$$

Assuming that such a value exists (in the case when there are several such values, we denote by ℓ_0 the largest of them), we see that (19) holds only when

$$U_{\ell_0}(y^0) - \psi(x^0, y^0) = 0 \quad \text{for some } (x^0, y^0) \in \mathbb{R} \times (0, \check{\Gamma})$$

because $U_{\ell_0} - \psi$ is separated from zero on $\overline{D}_\Gamma \setminus [\mathbb{R} \times (0, \check{\Gamma})]$. Moreover, (19) implies that

$$\nabla \psi = \nabla U_{\ell_0} \text{ at } (x^0, y^0), \text{ and so } (x^0, y^0) \in D_\Gamma \setminus \Gamma_\psi.$$

Since $D_\Gamma \setminus \Gamma_\psi$ is a domain, Theorem 1 is applicable in $D_\Gamma \setminus \Gamma_\psi$, which shows that ψ coincides with U_{ℓ_0} there. Hence these functions coincide in D_Γ because Γ_ψ has zero measure. However, this contradicts the fact that $U_{\ell_0} - \psi$ is separated from zero on $\overline{D}_\Gamma \setminus [\mathbb{R} \times (0, \check{\Gamma})]$.

The obtained contradiction shows that $U(y; \check{s})$ is greater then or equal to $\psi(x, y)$ on \overline{D}_Γ and vanishes when $y = 0$. Moreover, Theorem 1 implies that $U(\cdot; \check{s})$ and ψ cannot be equal at an inner point of $D_\Gamma \setminus \Gamma_\psi$ because the latter function is not a stream solution. Furthermore, $\nabla U \neq 0$ on Γ_ψ , and so the extended U is strictly greater than ψ on D_Γ , which completes the proof of (17).

To show that (A)–(C) are valid, we consider a point, say $(x^0, \check{\Gamma})$, at which the curve Γ is tangent to $y = \check{\Gamma}$, and so $U(\check{\Gamma}; \check{s}) - \psi(x^0, \check{\Gamma}) = 0$ because both terms on the left-hand side are equal to one. It has been proved that $U(y; \check{s}) - \psi(x, y) \geq 0$ on $\mathbb{R} \times [0, \check{\Gamma}]$, which implies

$$[U'(y; \check{s}) - \psi_y(x, y)]_{(x,y)=(x^0, \check{\Gamma})} \leq 0. \tag{20}$$

Since Bernoulli's equation at $(x_0, \check{\Gamma})$ has the form $\psi_y(x^0, \check{\Gamma}) = \sqrt{3r - 2\check{\Gamma}}$ — cf. formula (12) — inequality (20) gives

$$U'(\check{\Gamma}; \check{s}) \leq \sqrt{3r - 2\check{\Gamma}} \iff \check{s}^2 - 2\Omega(1) \leq 3r - 2\check{\Gamma}.$$

Hence $\mathcal{R}(\check{s}) \leq r$ in view of (16), and combining the latter inequality and $h(\check{s}) = \check{\Gamma}$, one obtains that (A) and (B) are true in assertion 1. Moreover, (C) is also true provided $r \leq r_0$.

Now we turn to assertion 2 and begin with the case when $\check{s} = s_0 > 0$; here equality is a consequence of the assumption that $\check{\Gamma} = h_0$. We introduce $\omega^{(\epsilon)}(\tau) = \omega(\tau) - \epsilon$, where $\epsilon > 0$ is small, and let $\Omega^{(\epsilon)}(\tau)$ and $U^{(\epsilon)}(y; s)$ be defined by formulas (13) with ω changed to $\omega^{(\epsilon)}$; similarly, we define $h^{(\epsilon)}(s)$ using (15), whereas to obtain $H_+^{(\epsilon)}$ and $H_-^{(\epsilon)}$ one has to combine (16) and (15).

Since $s_0 > 0$, we have

$$s_0^{(\epsilon)} = \sqrt{2 \max_{\tau \in [0,1]} \Omega^{(\epsilon)}(\tau)} < s_0.$$

Furthermore, it is straightforward to verify the inequalities

$$h^{(\epsilon)}(s) < h(s) \quad \text{and} \quad U^{(\epsilon)}(y; s) > U(y; s) \quad \text{for } s \geq s_0.$$

Therefore, $U^{(\epsilon)}(y; s_0)$ solves the problem on $(0, \check{\Gamma})$ analogous to (14), but with ω is changed to $\omega^{(\epsilon)}$ and with the value $U^{(\epsilon)}(\check{\Gamma}; s_0)$ greater than one. Moreover, in view of the inequality

$$U^{(\epsilon)''} + \omega(U^{(\epsilon)}) \geq 0 \quad \text{on } (0, \check{\Gamma}),$$

Theorem 1 is applicable, and so the considerations used at the beginning of the proof and involving $U(\cdot; \check{s})$ are valid for $U^{(\epsilon)}(\cdot; s_0)$ as well, thus yielding

$$\psi(x, y) < U^{(\epsilon)}(y; \check{s}) \quad \text{for } (x, y) \in \mathbb{R} \times (0, \check{\Gamma}), \quad (21)$$

which is similar to (17); here it is also taken into account that $\check{s} = s_0$. Letting $\epsilon \rightarrow 0$ in the last inequality, one obtains

$$\psi(x, y) \leq U(y; \check{s}) \quad \text{for } (x, y) \in \mathbb{R} \times (0, \check{\Gamma}),$$

form which the inequalities in (A)–(C) follow in the same way as above.

Now let $s_0 = 0$. First we assume that $\Omega(\tau) < 0$ for $\tau \in (0, 1]$, in which case we have

$$U'(y; s_0) > 0 \quad \text{for } y > 0 \quad \text{and} \quad U'(0; s_0) = 0$$

for the function defined by formulas (13). Then the considerations used in the case when $\check{\Gamma} < h_0$ are applicable. Otherwise, the considerations based on $U^{(\epsilon)}(y; s)$ yield (21), and the results follow by letting $\epsilon \rightarrow 0$.

3.5. Proof of Theorem 3. At its initial stage, the proof of this theorem is similar to that of Theorem 2. Namely, we consider the case when $\widehat{\Gamma} < h_0$ first. Since there exists $\widehat{s} > s_0$ such that $h(\widehat{s}) = \widehat{\Gamma}$, the function U given by formula (13) with $s = \widehat{s}$ solves problem (14) on $(0, \widehat{\Gamma})$. Moreover, the same formula defines this function for all $y \leq \widehat{\Gamma}$ provided $\omega(t)$ is extended to $t < 0$ by zero, in which case $\widehat{s}^2 > 2 \max \Omega(\tau)$. Then

$$U'(y; \widehat{s}) = \sqrt{\widehat{s}^2 - 2\Omega(U(y; \widehat{s}))} > 0 \quad \text{for } y \leq \widehat{\Gamma},$$

and so $U(y; \widehat{s})$ is a monotonically increasing function of y such that $U(y; \widehat{s}) < 0$ for $y < 0$.

Let $U_\ell(y) = U(y - \ell; \widehat{s})$ for $\ell \geq 0$, which implies that $U_\ell(y) < 0$ on $[0, \widehat{\Gamma}]$ provided $\ell > \widehat{\Gamma}$. Therefore, $U_\ell - \psi < 0$ on $\overline{D_\Gamma}$ for $\ell > \widehat{\Gamma}$. As in the proof of Theorem 2, one deduces that there is no $\ell_0 \in (0, \widehat{\Gamma})$ with

$$\max_{(x,y) \in \mathbb{R} \times [0, \widehat{\Gamma}]} \{U_{\ell_0}(y) - \psi(x, y)\} = 0.$$

Hence $U(y; \widehat{s}) - \psi(x, y)$ is nonpositive on $\overline{D_\Gamma}$ and vanishes when $y = 0$. Now, applying Theorem 1 in the same way as in the proof of Theorem 2, we arrive at inequality (18).

To prove that $\widehat{\Gamma} \geq H_+$, we argue by analogy with the proof of Theorem 2. In view of the periodicity of Γ , there exists $(x^0, y^0) \in \Gamma$ such that $y^0 = \widehat{\Gamma}$ (it is clear that Γ is tangent to $y = \widehat{\Gamma}$ at this point). Then $U(\widehat{\Gamma}; \widehat{s}) - \psi(x^0, \widehat{\Gamma}) = 0$, because both terms on the left-hand side are equal to one. Since ψ is a non-stream solution, we have

$$[U'(y; \widehat{s}) - \psi_y(x, y)]_{(x,y)=(x^0, \widehat{\Gamma})} \geq 0.$$

Using Bernoulli's equation for ψ at $(x^0, \widehat{\Gamma})$, we show that

$$U'(\widehat{\Gamma}; \widehat{s}) \geq \sqrt{3r - 2\widehat{\Gamma}}. \tag{22}$$

Indeed, from the boundary condition $\psi(x^0, \widehat{\Gamma}) = 1$ and the assumption that $\psi \leq 1$ on $\overline{D_\Gamma}$ it follows that $\psi_y(x^0, \widehat{\Gamma})$ is nonnegative, and so

$$\psi_y(x^0, \widehat{\Gamma}) = \sqrt{3r - 2\widehat{\Gamma}}.$$

Combining this and the inequality preceding (22), we see that (22) is true, which implies the required inequality.

3.6. Proof of Propositions 1 and 2. Since the proof of Proposition 2 is similar to that of Proposition 1, we restrict ourselves to proving the latter assertion only.

To prove Proposition 1, we notice that there exists $\check{s} > s_0$ such that $h(\check{s}) = \check{\Gamma}$ and $U'(y; \check{s}) > 0$ for $y \geq 0$; here $U(\cdot; \check{s})$ is defined by formula (13) provided $\omega(t)$ is extended to $t > 1$ by zero. (This follows in the same way as in the proof of Theorem 2.) Since inequality (20) is valid, it yields $\psi_y(x^0, \check{\Gamma}) > 0$.

Now we apply the partial hodograph transform in two neighborhoods of $(x^0, \check{\Gamma})$ so that in both cases the image of this point is $(q^0, 1)$ on the (q, p) -plane. In the first case, some $X_1 \subset D_\Gamma$ is mapped to a neighborhood

$$Q_1 \subset \{(q, p) : q \in \mathbb{R}, p < 1\},$$

and $h_1(q, p)$ in Q_1 corresponds to $\psi(x, y)$ defined in X_1 . In the second case, some $X_2 \subset \mathbb{R} \times (0, 1)$ is mapped to another neighborhood Q_2 on the same plane as Q_1 , whereas $h_2(q, p)$ in Q_2 corresponds to $U(y; \check{s})$. Since inequality (17) holds for ψ and $U(\cdot; \check{s})$, we see that $h_1 - h_2 > 0$ in some subset of $Q_1 \cap Q_2$ that is the intersection of $\{(q, p) : q \in \mathbb{R}, p < 1\}$ with a neighborhood of $(q^0, 1)$. Furthermore, $h_1 - h_2$ vanishes at $(q^0, 1)$ because $\psi(x^0, \check{\Gamma}) = U(\check{\Gamma}; \check{s}) = 1$. Then from Lemma 2 it follows that

$$[\partial_p(h_1 - h_2)]_{(q,p)=(q^0,1)} < 0,$$

which implies that inequality (20) is strict. Using this fact in the considerations that follow (20), we deduce that the inequalities in (A)–(C) are strict.

3.7. Discussion. In his renown book [2], the first edition of which was published in 1918, Carathéodory had proved quite a general theorem for the first order ordinary differential equation. It concerns the existence of a solution which satisfies the equation on an interval up to a set of Lebesgue measure zero. The proof is based on assumptions whose general form is now referred to as the Carathéodory condition (see [1, §1.4], for its discussion). In the framework of this condition, f is assumed to be continuous in almost all papers, dealing with equation (1), inequality (2) and their generalisations (see, for example, the notes [9] and [18] by Keller and Osserman, respectively, dating back to 1957, and numerous papers citing those notes). It should also be mentioned in this connection that nonuniqueness occurs for the first order ordinary differential equation when the smoothness of a nonlinear term is less than Lipschitz with respect to the unknown function.

Furthermore, considering equation (1) in [7] (see Theorem 1 on p. 209), the authors require even more, namely, that f be of class C^1 . This substantially simplifies the treatment of the equation compared to Theorem 1 in the

present paper, where $f \in L^p_{\text{loc}}(\mathbb{R})$ for $p > n$. On the other hand, the assumption imposed on solutions in our theorem, namely, that their gradients are nonvanishing, is essential. This condition allows us to avoid nonuniqueness even without the Carathéodory condition.

When we turn to the problem of periodic water waves with vorticity, the papers [4] and [17] should be mentioned. Discontinuous vorticity distributions from L^∞ are considered in the first of them, whereas the distribution is merely L^p -integrable with an arbitrary $p \in (1, \infty)$ in the second one. However, only unidirectional flows (they have no stagnation points within the fluid) were studied in both papers, and in this case the global partial hodograph transform can be applied to simplify the problem, thus reducing the effect of nonsmooth vorticity.

It should be mentioned that the assumptions on a solution of the water wave problem here are weaker not only than those imposed in our recent paper [12], but also than those in [13]. Indeed, the most restrictive condition in [13] is that the horizontal component of the velocity field is bounded from below by a positive constant.

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