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# INTEGRABLE SYSTEMS, POISSON PENCILS, AND HYPERELLIPTIC LAX PAIRS

In the modern approach to integrable Hamiltonian systems, their representation in the Lax form (the Lax pair or the $L$-$A$ pair) plays a key role. Such a representation also makes it possible to construct and solve multi-dimensional integrable generalizations of various problems of dynamics. The best known examples are the generalizations of Euler's and Clebsch's classical systems in the rigid body dynamics, whose Lax pairs were found by Manakov [10] and Perelomov [12]. These Lax pairs include an additional (spectral) parameter defined on the compactified complex plane or an elliptic curve (Riemann surface of genus one). Until now there were no examples of $L$-$A$ pairs representing physical systems with a spectral parameter running through an algebraic curve of genus more than one (the conditions for the existence of such Lax pairs were studied in [11]).

In given paper we consider a new Lax pair for the multidimensional Manakov system on the Lie algebra $\mathrm{so}(m)$ with a spectral parameter defined on a certain unramified covering of a hyperelliptic curve. An analogous $L$-$A$ pair for the Clebsch–Perelomov system on the Lie algebra $e(n)$ can be indicated.

In addition, the hyperelliptic Lax pair enables us to obtain the multidimensional generalizations of the classical integrable Steklov–Liapunov systems in the problem of a rigid body motion in an ideal fluid. The latter is known to be a Hamiltonian system on the algebra $e(3)$. It turns out that these generalized systems are defined not on the algebra $e(n)$, as one might expect, but on a certain product $\mathrm{so}(m) + \mathrm{so}(m)$. A proof of the integrability of the systems is based on the method proposed in [1].

## §1. The hierarchy of the Manakov systems and consistent Poisson brackets on $\mathrm{so}(m)$

Recall that the Euler–Frahm equations of free motion of an $m$-

dimensional rigid body have the Hamiltonian form

$$\dot{M} = [M, \Omega], \quad \Omega = \frac{\partial H}{\partial M}, \tag{1.1}$$

where $\Omega \in so(m)$ is the angular velocity, $M \in so^*(m)$ the angular momentum of the body ([5]). These equations are known to be integrable provided the operators $M$ and $g$ are connected as follows

$$[M, V] = [\Omega, U]. \tag{1.2}$$

Here $U, V$ are constant diagonal matrices

$$U = \operatorname{diag}(a_1, \dots, a_m), \quad V = \operatorname{diag}(b_1, \dots, b_m),$$

and all the eigenvalues of $U$ and $V$ are distinct.

This fact follows from the Lax pair of the system (1.1) with rational spectral parameter $\lambda$

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)], \qquad L(\lambda) = M + \lambda U, \quad A(\lambda) = \Omega + \lambda V. \tag{1.3}$$

This Lax pair was found by Manakov [10].

Now consider the case $V = U^2$. Then, in view of (1.2), we have $\Omega = UM + MU$, and equations (1.1) take the simple form

$$\dot{M}_{ij} = (a_j - a_i) \sum_{k=1}^{m} M_{ik} M_{kj}, \quad i, j = 1, \dots, m. \tag{1.4}$$

Apart from this "basic" system, there exists a whole hierarchy of "higher Manakov systems" commuting with (1.4). These systems are defined by the following hierarchy of $A$-operators in (1.3)

$$A_{l,r}(\lambda) = \{M^l, U^r\} + \lambda\{M^{l-1}, U^{r+1}\} + \cdots + \lambda^l U^k,$$
$$r \in \mathbb{N}, \quad l = 1, 3, 5, \dots,$$

where $\{M^l, U^r\}$ denotes a homogeneous symmetric matrix polynomial in $M$ and $U$ of degrees $s$ and $r$ respectively: for example: $\{M, U\} = MU + UM, \{M, U^2\} = MU^2 + UMU + U^2M$, etc. Expanding in powers of $\lambda$ both sides of the Lax equations

$$\frac{d}{dt}(M + \lambda U) = [M + \lambda U, A_{l,r}(\lambda)],$$

at $\lambda^0$ we obtain the higher Manakov systems

$$\dot{M} = [M, \{M^l, U^r\}], \tag{1.5}$$

and at other powers zero identities.

One can show that the hierarchy (1.5) is complete, i.e., any system commuting with (1.4) is defined by a corresponding linear combination of the right-hand sides of (1.5).

**Remark 1.** Once we have obtained the hierarchy of the systems (1.5), it is easy to see that the general integrable system

$$\dot{M} = [M, \Omega], \qquad \Omega_{ij} = \frac{b_i + b_j}{a_i + a_j} M_{ij} \tag{1.6}$$

can be included into this hierarchy: the right-hand side of (1.6) is a linear combination with constant coefficients of the right-hand side of (1.4) and "higher" quadratic flows from the hierarchy (1.5), i.e.,

$$[M, \Omega] = [M, \sum_{r=0}^{m-2} \alpha_r \{M, U^r\}], \qquad \alpha_r = \text{const}\,. \tag{1.7}$$

Indeed, using the relation between $\Omega$ and $M$ in (1.6), as well as the expressions

$$\{M, U^r\}_{ij} = \frac{a_i^l - a_j^l}{a_i - a_j} M_{ij}, \qquad l = r + 1,$$

from (1.7) we obtain the following system of $m(m-1)/2$ linear equations for determination of $\alpha_r$

$$\alpha_0(a_i - a_j) + \alpha_1(a_i^2 - a_j^2) + \cdots + \alpha_{m-2}(a_i^{m-1} - a_j^{m-1}) = b_i - b_j,$$
$$i < j = 1, \ldots, m,$$

of which only $m - 1$ are independent. Thus, the coefficients $\alpha_1, \alpha_1, \ldots, \alpha_{m-2}$, can be uniquely determined as functions of $b_1, \ldots, b_m$.

Recall that a hierarchy of Hamiltonian system can be generated by the Magri–Lenard recursion procedure which uses a pair of consistent Poisson brackets on a phase space [9]. Poisson brackets $\{\,,\,\}_0$ and $\{\,,\,\}_1$ are called *consistent* (compartiable) if for any three functions $f, g, h$

$$\{f, \{g, h\}_1\}_0 + \cdots + \{f, \{g, h\}_0\}_1 + \cdots = 0 \tag{1.8}$$

(the dots denote summation over cyclic permutations of $f, g, h$). Fulfilled condition (1.8), any linear combination $s\{\,,\,\}_1 + \{\,,\,\}_0$, $s \in \mathbb{R} \cup \infty$ is also a Poisson bracket, and all such combinations span a *Poisson pencil*, a line in the projective space of all Poisson brackets on the phase space. Clearly, any two bracket from the pencil are consistent.

Bolsinov [1] showed that the hierarchy of the systems (1.5) is related to the existence of two consistent Poisson brackets on the Lie algebra $\mathrm{so}(m) = \mathrm{so}^*(m) = \{M\}$: the first corresponds to the standard commutator $[M_1, M_2]_0 = M_1M_2 - M_2M_1$, the second to commutator

$$[M_1, M_2]_1 = M_1UM_2 - M_2UM_1, \qquad M_1, M_2, [M_1, M_2]_1 \in \mathrm{so}(m)$$

(here $U$ is the same matrix as above). These brackets generate the Hamiltonian systems

$$\begin{aligned} \dot{M} &= w_0\left(\frac{\partial H}{\partial M}, \cdot\right) = \left[M, \frac{\partial H}{\partial M}\right], \\ \dot{M} &= w_1\left(\frac{\partial H}{\partial M}, \cdot\right) = M\frac{\partial H}{\partial M}U - U\frac{\partial H}{\partial M}M, \end{aligned} \tag{1.9}$$

where $w_0(\cdot,\cdot), w_1(\cdot,\cdot)$ are the corresponding Poisson structures. The form for $w_1(\cdot,\cdot)$ follows from the commutator $[\,,\,]_1$ by the definition of the coadjoint action of a Lie algebra:

$$\begin{aligned} \forall M \in \mathrm{so}^*(m),\ Y, Z \in \mathrm{so}(m) : \langle \mathrm{ad}_Y^* M, Z\rangle = \langle MYU - UYM, Z\rangle = \\ = \langle M, [Z, Y]_1\rangle, \end{aligned}$$

$\langle\,,\,\rangle$ being the Killing form of the Lie algebra $\mathrm{so}(m)$. A complete set of invariants of the brackets $\{\,,\,\}_0$ and $\{\,,\,\}_1$ is given by the functions

$$\mathcal{I}_k = \mathrm{tr}(M^k), \ \text{ respectively } \ \mathcal{J}_k = \mathrm{tr}(MU^{-1})^k, \quad k = 2, 4, \dots, 2[m/2].$$

Sunstituing to $w_1(\,,\,)$ the annulators of $w_0(\,,\,)$, the powers $M^l$, $l = 1, 3, \dots$, for each $l$ we obtain the series of Hamiltonian vector fields

$$\begin{aligned} v_{1l} &= w_1(M^l, \cdot) &&= [M, \{M^l, U\}], \\ v_{2l} &= w_1(\{M^l, U\}, \cdot) &&= [M, \{M^l, U^2\}], \\ &\dots \quad \dots && \dots \quad \dots \\ v_{\nu l} &= w_1(\{M^l, U^{\nu-1}\}, \cdot) &&= [M, \{M^l, U^\nu\}], \end{aligned}$$

which gives a subhierarchy of the Manakov hierarchy (1.5).

## §2. Hyperelliptic Lax pairs

Now turn to another Lax representation of the Euler–Manakov equations (1.4). It was found in [4] and has the form

$$\dot{L}(s) = [L(s), A(s)] \tag{2.1}$$

$$L(s)_{ij} = \frac{\sqrt{\Phi(s)}}{\sqrt{(s-a_i)(s-a_j)}} M_{ij}, \quad A(s)_{ij} = \sqrt{(s-a_i)(s-a_j)} M_{ij},$$
$$\Phi(s) = (s-a_1)(s-a_2)\cdots(s-a_m), \qquad i,j = 1,\dots,m,$$

where $s \in \mathbb{C}$ is a parameter. The functions $w_{ij} = \sqrt{(s-a_i)(s-a_j)}$ called *biradicals* are assumed to satisfy the relations

$$w_{ik}w_{kj} = (s-a_k)w_{ij}. \tag{2.2}$$

**Proposition 1.** *Biradicals $w_{ij}, i,j = 1,\dots,m$ that satisfy conditions (2.2), as well as the quotients $w/w_{ij}$, are single-valued functions on a $4^g$-sheeted unramified covering $\widehat{\Gamma}_m$ of the hyperelliptic curve $\Gamma : \{w^2 = \Phi(s)\}$ of genus $g$.*

In this connection we shall refer to the Lax pair (2.1) and the parameter $s$ as *hyperelliptic*. Writing out equation (2.1) in the scalar form and taking into account (2.2), after elimination of the radicals we obtain exactly equations (1.4). The other systems of the hierarchy on so$(m)$ are generated by the operators

$$A_{l,r}(s) = SP_{l,r}(s)S, \qquad l = 1,3,\dots, \quad r \in \mathbb{N} \tag{2.3}$$
$$S = \mathrm{diag}(\sqrt{s-a_1},\dots,\sqrt{s-a_m}),$$
$$P_{l,r}(s) = s^{r-1}M^l + s^{r-2}\{M^l, U\} + \dots + s\{M^l, U^{r-2}\} + \{M^l, U^{r-1}\}.$$

Substituting the operators (2.3) in (2.1), we obtain precisely the higher Manakov systems (1.5). Indeed, taking into account the relation $S = sE - U$, $E$ being the identity matrix, we find

$$\dot{M} = s[M, P_{l,r}(s)] - (MP_{l,r}(s)U - UP_{l,r}(s)M). \tag{2.4}$$

The coefficient at $s^\nu$ ($\nu = 1,\dots,r-1$) in the right-hand side of (2.4) equals

$$[M, \{M^l, U^{r-\nu}\}] - (M\{M^l, U^{r-\nu-1}\}U - U\{M^l, U^{r-\nu-1}\}M) = 0.$$

The free term in (2.4) coincides with the right-hand side of (1.5).

**Remark** 2. Equations (2.4) are obviously Hamiltonian with a Hamilton function $H$ such that $\partial H/\partial M = P_{l,r}(s)$ with respect to the Poisson bracket $s\{\,,\,\}_0 - \{\,,\,\}_1$, where $\{\,,\,\}_0$ and $\{\,,\,\}_0$ are the consistent Poisson brackets defined in (1.9) and forming the Poisson pencil for the Manakov systems. Thus, the hyperelliptic Lax pairs (2.1), (2.3) can be regarded as the representation of (2.4) in a matrix commutator form, while the parameter $s$ as a coordinate on the Poisson pencil.

Now write out the characteristic polynomial of the $L$-matrix

$$|L(s) - zE| = z^m + \sum_{k=2}^{m} z^{m-k} (\Phi(s))^{k/2-1} \widetilde{\mathcal{I}_k}(s, M) \quad (k \text{ is even}), \tag{2.5}$$

$$\widetilde{\mathcal{I}_k}(s, M) = \sum_I \frac{\Phi(s)}{(s - a_{i_1}) \dots (s - a_{i_k})} |M|_I^I = \sum_{\nu=0}^{m-k} s^u H_{k\nu}(M),$$

where $|M|_I^I$ are diagonal minors of order $k$ corresponding to the multi-indicies $I = \{i_1 \dots i_k\} \subset \{1 \dots m\}$, $1 \leqslant i_1 < \dots < i_k \leqslant m$. More specifically,

$$\begin{aligned}
\widetilde{\mathcal{I}}_2(s, M) &= \sum_{i<j}^{m} \frac{\Phi(s)}{(s - a_i)(s - a_j)} M_{ij}^2 = s^{m-2} \sum_{i<j}^{m} M_{ij}^2 \\
&\quad + s^{m-3} \sum_{i<j}^{m} (a_i + a_j - \Delta_1) M_{ij}^2 + \dots + \sum_{i<j}^{m} \frac{a_1 a_2 \cdots a_m}{a_i a_j} M_{ij}^2, \\
\widetilde{\mathcal{I}}_4(s, M) &= \sum_{i<j<k<l}^{m} \Phi(s) \frac{(M_{ij} M_{kl} - M_{ik} M_{jl} + M_{il} M_{kj})^2}{(s - a_i)(s - a_j)(s - a_k)(s - a_l)}, \quad \dots; \\
\Delta_1 &= a_1 + a_2 + \dots + a_m, \quad \dots.
\end{aligned}$$

One sees easily that for a fixed parameter $s$ the polynomials $\widetilde{\mathcal{I}}_4(s, M)$, $k = 2, \dots, 2[m/2]$ form presicely a complete set of annulators of the Poisson bracket $s\{\,,\,\}_0 - \{\,,\,\}_1$ of the Poisson pencil on so$(m)$. Their coefficients, the polynomials $H_{k\nu}(M)$, $k = 2, 4, \dots \leqslant m, \mu = 0, 1, \dots, m$ form a complete set of first integrals of the Manakov system in involution. Note that all of them are independent. On the contrary, the coefficients of the characteristic polynomial for the Lax pair (1.3) with the rational spectral parameter $\lambda$ are not all independent.

For example, the Manakov systems on the 15-dimensional algebra so (6) possess 9 independent integrals $H_{k\nu}(M)$ such that

$$\begin{aligned}\widetilde{\mathcal{I}}_2(s,M) &= s^4H_{24}+s^3H_{23}+s^2H_{22}+sH_{21}+H_{20},\\ \widetilde{\mathcal{I}}_4(s,M) &= \qquad\qquad\qquad +s^2H_{42}+sH_{41}+H_{40},\\ \widetilde{\mathcal{I}}_6(s,M) &= \qquad\qquad\qquad\qquad\qquad\qquad +H_{60}.\end{aligned}$$

Therefore, the dimension of general invariant tori of this system equals $15-9=6$. This is consistent with the general formula

$$\begin{aligned}\text{dim of general tori} &= \frac{1}{2}[\dim \mathrm{so}\,(m) - \operatorname{corank} \mathrm{so}\,(m)]\\ &= \frac{1}{2}\left(\frac{m(m-1)}{2} - \left[\frac{m}{2}\right]\right).\end{aligned}$$

## §3. Remarks on uniqueness of solutions. The gyroscopic Zhykovsky–Volterra system

According to the finite-gap integration method (FGI) [2], a representation of a dynamical system in the Lax form enables one to find its explicit solutions in terms of theta-functions of the Jacobi variety $\mathrm{Jac}(\mathcal{C})$ of the spectral curve

$$\mathcal{C}: \{P(\lambda,\mu) = \det(L(\lambda) - \mu I) = 0\},$$

or on its subabelian variety $\mathrm{Prym}(\mathcal{C}) \subset \mathrm{Jac}(\mathcal{C})$.

However, as was shown by Hence [6], after application of the FGI method to Manakov's $L$-$A$ pair (1.3) with the rational parameter for the cases $m=3,4$ the angular momentum matrix $M$ can be determined only up to an action of a finite discrete group, which flips signs of some of the $M_{ij}$'s. Thus, this approach gives rise to solutions for squares of the variables $M_{ij}$ but not for the variables themselves. The same situation occurs for the higher-dimensional Frahm–Manakov systems.

From the other hand, one can show that for the algebras so (3) and so (4) these variables can be uniquely expressed in terms of theta-functions on $\mathrm{Prym}(\mathcal{D})$, where $\mathcal{D}$ is the spectral curve for the hyperelliptic Lax pair (2.1). Thus, the latter provide more information about the algebraic structure of the integrable systems.

To illustrate this phenomenon, we consider another generalization of the Euler top problem, namely the Zhukovsky–Volterra system describing free motion of a rigid body carrying an axisymmetric rotor (gyrostat). The equations of motion can be written in the vector form

$$\dot{M} = M\times\omega, \qquad \omega = (a_1M_1-g_1, a_2M_2-g_2, a_3M_3-g_3), \qquad (3.1)$$

where $M, \omega, g \in \mathbb{R}^3$ are respectively the total angular momentum of the body with the rotor, the angular velocity of the body, and an arbitrary constant vector related to the angular momentum of the rotor.

Aside from quadratic terms in $M_\alpha$, these equations include linear terms in the right-hand side. Such an elementary generalization of Euler's equations implies, however, much more complicated procedure of its explicit integration (V. Volterra [15]).

Note that, in contrast to the homogeneous Euler equations, neither generalized system (3.1) nor its first integrals

$$\begin{aligned} Q_1 &= M_1^2 + M_2^2 + M_3^2, \\ Q_0 &= a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 - 2(M_1 g_1 + M_2 g_2 + M_3 g_3) \end{aligned} \tag{3.2}$$

admit transformations of the type $M_\alpha \to M_\alpha, M_\beta \to -M_\beta, M_\gamma \to -M_\gamma$.

A Lax pair with a rational spectral parameter for the system (3.1) is not known and the asymmetry just mentioned above seems to obstruct the existence of such a Lax pair. However, an $L$-$A$ pair with an elliptic spectral parameter for this system does exist and has the form

$$\dot{L}(s) = [L(s), A(s)], \qquad L(s), A(s) \in \mathrm{so}(3), \tag{3.3}$$
$$L(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \sqrt{s - a_\gamma} M_\gamma - g_\gamma / \sqrt{s - a_\gamma},$$
$$A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \sqrt{(s - a_\alpha)(s - a_\beta)} M_\gamma,$$
$$(\alpha, \beta, \gamma) = (1, 2, 3),$$

$\varepsilon_{\alpha\beta\gamma}$ being the Levi-Civita tensor. The characteristic polynomial $|L(s) - rI|$ gives rise to the family of quadratic integrals

$$Q(s) = \sum_{\alpha=1}^{3} \left[\sqrt{s - a_\alpha} M_\alpha - g_\alpha / \sqrt{s - a_\alpha}\right]^2 =$$
$$= sQ_1(M) - Q_0(M) + \sum_{\alpha=1}^{3} g_\alpha^2 / (s - a_\alpha).$$

It would be interesting to find multidimensional generalizations of the Zhukovski–Volterra system by using the structure of the elliptic Lax pair (3.3).

## §4. The hierarchy of the Steklov–Liapunov–Rubanovsky systems

Recall that the Kirchhoff equations describing motion of a rigid body in an ideal fluid are Hamiltonian with respect to the standard Lie-Poisson bracket of the Lie algebra $e(3)$

$$\{M_\alpha, M_\beta\} = \varepsilon_{\alpha\beta\gamma} M_\gamma, \quad \{M_\alpha, p_\beta\} = \varepsilon_{\alpha\beta\gamma} p_\gamma, \quad \{p_\alpha, p_\beta\} = 0, \quad (4.1)$$
$$(\alpha, \beta, \gamma) = (1, 2, 3),$$

where $M = (M_1, M_2, M_3)$ and $p = (p_1, p_2, p_3)$ play the role of the impulsive momentum and the impulsive force.

It is well-known that, apart from the Hamiltonian of the problem, the Kirchhoff equations always have two "trivial" integrals

$$(M, p), \quad (p, p), \tag{4.2}$$

which are Casimir functions of the bracket (4.1). Steklov [14] found a Hamiltonian for which these equations possess a fourth additional integral, and therefore are integrable by the Jacoby theorem. Later Liapunov [8] independently discovered an integrable case of the Kirchhoff equations whose integral of the kinetic energy turned out to be a linear combination of Steklov's additional integral and the two trivial integrals. Thus, the Steklov and Liapunov systems possess one and the same set on first integrals and define, in fact, different trajectories on the same invariant manifolds, the two-dimensional tori.

After papers of N. Zhykovsky and V. Volterra devoted to the problem of a free motion of a gyrostat, mathematicians tried to find generalizations of known integrable systems of dynamics which would include additional gyroscopic (linear) terms. For the Steklov–Liapunov systems such integrable generalizations were found by Rubanovsky [13]. These are represented by the Kirchhoff equations on $e(3)$ with the Hamiltonians

$$H_1 = \frac{1}{2} \sum_{\alpha=1}^{3} \big( b_\alpha (M_\alpha - 2g_\alpha)^2 + 2\nu b_\beta b_\gamma M_\alpha p_\alpha$$
$$+ \nu^2 b_\alpha (b_\beta - b_\gamma)^2 p_\alpha^2 + 4\nu (b_\beta + b_\gamma) g_\alpha p_\alpha \big),$$

$$H_2 = \frac{1}{2} \sum_{\alpha}^{3} \big( M_\alpha^2 - 2\nu b_\alpha M_\alpha p_\alpha + \nu^2 (b_\beta - b_\gamma)^2 p_\alpha^2 + 8 g_\alpha p_\alpha \big),$$
$$b_1, b_2, b_3, g_1, g_2, g_3, \nu = \text{const}, \quad (\alpha, \beta, \gamma) = (1, 2, 3). \tag{4.3}$$

In hydrodynamics these Hamiltonians describe a motion of a gyrostat in an ideal fluid under the action of an Archimedus torque, which arises when the barycenter of the gyrostat does not coincide with its volume center. As in the previous section, the constant vector $g = (g_1, g_2, g_3)$ is related to the angular momentum of the rotor of the gyrostat. On putting $g = 0$ the functions $H_1$ and $H_2$ are reduced respectively to the classical Steklov and Liapunov Hamiltonians.

It can be checked that $\{H_1, H_2\} = 0$ with respect to the Lie–Poisson bracket on $e^*(3)$. Therefore, the Rubanovsky systems are indeed integrable.

In the sequel, without loss of generality, we put $\nu = 1$ (this can always be achieved by an appropriate transformation $p_\alpha \to p_\alpha/\nu$).

For $g = 0$ the Kirchhoff equations with the Hamiltonians (4.3) were explicitly solved by brilliant analyst F.Kötter [7], who used the change $(M, p) \to (z, p)$:

$$2z_\alpha = M_\alpha - (b_\beta + b_\gamma)p_\alpha, \qquad \alpha = 1, 2, 3, \tag{4.4}$$

and, as a matter of fact, represented the Hamilton equations in a Lax form. It turns out that after the same change the gyroscopic Rubanovsky systems with the Hamiltonians (4.3) admit the following generalizations of Kotter's $L$-$A$ pair

$$\dot{L}(s) = [L(s), A(s)], \qquad L(s), A(s) \in \mathrm{so}(3), \quad s \in \mathbb{C} \tag{4.5}$$
$$L(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\left(\sqrt{s - b_\gamma}(z_\gamma + sp_\gamma) + g_\gamma/\sqrt{s - a_\gamma}\right),$$

where, as above, $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor. To the systems with the Hamiltonians $H_1$ and $H_2$ in (4.3) there correspond the operators

$$A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\frac{1}{s}\sqrt{(s - b_\alpha)(s - b_\beta)}(b_\gamma z_\gamma - g_\gamma), \tag{4.6}$$

and, respectively,

$$A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\sqrt{(s - b_\alpha)(s - b_\beta)}p_\gamma. \tag{4.7}$$

It is obvious that the $L$-$A$ pairs (4.5), (4.6), (4.7) generalize the Lax equations (3.3) for the Zhykowski–Volterra system. According to Proposition 1, the radicals in (4.5)–(4.7) are single-valued functions on the elliptic curve $\hat{\Gamma}$, the 4-sheeted unramified covering of the plane curve $\Gamma : \{w^2 = (s - b_1)(s - b_2)(s - b_3)\}$. Writing out the characteristic equation for $L(s)$ we arrive at the following family of quadratic integrals

$$Q(s) = \sum_{\gamma=1}^{3} \left[\sqrt{s-b_\gamma}(z_\gamma + sp_\gamma) + g_\gamma/\sqrt{s-b_\gamma}\right]^2$$

$$= s^3 I_2 + s^2 I_1 + 2sH_2 - 2H_1 + \sum_{\gamma=1}^{3} g_\gamma^2/(s-b_g a),$$

where

$$H_1 = \frac{1}{2}(z, Bz) - (z, g), \quad H_2 = \frac{1}{2}(z,z) - (Bz, p) + (p, g),$$
$$I_1 = 2(z,p) - (Bp,p), \quad I_2 = (p,p), \qquad B = \operatorname{diag}(b_1, b_2, b_3).$$

It is easy to see that under the change (4.4) the functions $I_1, I_2$ transform to the invariants (4.2) of the algebra $e(3)$, while the integrals $H_1(z,p)$, $H_2(z,p)$ (up to addition of these invariants) transform to the Hamiltonians (4.3).

The explicit form of the multidimensional Frahm–Manakov systems was known before the corresponding Lax pair was found. For multidimensional generalizations of the Steklov–Liapunov systems the situation is different, and we have to constuct them by relying upon the elliptic Lax pair (4.5).

It turns out that such generalizations do exist, but they are defined not on the Lie algebra $e(n)$, as one might expect, but on a certain product $\mathrm{so}(m) + \mathrm{so}(m)$. Namely, we regard $p$ not as a vector in $\mathbb{C}^m$, but as a vector in the algebra $\mathrm{so}(m)$ (see [2]). Let us introduce new matrix variables $Z, P \in \mathrm{so}(m)$. Then the generalization of the Lax pair (4.5) has the form

$$\dot{L}(s) = [L(s), A(s)], \qquad L(s), A(s) \in \mathrm{so}(m), \quad s \in \mathbb{C},$$
$$L(s)_{ij} = \frac{\sqrt{\Phi(s)}}{\sqrt{(s-b_i)(s-b_j)}}(Z+sP)_{ij}, \qquad i,j = 1,\dots,m, \tag{4.8}$$
$$\Phi(s) = (s-b_1)(s-b_2)\cdots(s-b_m).$$

Here $b_1, \dots, b_m$ are distinct constants. To obtain the generalized Steklov and Liapunov systems we must put

$$A(s)_{ij} = -\frac{1}{s}\sqrt{(s-b_i)(s-b_j)}\frac{\det B}{b_i b_j}Z_{ij}, \qquad B = \operatorname{diag}(b_1, \dots, b_m), \tag{4.9}$$

and, respectively,

$$A(s)_{ij} = \sqrt{(s-b_i)(s-b_j)}P_{ij}. \tag{4.10}$$

It is seen easily that this Lax pair construction also generalizes the hypereliptic Lax pair (2.1) for the Frahm–Manakov system. Just as in (2.1), all the biradicals $\sqrt{(s-b_i)(s-b_j)}$, as well as $\sqrt{\Phi(s)}$, are single-valued functions on the $4^g$-sheeted unramified covering of the hyperelliptic curve $\Gamma_m$.

Apart from these generalized systems these exists a whole hierarchy of the "higher" Steklov–Liapunov systems generated by the $A$-operators

$$A_{l,r}(s) = -S\widetilde{A}_{l,r}(s)S, \qquad S = \operatorname{diag}(\sqrt{s-b_1},\ldots,\sqrt{s-b_m}),$$
$$l = 1,3,5,\ldots, \quad r = 0,1,2,\ldots,$$

where $\widetilde{A}_{l,r}(s)$ is the result of dividing the polynomial

$$s^r(Z+sP)^l + s^{r-1}\{B,(Z+sP)^l\} + \cdots + s \\ \{B^{r-1},(Z+sP)^l\} + \{B^r,(Z+sP)^l\} \tag{4.11}$$

by $s^l$ and rejecting the terms at negative powers of $s$ in obtained expression. Note that the polynomial (4.11) arises from the polynomial $P_{l,r}$ in (2.3) under the substitution $M \to Z+sP$, $U \to B$.

In particular, putting $A = A_{1,r}$ with

$$\widetilde{A}_{1,r}(s) = s^r P + s^{r-1}\{B,P\} + \cdots + \{B^r,P\} + s^{r-1}Z + \cdots + \{B^{r-1},Z\},$$

we obtain the following subhierarchy of quadratic systems

$$\begin{aligned} \dot Z &= [Z,\{Z,B^r\}] + (Z\{P,B^r\}B - B\{P,B^r\}Z), \\ \dot P &= [P,\{P,B^{r+1}\}] + [P,\{Z,B^r\}]. \end{aligned} \tag{4.12}$$

The matrix $A_{1,0}$ coinsides with the $A$-operator (4.10) defining the multidimensional generalization of the Liapunov system.

The generalized Steklov system defined by (4.9) has the form

$$\begin{aligned} \dot Z &= [Z, \det BB^{-1}ZB^{-1}] + \det B(ZB^{-1}P - PB^{-1}Z), \\ \dot P &= [P, \det BB^{-1}ZB^{-1}]. \end{aligned} \tag{4.13}$$

The right-hand side of this system is a linear combination of those of (4.12).

It turns out that all the systems from the Steklov–Liapunov hierarchy have the Hamiltonian form

$$\dot{Z} = \left[Z, \frac{\partial \mathcal{H}}{\partial Z}\right] + B\frac{\partial \mathcal{H}}{\partial Z}P - P\frac{\partial \mathcal{H}}{\partial Z}B + \left[P, \frac{\partial \mathcal{H}}{\partial P}\right], \quad \dot{P} = \left[P, \frac{\partial \mathcal{H}}{\partial Z}\right]$$

with respect to the following Lie–Poisson bracket on the space so $(m)$ + so $(m)$

$$\begin{aligned} \{f,h\}_1 = \langle Z, [d_Z f, d_Z h]\rangle + \langle P, [d_Z f, d_P h] + [d_P f, d_Z h]\rangle \\ -\langle P, (d_Z f B d_Z h - d_Z h B d_Z f)\rangle, \end{aligned} \tag{4.14}$$

$$\langle X, Y\rangle = -\frac{1}{2}\operatorname{tr}(X, Y), \quad (d_Z f)_{ij} = \partial f/\partial Z_{ij}, \quad (df)_{ij} = \partial f/\partial P_{ij}.$$

The fact that this is indeed a Poisson bracket follows from its decomposition in the sum of two consistent Poisson brackets: the first one is the Lie–Poisson bracket of the semi-direct product so $(m)$ + so $(m)$, and the second is the bracket of the nilpotent Lie algebra defined by the follwing commutator

$$\begin{gathered} \text{for } X, Y \in \text{so}\,(m), \text{ and } (X, Y) \in \text{so}\,(m) + \text{so}\,(m) : \\ [(X_1, Y_1), (X_2, Y_2)] = (0, X_1 B X_2 - X_2 B X_1). \end{gathered}$$

Note that up to multiplication by 1/2 Kötter's substitution (4.4) takes the Poisson bracket (4.1) on $e(3)$ to the Poisson bracket (4.14) for $m = 3$.

One can easily verify that the Poisson bracket (4.14) has exactly 2[m/2] independent annulators

$$\mathcal{P}_k = \operatorname{tr}(P^k), \quad \mathcal{Q}_k = \operatorname{tr}(ZP^{k-1} + P^k B), \ k = 2, 4, \ldots, 2[m/2]. \tag{4.15}$$

Hamiltonians of the systems (4.12) (for $r = 0$) and (4.13) with respect to this bracket are respectively

$$\mathcal{H} = \frac{1}{4}\operatorname{tr}(Z^2 + Z(BP + PB) + P(B^2 P + BPB + PB^2)),$$

$$\mathcal{H} = \frac{1}{4}\det B \operatorname{tr}(ZB^{-1})^2.$$

Besides, these, as well as the higher systems of the hierarchy are also Hamiltonian with respect to the other Poisson bracket on so $(m)$+so $(m)$:

$$\{f,h\}_0 = \langle Z, d_Z f B d_Z h - d_Z h B d_Z f\rangle + \langle P, [d_P f, d_P h]\rangle, \tag{4.16}$$

i.e., they have the Hamiltonian form

$$\dot{Z} = Z\frac{\partial \mathcal{K}}{\partial Z}B - B\frac{\partial \mathcal{K}}{\partial Z}Z, \quad \dot{P} = \left[P, \frac{\partial \mathcal{K}}{\partial P}\right],$$

where for the systems (4.12) ($r = 0$) and (4.13) one must put respectively

$$\mathcal{K} = \frac{1}{4}\mathcal{P}_2, \quad \mathcal{K} = -\frac{1}{4}\det B(\operatorname{tr}(B^{-1}ZB^{-1}P) + \operatorname{tr}(B^{-1}ZB^{-1}ZB^{-1})).$$

The complete set of independent annulators of the bracket (4.16) is represented by $2[m/2]$ functions

$$\mathcal{P}_k(P) \quad \text{and} \quad \mathcal{R}_k(Z) = \operatorname{tr}(ZB^{-1})^{2k}, \qquad k = 2, 4, \ldots, 2[m/2].$$

One may check by hand that the brackets (4.14) and (4.16) are consistent. Thus in the phase space $\operatorname{so}(m) + \operatorname{so}(m)$ we have the Poisson pencil $s\{\,,\,\}_1 - \{\,,\,\}_0$, $s \in \mathbb{C} \cup \infty$.

Now consider the characteristic polynomial

$$|L(s) - wI| = w^m + \sum_{k=2}^{m} w^{m-k}(\Phi(s))^{k/2-1}\widetilde{\mathfrak{J}}_k(s, Z, P),$$
$$k = 2, \ldots, 2[m/2],$$

$$\widetilde{\mathfrak{J}}_k(s, Z, P) = \sum_I \frac{\Phi(s)}{(s - b_{i_1}) \ldots (s - b_{i_k})} |Z + sP|_I^I = \sum_{\mu=0}^{m-k} s^\mu H_{k\mu}(Z, P),$$

where, as above, $|Z + sP|_I^I$ denote the $k$-order diagonal minor corresponding to the multi-index $I = \{i_1 \ldots i_k\}$.

Again, as in the characteristic polynomial (2.5) for the Manakov systems, one can show that the functions $\mathcal{P}_k(P)$, $\widetilde{\mathfrak{J}}_k(s, Z, P)$, $k = 2, \ldots, 2[m/2]$ represent a complete set of annulators of the Poisson bracket $s\{\,,\,\}_1 - \{\,,\,\}_0$ on $\operatorname{so}(m) + \operatorname{so}(m)$. In particular, $\widetilde{\mathfrak{J}}_k(0, Z, P)$ are functions of the annulators $\mathcal{R}_k(Z)$ of the bracket $\{\,,\,\}_0$. Thus, the hyperelliptic Lax pair (4.8) for the multidimenisional Steklov–Liapunov systems turns out to be closely related to the Poisson pencil on $\operatorname{so}(m) + \operatorname{so}(m)$.

For odd dimension $m$ the polynomials $H_{k\nu}(Z, P)$, $k = 2, 4, \cdots \leqslant m$, $\nu = 0, 1, \ldots, m$ form a complete set of $(m+1)[m/2]$ independent first integtals in involution of the Steklov–Liapunov systems, which is sufficient for their integrability (the two major coefficients $H_{km}(P)$, $H_{k,m-1}(Z, P)$ are annulators of the bracket (4.14), which depend on the Casimir functions (4.15)). The same holds for even $m$ with the only exeption: the polynomial $\widetilde{\mathfrak{I}}_m(s)$ is the full square of a polynomial $\widetilde{\mathfrak{I}}'_m(s)$ of degree $m/2$ in $Z, P$. The coefficients of the latter are independent of each other and the integrals $H_{k\nu}$, with $k = 2, \dots, m-2$ (two major coefficients in $\widetilde{\mathfrak{I}}'_m(s)$ are again annulators of the bracket (4.14)).

For each fixed $s \in \mathbb{C} \cup \infty$ the restriction of the Poisson bracket $s\{\,,\}_1 - \{\,,\}_0$ to a regular symplectic leave, a joint level manifold $\mathcal{F}$ of the functions $\mathcal{P}_k(P)$, $\widetilde{\mathfrak{I}}_k(s, Z, P)$, $k = 2, \dots, 2[m/2]$ is nondegenerate. Since $\dim \mathcal{F} = m(m-1) - 2[m/2]$, the dimension of the general invariant tori equals $m(m-1)/2 - [m/2]$.

As a result of our analysis, we arrive at the unexpected conclusion: *the Steklov–Liapunov systems are the generalizations of Frahm–Manakov systems on* so$\,(m)$. In this connection the following question arises: are there such integrable generalizations of the latter systems that give rise to some integrable generalizations of the Steklov–Liapunov systems? It turns out that there exists a whole series of such generalizations. For example, consider a "hybrid" system on the phase space $(Z, P, \gamma)$, $Z, P \in \mathrm{so}\,(m)$, $\gamma \in \mathbb{R}^m$ defined by the following Lax pair with $(m+1) \times (m+1)$ matrices

$$\dot{L}(t) = [L(t), A(t)], \qquad t \in \mathbb{C}, \tag{4.17}$$

$$L(t) = \begin{pmatrix} W(t) & G(t) \\ -G^T(t) & 0 \end{pmatrix}, \qquad W(t) \in \mathrm{so}\,(m), \quad G(t) \in \mathbb{R}^m,$$

$$W_{ij}(t) = \frac{\sqrt{\Phi(t)}}{\sqrt{(t-b_i)(t-b_j)}}(Z + tP)_{ij}, \quad G_i(t) = \frac{\sqrt{\Phi(t)}}{\sqrt{t-b_i}}\gamma_i,$$
$$i, j = 1, \dots, m.$$

Putting in (4.17)

$$A(t) = \begin{pmatrix} \hat{W}(t) & \hat{G}(t) \\ -\hat{G}^T(t) & 0 \end{pmatrix}, \qquad \hat{W}(t) \in \mathrm{so}\,(m), \quad \hat{G}(t) \in \mathbb{R}^m,$$
$$\hat{W}_{ij}(t) = \sqrt{(t-b_i)(t-b_j)}P_{ij}, \quad \hat{G}_i(t) = \sqrt{t-b_i}\gamma_i,$$

and introducing the matrix $\Gamma_{ij} = \gamma_i\gamma_j$, we arrive at the following integrable equations

$$\begin{aligned} \dot{Z} &= ZPB - BPZ + [\Gamma, B], \\ \dot{P} &= [P, PB + BP] + [P, Z], \\ \dot{\Gamma} &= [\Gamma, Z] + \Gamma PB - BP\Gamma, \end{aligned} \tag{4.18}$$

which for $\Gamma \to 0$ are reduced to the generalized Liapunov system (4.12) with $r = 0$, and for $P \to 0$ they turn to the equations

$$\dot{Z} = [\Gamma, B], \quad \dot{\Gamma} = [\Gamma, Z],$$

or, in scalar form,

$$\dot{Z}_{ij} = (b_j - b_i)\gamma_i\gamma_j, \quad \dot{\gamma}_i = -\sum_{j=1}^{m} Z_{ij}\gamma_j,$$

which represents an integrable system on the algebra $e(m) = \{Z, \gamma\}$, namely, the multidimensional generalization of the second Clebsch case in hydrodynamics (see [4, 12]).

As in the previous examples, apart from (4.18) one may generate a whole hierarchy of higher systems on $\mathrm{so}(m) + \mathrm{so}(m) + \mathbb{R}^m$ which have the same first integrals. We only note that for the "classical" case $m = 3$ their general invariant tori are three-dimensional.

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