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REAL ALGEBRAIC CURVES OF BIDEGREE (5,5) ON THE QUADRIC ELLIPSOID

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In this paper, the topological classification of nonseparating (respectively, separating) real algebraic nonsingular $(M-i)$-curves of bidegree $(5,5)$ on the quadric ellipsoid is completed. In particular, it is shown that previously known restrictions form a complete system for this bidegree. Therefore, the main part of the paper concerns the construction of real algebraic curves. The strategy is to reduce the problem of construction of curves on the quadric ellipsoid to construction of curves on the second Hirzebruch surface by degenerating the quadric ellipsoid to the quadratic cone. Next, various classical construction methods on toric surfaces are combined, such as *dessins d'enfants* and Viro's patchworking method.

§1. Introduction

A real algebraic variety is a real algebraic compact complex variety X equipped with an antiholomorphic involution $\sigma : X \to X$, called real structure on X. The real part $\mathbb{R}X$ of X is the set of points fixed by σ. A subject of interest about real algebraic varieties is the study of the topology of their real part and it is related to Hilbert's classical 16th problem whose first part is about the classification of the oval arrangements of nonsingular real algebraic plane curves. In this paper, we are interested in the study of the topology of real algebraic curves in the quadric ellipsoid. We give some definitions and notation in Subsection 1.1, we state the main results in Subsection 1.2, and explain the structure of the paper in Subsection 1.3.

First of all, let us present some general definitions and known results about real algebraic curves.

Let (A, σ) be a compact nonsingular real algebraic curve and let l denote the number of connected components of the set of real points of A. The following inequality was proved by Harnack for real algebraic plane curves and, after, generalized by Klein.

Ключевые слова: real algebraic variety, Hilbert's 16th problem, topology of real algebraic curves, quadric ellipsoid.

Proposition 1.1 (see [Har76, Kle22]). *The number l is bounded by $g+1$, where g is the genus of A.*

Definition 1.2. If $l = g+1$, we say that A is an *M-curve* or a *maximal* curve. Otherwise, for $0 \leqslant l \leqslant g$, we say that A is an *$(M-i)$-curve* with $i = g+1-l$.

Definition 1.3 (see [Kle22]). If $A \setminus \mathbb{R}A$ is connected, we say that A is of *type* II or *nonseparating*, otherwise of *type* I or *separating*.

Looking at the real part of the curve and its position with respect to its complexification gives us information about l and *vice versa*. For example, we know that if A is maximal, then A is of type I. Or, if A is of type I, then l has the parity of $g+1$.

Definition 1.4 (see [Rok72]). If A is of type I, the two halves of $A \setminus \mathbb{R}A$ induce two opposite orientations on $\mathbb{R}A$ called the *complex orientations* of the curve.

Variations of the 16th Hilbert problem propose to study topological classifications of real algebraic curves up to various equivalence relations (up to homeomorphism, respectively, isotopy, respectively, rigid isotopy). Let (X, σ) be a real algebraic surface and let A and B be real algebraic curves realizing a certain class α in $H_2(X; \mathbb{Z})$.

Definition 1.5. We say that the pairs $(\mathbb{R}X, \mathbb{R}A)$ and $(\mathbb{R}X, \mathbb{R}B)$ are homeomorphic if there exists a homeomorphism $f\colon \mathbb{R}X \to \mathbb{R}X$ such that $f(\mathbb{R}A) = \mathbb{R}B$.

Definition 1.6. We say that the pairs $(\mathbb{R}X, \mathbb{R}A)$ and $(\mathbb{R}X, \mathbb{R}B)$ are isotopic if the pairs $(\mathbb{R}X, \mathbb{R}A)$ and $(\mathbb{R}X, \mathbb{R}B)$ are homeomorphic via a homeomorphism f and there exists a 1-parameter family of homeomorphisms $\phi_t\colon \mathbb{R}X \to \mathbb{R}X$, $t \in [0,1]$, such that $\phi_0 = id$ and $\phi_1 = f$.

Definition 1.7. We say that the pairs $(\mathbb{R}X, \mathbb{R}A)$ and $(\mathbb{R}X, \mathbb{R}B)$ are rigid isotopic if the pairs $(\mathbb{R}X, \mathbb{R}A)$ and $(\mathbb{R}X, \mathbb{R}B)$ are isotopic via a homeomorphism f and a 1-parameter family of homeomorphisms ϕ_t such that $\phi_t(\mathbb{R}A)$ is the set of real points of a real algebraic curve of class α in $H_2(X; \mathbb{Z})$, for all $t \in [0,1]$.

1.1. Quadric ellipsoid. Let X be $\mathbb{C}P^1 \times \mathbb{C}P^1$, equipped with the anti-holomorphic involution

$$\begin{array}{rccc} \sigma\colon & X & \longrightarrow & X \\ & (x,y) & \longmapsto & (\overline{y}, \overline{x}), \end{array}$$

where $x = [x_0 : x_1]$ and $y = [y_0 : y_1]$ are in $\mathbb{C}P^1$ and $\overline{x} = [\overline{x_0} : \overline{x_1}]$ and $\overline{y} = [\overline{y_0} : \overline{y_1}]$ are, respectively, the images of x and y via the standard complex conjugation on $\mathbb{C}P^1$. The real part of X is homeomorphic to S^2. It is well known that X is isomorphic to the quadric ellipsoid in $\mathbb{C}P^3$, in the category of real algebraic varieties. A nonsingular real algebraic curve A on X is defined by a bihomogeneous polynomial of bidegree (d,d)

$$P(x_0,x_1,y_0,y_1)=\sum_{i,j=1}^{d,d} a_{i,j}x_1^i x_0^{d-i}y_1^j y_0^{d-j}$$

where d is a positive integer and the coefficients satisfy $a_{i,j}=\overline{a_{j,i}}$. Let $\mathbb{R}A$ be the set of real points of A. The connected components of $\mathbb{R}A$ are called *ovals*. We are interested in the classification of the oval arrangements of nonsingular real algebraic curves on X. Thanks to the adjunction formula, for a nonsingular real algebraic curve A of bidegree (d,d) on X we have $g=(d-1)^2$. It follows that the number of ovals of $\mathbb{R}A$ is bounded by $(d-1)^2+1$ (Proposition 1.1).

The rigid isotopy classification of nonsingular real algebraic curves of bidegree (d,d) is known for $d<5$ (see [GS80]). For a more general idea on topological classifications of real curves on quadric surfaces, see [Zvo91, NS05a, NS07, DZ99, Nik85, Mik94] and [DK00]. Moreover, Mikhalkin in [Mik94] gave a partial classification, up to homeomorphism, of real algebraic M-curves of bidegree $(5,5)$ on the quadric ellipsoid. In this paper, for all $0\leqslant l\leqslant 17$, we give the complete classification, up to homeomorphism, of the topological types realized by the pair $(\mathbb{R}X,\mathbb{R}A)$, where A is a type II or a type I nonsingular real algebraic $(M-i)$-curve of bidegree $(5,5)$ on X, with $i=g+1-l$ (Theorems 1.9, 1.10, 1.11, and 1.12).

Before stating the main results of this paper, we introduce some notation. Given a collection $\bigsqcup_{i=1,\dots l}B_i$ of l disjoint circles embedded in S^2, we encode the topological pair $\left(S^2,\bigsqcup_{i=1,\dots l}B_i\right)$ as follows. Let p be any point in $S^2\setminus\bigsqcup_{i=1,\dots l}B_i$. Each oval B_i bounds two nonhomeomorphic parts in $S^2\setminus\{p\}$. We call the *interior* the part homeomorphic to a disk and the *exterior* the other one. For each pair of ovals, if one is in the interior of the other we talk about an *injective pair*, otherwise of a *noninjective* one. We shall adopt the following notation to encode a given topological pair $\left(S^2\setminus\{p\},\bigsqcup_{i=1,\dots l}B_i\right)$. An empty union of ovals is denoted by 0. We say that a union of l ovals realizes l if there are no injective pairs. The symbol $\langle\mathcal{S}\rangle$ denotes the disjoint union of a collection of ovals realizing $\mathcal{S}$ and an oval forming an injective pair with each oval of the collection. Finally, the disjoint union of any two collections of ovals realizing, respectively, $\mathcal{S}'$ and $\mathcal{S}''$ in $S^2\setminus\{p\}$, is denoted by $\mathcal{S}'\sqcup\mathcal{S}''$ if none of the ovals of one collection forms an injective pair with ovals of the other one.

We say that the pair $\left(S^2,\bigsqcup_{i=1,\dots,l}B_i\right)$ realizes $\mathcal{S}$ if there exists a point $p\in S^2\setminus\bigsqcup_{i=1,\dots,l}B_i$ such that $\left(S^2\setminus\{p\},\bigsqcup_{i=1,\dots,l}B_i\right)$ realizes $\mathcal{S}$. As an example, we have depicted in a) of Figure 1 an arrangement of 11 ovals in S^2 projected on a plane from some point $p \in S^2$. The pair $\left(S^2, \bigsqcup_{i=1,\dots,11} B_i\right)$ realizes $1 \sqcup \langle 2\rangle \sqcup \langle 1\rangle \sqcup \langle\langle 3\rangle\rangle$.

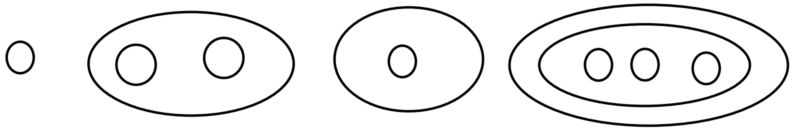

Figure 1. Example of an arrangement of embedded circles in $S^2 \setminus \{p\}$.

Definition 1.8. An arrangement of disjoint circles embedded in $\mathbb{R}X$ is called a real scheme. Let A be a real algebraic curve of bidegree (d,d) on X, we say that A has real scheme $\mathcal{S}$ if the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $\mathcal{S}$.

1.2. Main results. We show that the previously known results on the topological obstructions for bidegree (d,d) real algebraic curves of type II or of type I form a complete system of restrictions for bidegree $(5,5)$. Indeed, all real schemes listed in Theorems 1.9, 1.10, 1.11, and 1.12 are those nonprohibited by Propositions 1.1, 2.1, and 2.4 and Theorem 2.3; moreover, such real schemes are all realizable by type II or type I real algebraic $(M-i)$-curves of bidegree $(5,5)$ in X. The main results of this paper are Theorems 1.9, 1.10, 1.11, and 1.12.

Theorem 1.9 (M-curves)**.** *Let A be a nonsingular real algebraic M-curve of bidegree $(5,5)$ on X. Then the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:*

$$\alpha \sqcup \langle\beta\rangle \sqcup \langle\gamma\rangle,\ \alpha \equiv 1 \pmod 4,\ \textit{with } \alpha+\beta+\gamma = 15.$$

Moreover, all such real schemes are realizable by nonsingular real algebraic M-curves of bidegree $(5,5)$.

Proof. The real schemes listed above are those nonprohibited by Propositions 1.1, 2.1 and (1) of Theorem 2.3. In Propositions 4.8, 4.9, and 4.10, we construct nonsingular real algebraic M-curves of bidegree $(5,5)$ on X realizing all real schemes listed above. □

Theorem 1.10 ($(M-1)$ curves)**.** *Let A be a nonsingular real algebraic $(M-1)$-curve of bidegree $(5,5)$ on X. Then the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:*

$$\alpha \sqcup \langle\beta\rangle \sqcup \langle\gamma\rangle,\ \alpha \equiv 0 \textit{ or } 1 \pmod 4,\ \textit{with } \alpha+\beta+\gamma = 14.$$

Moreover, all such real schemes are realizable by nonsingular real algebraic $(M-1)$-curves of bidegree $(5,5)$.

Proof. The real schemes listed above are those nonprohibited by Propositions 1.1 and 2.1, and (2) of Theorem 2.3. In Propositions 4.8, 4.9, and 4.10, we construct nonsingular real algebraic $(M-1)$-curves of bidegree $(5,5)$ on X realizing all real schemes listed above. □

Theorem 1.11 ($(M-2)$-curves). *Let A be a nonsingular real algebraic $(M-2)$-curve of bidegree $(5,5)$ on X. If A is of type* I, *then the pair* $(\mathbb{R}X, \mathbb{R}A)$ *realizes one of the following real schemes*:

$$(1)\ \alpha \ \sqcup\ \langle\beta\rangle\ \sqcup\ \langle\gamma\rangle,\ \alpha \equiv 0 \pmod 2,\ \textit{with}\ \alpha+\beta+\gamma = 13;$$

if A is of type II, *then one of the following schemes:*

$$(2)\ \alpha\ \sqcup\ \langle\beta\rangle\ \sqcup\ \langle\gamma\rangle,\quad \alpha \not\equiv 2 \pmod 4,\quad \textit{with}\quad \alpha+\beta+\gamma = 13.$$

Moreover, all the real schemes in (1) *and* (2) *are realizable by nonsingular real algebraic $(M-2)$-curves of bidegree* $(5,5)$, *respectively, of type* I *and* II.

Proof. The real schemes listed above are those nonprohibited by Propositions 1.1 and 2.1, and (3) of Theorem 2.3. In Propositions 4.8 and 4.9, we construct nonsingular real algebraic $(M-2)$-curves of bidegree $(5,5)$ on X realizing all real schemes listed above. □

Theorem 1.12 (Type I and II curves). *Let A be a nonsingular real algebraic $(M-i)$-curve of bidegree* $(5,5)$ *on X, where $3 \leqslant i \leqslant 17$. If A is of type* II, *then the pair* $(\mathbb{R}X, \mathbb{R}A)$ *realizes one of the following real schemes*:

$$(1)\ 0\ \textit{and}\ 1,$$

$$(2)\ \alpha\ \sqcup\ \langle\beta\rangle\ \sqcup\ \langle\gamma\rangle,\quad \textit{with}\quad \alpha+\beta+\gamma = 17-(i-2);$$

if A is of type I, *then $i = 4, 6, 8, 10, 12$ and the pair* $(\mathbb{R}X, \mathbb{R}A)$ *realizes one of the following real schemes*:

$$(3)\ \langle\langle\langle\langle 1\rangle\rangle\rangle\rangle,$$

$$(4)\ \alpha\ \sqcup\ \langle\beta\rangle\ \sqcup\ \langle\gamma\rangle,\quad \textit{with}\quad \alpha\equiv 0 \pmod 2\ \textit{when}\ \alpha+\beta+\gamma = 5, 9,$$

$$\textit{and with}\ \alpha \equiv 1 \pmod 2\ \textit{when}\ \alpha+\beta+\gamma = 7, 11.$$

Moreover, the real schemes in (1), (2) *and* (3), (4) *are realizable by nonsingular real algebraic curves of bidegree* $(5,5)$, *respectively, of type* II *and* I.

Proof. The real schemes listed above are those nonprohibited by Propositions 1.1, 2.1 and 2.4, and (4) of Theorem 2.3. In Propositions 4.8, 4.9, and 4.10, we construct nonsingular real algebraic $(M-i)$-curves of bidegree $(5,5)$ on X realizing all real schemes listed above, with $3 \leqslant i \leqslant 17$. □

Remark 1.13. We claim that the classification up to homeomorphism of the pairs $(\mathbb{R}X, \mathbb{R}A)$ is equivalent to the classification up to isotopy, where $A \subset X$ is a nonsingular real algebraic curve of bidegree (d, d). In order to prove the claim, it suffices to show that $(\mathbb{R}X, \mathbb{R}A)$ is isotopic to $(\mathbb{R}X, g(\mathbb{R}A))$, where $g\colon \mathbb{R}X \to \mathbb{R}X$ is the homeomorphism reflecting $\mathbb{R}X$ with respect to an oval O of $\mathbb{R}A$. For every two distinct points p_1, p_2 on O, there exists a pencil Π_{p_1,p_2} of hyperplanes on X passing through p_1, p_2. Choose one more point p_3 on O and another one p_4 on $\mathbb{R}X \setminus O$, then the two hyperplanes of Π_{p_1,p_2} passing, respectively, through p_3 and p_4, intersect along a line L_{p_1,p_2}. Finally, take the isotopy f that rotates $\mathbb{R}X$ through 180° with respect to L_{p_1,p_2} and sends the pair $(\mathbb{R}X, \mathbb{R}A)$ to the pair $(\mathbb{R}X, g(\mathbb{R}A))$.

1.3. Structure of the paper. In §2, we present previously known results that give topological restrictions on type II, respectively, type I, real algebraic $(M - i)$-curves of bidegree (d, d) on X. The content of this paper is the proof that such system of restrictions is complete for bidegree $(5, 5)$. Indeed, the main part of the paper concerns the construction of type II, respectively, type I, real algebraic $(M - i)$-curves of bidegree $(5, 5)$ in X realizing all real schemes that are not prohibited by the restrictions in §2.

In Subsection 3.2, we explain how to construct a real algebraic curve of bidegree (d, d) on the quadric ellipsoid in $\mathbb{C}P^3$ with topology prescribed by the topology of a real algebraic curve of bidegree $(d, 0)$ constructed on the second Hirzebruch surface. In the rest of §3, we introduce Hirzebruch surfaces (Subsection 3.1) and we give some construction tools we have on such surfaces:

- particular birational transformations (Subsection 3.1);
- Orevkov's method via *dessins d'enfants* (Subsection 3.3).

In §4 we end the proof of Theorems 1.9, 1.10, 1.11, and 1.12:

- in Subsection 4.1, we realize some intermediate constructions on Hirzebruch surfaces using the construction tools of §3;
- in Subsection 4.2, using Viro's patchworking method ([Vir06]) and a variant of it developed by Shustin ([Shu05]), we finally construct real algebraic curves of bidegree $(5, 5)$ on X realizing all real schemes listed in Theorems 1.9, 1.10, 1.11, and 1.12.

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§2. Restrictions

2.1. Restriction on the depth of nests. A collection N_h of h disjoint embedded circles in S^2 is called a *nest of depth h* if any connected component of $S^2 \setminus N_h$ is either a disk or an annulus. Two nests are said to be disjoint if each of them lies on one of the disks bounded by the other.

Proposition 2.1 (see [DK00, Proposition 4.9.2]). *Let A be a nonsingular real algebraic curve of bidegree (d, d) on X. Then the total number of ovals in any collection of three pairwise disjoint nests of $\mathbb{R}A$ does not exceed d.*

Proposition 2.1 implies in particular that the maximal depth for a nest of such a curve A is d. Furthermore, it is well known that if A is of type I and has d ovals, it has a nest of maximal depth d (see Proposition 2.4).

Combining Proposition 1.1 with Proposition 2.1, one obtains the following immediate result.

Corollary 2.2. *Let A be a nonsingular real algebraic curve of bidigree $(5, 5)$ on X. Then the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the following real schemes:*

- 0 *and* 1,
- $\alpha \sqcup \langle\beta\rangle \sqcup \langle\gamma\rangle$, *for* $0 \leqslant \alpha + \beta + \gamma \leqslant 15$,
- $\langle\langle\langle\langle 1\rangle\rangle\rangle\rangle$.

2.2. Congruences and a complex orientation formula on the ellipsoid. As an application of Guillou–Marin congruences [GM77] (which are a generalization of Rokhlin's congruences), Mikhalkin proved the following theorem.

Theorem 2.3 (see [Mik94, Theorem 1]). *Let A be a nonsingular real algebraic $(M - i)$-curve of bidegree (d, d) on X, with d odd. Let B be a disjoint union of connected components of $\mathbb{R}X \setminus \mathbb{R}A$ such that $\mathbb{R}A$ bounds B.*

(1) *If A is an M-curve, then*

$$\chi(B) \equiv \frac{d^2 + 1}{2} \pmod 8.$$

(2) *If A is an $(M - 1)$-curve, then*

$$\chi(B) \equiv \frac{d^2 + 1}{2} \pm 1 \pmod 8.$$

(3) *If A is an $(M - 2)$-curve and*

$$\chi(B) \equiv \frac{d^2 - 7}{2} \pmod 8,$$

then A is of type I.

(4) *If A is of type* I, *then* $\chi(B) \equiv 1 \pmod 4$.

Let A be a nonsingular real algebraic curve of type I on X. Fix a complex orientation on $\mathbb{R}A$. Since any pair of ovals of $\mathbb{R}A$ bounds an annulus in $\mathbb{R}X$, we distinguish two types of pairs: denote by Π_- (respectively, Π_+) the number of pairs of ovals realizing the same (respectively, different) first homology class of the corresponding annulus. As an application of the generalizations of Rokhlin's formula of complex orientations, Zvonilov in [Zvo83] gave a complex orientation formula for type I nonsingular real algebraic curves on X. This formula depends on the choice of an auxiliary point in $\mathbb{R}X \setminus \mathbb{R}A$. Afterwards, Orekvov in [Ore07] reformulated it with no dependence on the choice of an auxiliary point.

Proposition 2.4 (see [Zvo83] and [Ore07, Proposition 1.2]). *Let A be a nonsingular real algebraic type* I *curve of bidegree (d,d) on X. Denoting by l the number of connected components of $\mathbb{R}A$, one has the following complex orientation formula*:

$$2(\Pi_+ - \Pi_-) = l - d^2. \tag{1}$$

Corollary 2.5 (see [Ore07, Proposition 1.3]). *Let A be a nonsingular real algebraic type* I *curve of bidegree (d,d) on X. Then $\mathbb{R}A$ has at least d connected components. Furthermore, in the case where $\mathbb{R}A$ has d connected components, it consists of a nest of maximal depth d.*

Corollary 2.2 and Theorem 2.3 give a complete system of restrictions for real schemes of nonsingular real algebraic curves of bidegree $(5,5)$ on X. Moreover, Theorem 2.3 and Proposition 2.4 allow us to give even finer restrictions on the real schemes, listed in Corollary 2.2, that may be realized by type I (respectively, type II) nonsingular real algebraic curves of bidegree $(5,5)$ on X. Therefore, given a nonsingular real algebraic curve A of bidegree $(5,5)$ on X, the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the real schemes listed in Theorems 1.9, 1.10, 1.11, and 1.12.

In the next sections we pass to the construction part of the classification.

§3. Construction tools

3.1. Hirzebruch surfaces. A Hirzebruch surface is a compact complex surface that admits a holomorphic fibration over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^1$. Every Hirzebruch surface is biholomorphic to exactly one of the surfaces $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$ for $n \geqslant 0$. The surface Σ_n admits a natural fibration

$$\pi_n \colon \Sigma_n \to \mathbb{C}P^1$$

with fiber $\mathbb{C}P^1 =: F_n$. Denote by B_n, respectively, E_n, the section $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \{0\})$, respectively, $\mathbb{P}(\{0\} \oplus \mathbb{C})$. The self-intersection of B_n (respectively, E_n and F_n) is n (respectively, $-n$ and 0). When $n \geqslant 1$, the exceptional divisor E_n

determines uniquely the Hirzebruch surface because it is the only irreducible and reduced algebraic curve in Σ_n with negative self-intersection.

For example $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$. The Hirzebruch surface Σ_1 is the complex projective plane blown-up at a point, and Σ_2 is the quadratic cone with equation $Q_0 : X^2 + Y^2 - Z^2 = 0$ blown-up at the node in $\mathbb{C}P^3$. The fibration of Σ_2 (respectively, of Σ_1) is the extension of the projection from the blown-up point to a hyperplane section (respectively, to a line) which does not pass through the blown-up point.

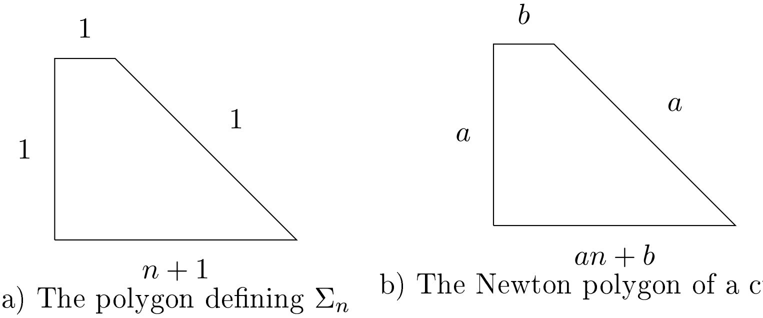
The group $H_2(\Sigma_n;\mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and is generated by the classes $[B_n]$ and $[F_n]$. An algebraic curve C in Σ_n is said to be of bidegree (a, b) if it realizes the homology class $a[B_n] + b[F_n]$ in $H_2(\Sigma_n;\mathbb{Z})$. Note that $[E_n] = [B_n] - n[F_n]$ in $H_2(\Sigma_n;\mathbb{Z})$. An algebraic curve of bidegree $(3, 0)$ on Σ_n is called a *trigonal curve.*

We can obtain Σ_{n+1} from Σ_n via a birational transformation

$$\beta_n^p : \Sigma_n - - \to \Sigma_{n+1}$$

that is the composition of a blow-up at a point $p \in E_n \subset \Sigma_n$ and a blow-down of the strict transform of the fiber $\pi_n^{-1}(\pi_n(p))$.

The surface Σ_n is also the projective toric surface that corresponds to the polygon of vertices $(0, 0), (0, 1), (1, 1), (n + 1, 0)$, depicted in Figure 2a) where the number labeling an edge corresponds to its integer length. The Newton polygon of an algebraic curve C of bidegree (a, b) on Σ_n, lies inside the trapezoid with vertices $(0, 0), (0, a), (b, a), (an+b, 0)$ as in Figure 2b). The surface Σ_n is canonically endowed with a real structure induced by the standard complex conjugation in $(\mathbb{C}^*)^2$. For this real structure, the real part of Σ_n, denoted by $\mathbb{R}\Sigma_n$, is a torus if n is even and a Klein bottle if n is odd. We will depict $\mathbb{R}\Sigma_n$ as a quadrangle whose opposite sides are identified in a suitable way. Moreover, the horizontal sides will represent $\mathbb{R}E_n$.



a) The polygon defining Σ_n b) The Newton polygon of a curve of bidegree (a, b) on Σ_n

Figure 2

The restriction of π_n to $\mathbb{R}\Sigma_n$ defines an S^1-bundle over S^1, which we denote by $\mathcal{L}$. We are interested in the isotopy types with respect to $\mathcal{L}$ of real algebraic curves in $\mathbb{R}\Sigma_n$.

Definition 3.1.
- An arrangement η of circles and points immersed in $\mathbb{R}\Sigma_n$ is called a real scheme. We say that a real algebraic curve $C \in \mathbb{R}\Sigma_n$ has real scheme η if the pair $(\mathbb{R}\Sigma_n, \mathbb{R}C)$ realizes η.
- Two arrangements of immersed circles and points in $\mathbb{R}\Sigma_n$ are *$\mathcal{L}$-isotopic* if there exists an isotopy of $\mathbb{R}\Sigma_n$ that brings one arrangement to the other, each line of $\mathcal{L}$ to another line of $\mathcal{L}$, and whose restriction to $\mathbb{R}E_n$ is an isotopy of $\mathbb{R}E_n$.
- An arrangement of circles and points immersed in $\mathbb{R}\Sigma_n$ up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_n$ is called an *$\mathcal{L}$-scheme*.
- An $\mathcal{L}$-scheme is *realizable* by a real algebraic curve of bidegree (a, b) in Σ_n if there exists such a curve whose real part is $\mathcal{L}$-isotopic to the arrangement of circles and points in $\mathbb{R}\Sigma_n$.
- A *trigonal* $\mathcal{L}$-scheme is an $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ that intersects each fiber at 1 or 3 real points counted with multiplicities and that does not intersect $\mathbb{R}E_n$.
- A trigonal $\mathcal{L}$-scheme η in $\mathbb{R}\Sigma_n$ is *hyperbolic* if it intersects each fiber at 3 real points counted with multiplicities.

3.2. The quadric ellipsoid and the second Hirzebruch surface. We explain in this section how to construct a real algebraic curve of bidegree (d, d) on the quadric ellipsoid in $\mathbb{C}P^3$ with topology prescribed by the topology of a real algebraic curve of bidegree $(d, 0)$ in Σ_2, endowed with the canonical real structure. The advantage of working with the real toric surface Σ_2 is that, on such a surface, various construction tools are available, such as: the birational transformations presented in Subsection 3.1, Orevkov's construction method via *dessins d'enfants* (Subsection 3.3) and Viro's patchworking method (see for example [Vir06]).

Let C be a real algebraic curve of degree $(d, 0)$ in Σ_2. Let η be the real scheme realized by $\mathbb{R}C$ in $\mathbb{R}\Sigma_2$ (a torus). Now, cut $\mathbb{R}\Sigma_2$ along $\mathbb{R}E_2$, as depicted in a) of Figure 3, and glue two disks D_1, D_2 as depicted in b) of Figure 3. By this construction we obtain a 2-sphere S^2. Moreover, from the arrangement of the triplet $(\mathbb{R}\Sigma_2, \mathbb{R}E_2, \eta)$ we obtain an arrangement B of embedded circles in S^2. As an example, look at Figure 4 where we obtain the arrangement $1 \sqcup \langle 1 \rangle$ in S^2.

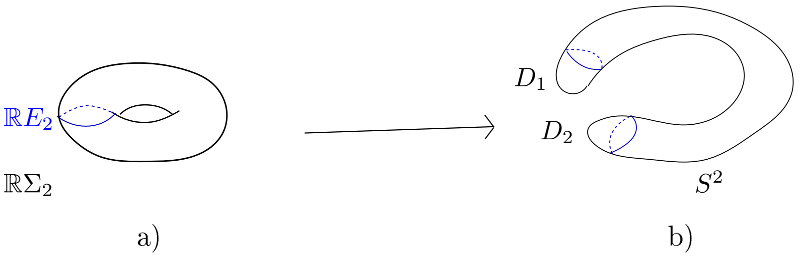


Figure 3. From a torus to a 2-sphere

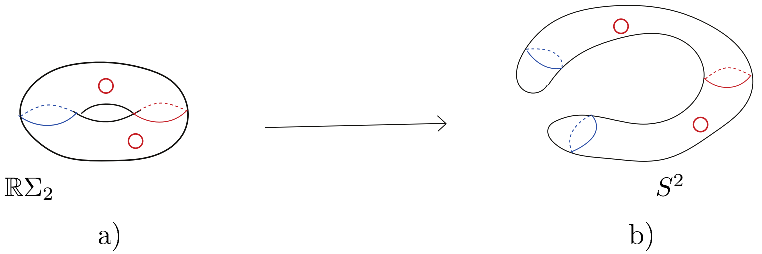


Figure 4. Example: from an arrangement of embedded circles in $\mathbb{R}\Sigma_2$ to an arrangement in S^2

Proposition 3.2. *Let C be a nonsingular real algebraic curve of bidegree $(d, 0)$ in Σ_2. Let B be the real scheme on the sphere S^2 obtained from the pair $(\mathbb{R}\Sigma_2, \mathbb{R}C)$ by the construction above. Then, B is realizable by a nonsingular real curve of bidegree (d, d) on the quadric ellipsoid in $\mathbb{C}P^3$.*

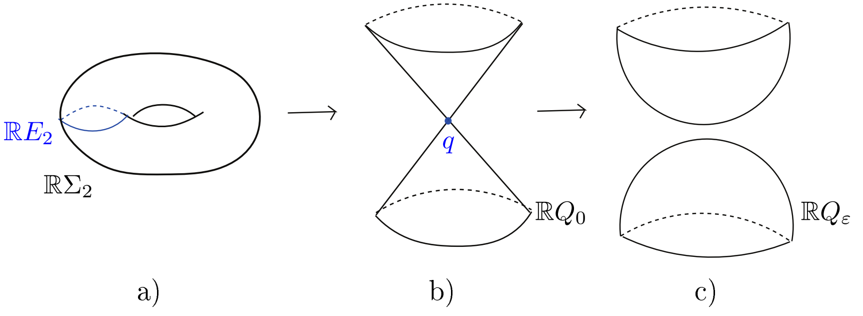


Figure 5

Proof. Let $[X : Y : Z : W]$ be the homogeneous coordinates in the 3-dimensional complex projective space. Let Q_0 be the quadratic cone with equation $X^2 + Y^2 - Z^2 = 0$ in $\mathbb{C}P^3$. Recall that we obtain Σ_2 blowing-up Q_0 at the point $q = [0 : 0 : 0 : 1]$. The image of C via the blow-down is a real algebraic curve $\widetilde{C}$ of degree $2d$ in $\mathbb{C}P^3$. Since the dimension of the space of curves of bidegree $(d,0)$ in Σ_2 is equal to the dimension of the space of complete intersections of the surfaces of degree d in $\mathbb{C}P^3$ with Q_0, the curve $\widetilde{C}$ is the intersection of a nonsingular real algebraic surface S_d of degree d (not passing through the node of Q_0) and Q_0. Observe that we can perturb Q_0 to the quadric ellipsoid Q_ε of equation $X^2 + Y^2 - Z^2 = -\varepsilon W^2$, where $\varepsilon > 0$ (see Figure 5). Since a real algebraic curve of bidegree (d,d) on Q_ε is the intersection of the quadric ellipsoid and a surface of degree d, the intersection of S_d and Q_ε is a real algebraic curve A of bidegree (d,d). Moreover, the pair $(\mathbb{R}Q_\varepsilon, \mathbb{R}A)$ realizes B. □

3.3. *Dessins d'enfants.* Orevkov in [Ore03] formulated the existence of real algebraic trigonal curves realizing a given trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ in terms of the existence of a real rational graph on $\mathbb{C}P^1$. Later on, Degtyarev, Itenberg, and Zvonilov in [DIZ14] gave a general way to determine if such real algebraic curves are of type I or II.

In Subsection 4.1, we exploit this construction technique. Therefore, we present here some results of [Ore03] and [DIZ14].

Definition 3.3. Let n be a fixed positive integer. We say that a graph Γ is a *real trigonal graph of degree* n if

- it is a finite oriented connected graph embedded in $\mathbb{C}P^1$ and invariant under the standard complex conjugation of $\mathbb{C}P^1$;
- it is decorated with the following additional structure:
  - every edge of Γ is colored solid, bold, or dotted;
  - every vertex of Γ is $\bullet$, $\circ$, $\times$ (said *essential vertices*) or monochrome

  and satisfies the following conditions:
  1. any vertex is incident to an even number of edges; moreover, any $\circ$-vertex (respectively, $\bullet$-vertex) is incident to a multiple of 4 (respectively, 6) number of edges;
  2. for each type of essential vertices, the total sum of edges incident to the vertices of a certain fixed type is $12n$;
  3. the orientations of the edges of Γ form an orientation of $\partial(\mathbb{C}P^1 \setminus \Gamma)$ that is compatible with an orientation of $\mathbb{C}P^1 \setminus \Gamma$ (see Figure 8);
  4. all edges incident to a monochrome vertex have the same color;
  5. $\times$-vertices are incident to incoming solid edges and outgoing dotted edges;
  6. $\circ$-vertices are incident to incoming dotted edges and outgoing bold edges;
  7. $\bullet$-vertices are incident to incoming bold edges and outgoing solid edges.

Let n be a positive integer and let $c(x,y) = y^3 + b_2(x)y + b_3(x)$ be a real polynomial, where $b_i(x)$ has degree in in x. By a suitable change of coordinates in Σ_n, any trigonal curve C in Σ_n can be put into this form. Denote by $\Delta = -4b_2^3 + 27b_3^2$ the discriminant of $c(x,y)$ with respect to the variable y. The knowledge of the arrangement of the real roots of the real polynomials $\Delta = -4b_2^3 + 27b_3^2$, $27b_3^2$ and $-4b_2^3$ in $\mathbb{R}\Sigma_n$ allows us to recover the trigonal $\mathcal{L}$-scheme realized by C in $\mathbb{R}\Sigma_n$. Let $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ be the homogenized discriminant, i.e., the rational function defined by $f := \frac{\Delta}{27b_3^2}$. Orevkov's method allows us to construct real polynomials $c(x,y)$ that have prescribed arrangements of the real roots and the construction is based on consideration of the arrangement of the graph given by $f^{-1}(\mathbb{R}P^1)$ with the coloring and orientation induced by those of $\mathbb{R}P^1$ as depicted in a) of Figure 6. In this section, we only give an algorithmic way to encode any trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ into a colored oriented graph on $\mathbb{R}P^1 \subset \mathbb{C}P^1$ looking merely, at the intersections of the fibers of $\mathcal{L}$ with η; for details, see [Bru07, Ore03].

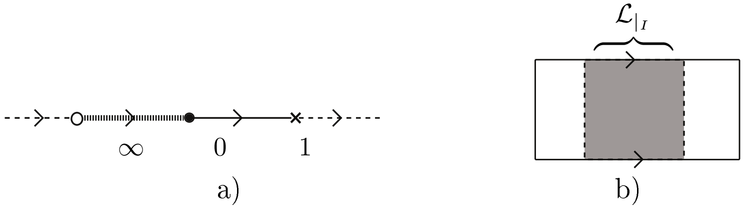


Figure 6. a) Colored oriented $\mathbb{R}P^1$. b) $\mathcal{L}_{|_I}$ in $\mathbb{R}\Sigma_n$.

Definition 3.4. Let η be a trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$. For any fixed interval of points $I := \{(x,\overline{y}) : \overline{y} \in \mathbb{R},\ x \in [a, a+b] \subset \mathbb{R},\ a \neq b\} \subset \mathbb{R}\Sigma_n$, we denote with $\mathcal{L}_{|_I}$ the fibers of $\mathcal{L}$ containing the points of I (see b) of Figure 6). Thanks to $\pi_{n|_{\mathbb{R}\Sigma_n}}$, we can encode η into a colored oriented graph $\overline{\Gamma}$ on $\mathbb{R}P^1 \subset \mathbb{C}P^1$ as follows (in Figure 7 the dashed lines denote fibers of $\mathcal{L}$).

(1) With each fiber of $\mathcal{L}$ intersecting η at two points, we associate a $\times$-vertex on $\mathbb{R}P^1$.
(2) Given an interval I, let F_1, F_2 be two fibers of $\mathcal{L}_{|_I}$ intersecting η at two points such that η, up to $\mathcal{L}$-isotopy, is locally as depicted in b) or c) of Figure 7. Let F_3 be another fiber between F_1, F_2. Then, we associate a $\circ$-vertex on $\mathbb{R}P^1$ with F_3. Moreover, if each other fiber between F_1 and F_2 intersects η at only one real point (as in b) of Figure 7), then, with a fiber between F_1 and F_3 (respectively, F_3 and F_2), we associate a $\bullet$-vertex on $\mathbb{R}P^1$. We put bold edges between $\bullet$ and $\circ$-vertices.

(3) For all intervals I, except for the fibers of $\mathcal{L}_{|_I}$ with which we associate essential vertices and bold edges, we associate dotted (respectively, solid) edges on $\mathbb{R}P^1$ with the fibers of $\mathcal{L}_{|_I}$ that intersect η at three distinct real points (respectively, only one real point).

(4) The orientations of the edges incident to a vertex are in an alternating order. In particular, the orientations of the edges incident to an essential vertex are, respectively, as described in items 5, 6, and 7 of Definition 3.3.

The graph $\overline{\Gamma}$, called the real graph, is considered up to isotopy of $\mathbb{R}P^1$, namely, it is determined by the order of its colored vertices because the edges are determined by the color of their adjacent vertices.

We say that $\overline{\Gamma}$ is completable in degree n if there exists a complete real trigonal graph Γ of degree n such that $\Gamma \cap \mathbb{R}P^1 = \overline{\Gamma}$.

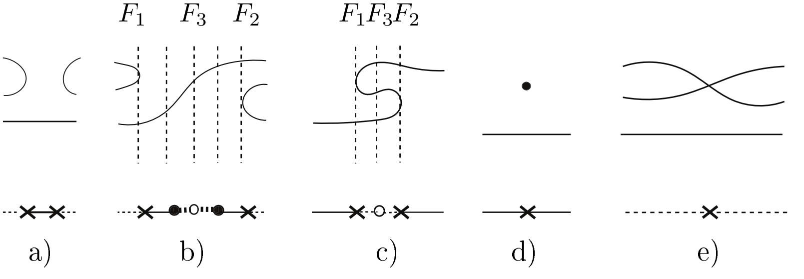


Figure 7. The local topology of trigonal $\mathcal{L}$-schemes and their corresponding real graphs.

Theorem 3.5 (see [Ore03])**.** *A trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ is realizable by a real algebraic trigonal curve if and only if its real graph is completable in degree n.*

Given a real graph $\overline{\Gamma}$, we depict only the completion to a real trigonal graph Γ on a hemisphere of $\mathbb{C}P^1$ because Γ is symmetric with respect to the standard complex conjugation. Moreover, we can omit orientations in figures representing real trigonal graphs because each vertex is adjacent to an even number of edges oriented in an alternating order as, for example, depicted in Figure 8, and such orientations are compatible with each others. Theorem 3.5 was improved in [DIZ14] in order to check if a given trigonal $\mathcal{L}$-scheme is realizable by a real trigonal curve of type I. We say that a real algebraic singular curve is of type I (respectively, of type II) if its normalization is of type I (respectively, of type II).

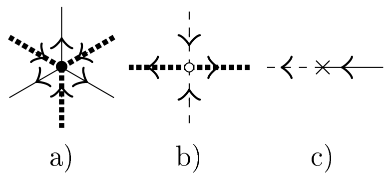


Figure 8. Colored vertices of a real trigonal graph.

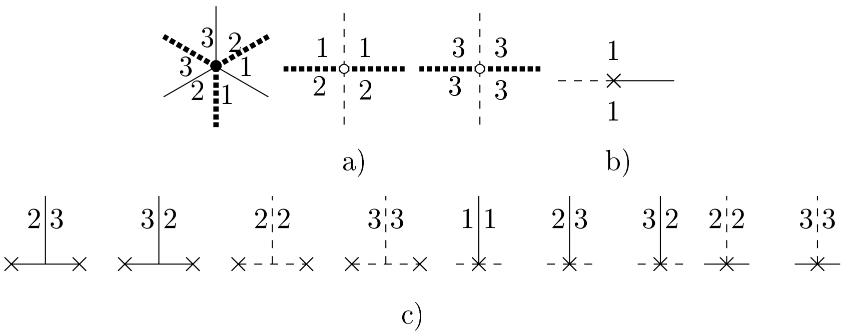


Figure 9. Type I labeling.

Definition 3.6. Let Γ be a real trigonal graph of degree n. We say that Γ is of *type* I if we can label each connected component of $\mathbb{C}P^1 \setminus \Gamma$, with the numbers $1, 2$ or 3, in such a way that:

- neighboring connected components of a $\bullet$-vertex, or a $\circ$-vertex of Γ, are labeled as depicted in one of the pictures in a) of Figure 9;
- neighboring connected components of a $\times$-vertex that does not belong to $\Gamma \cap \mathbb{R}P^1$ are labeled as depicted in b) of Figure 9;
- neighboring connected components of $\times$-vertices belonging to $\Gamma \cap \mathbb{R}P^1$ are labeled as depicted in in c) of Figure 9.

Otherwise, we say that Γ is of *type* II.

The original statement in [DIZ14] of the following theorem treats only the case of a nonsingular real trigonal curve, but it is possible to extend it to real nodal trigonal curves ([Jar18]).

Theorem 3.7 (see [DIZ14])**.** *A nonhyperbolic trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ is realizable by a real trigonal curve of type* I (*respectively, of type* II) *if and only if its real graph has a completion in degree n that is of type* I (*respectively, type* II)*.*

Remark 3.8. For each nonhyperbolic trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ realizable by an irreducible real trigonal curve of type I, each completion in degree n of its real graph has a unique type I labeling (see [DIZ14]). So, later on, each time we have to assign a labeling to a real trigonal graph of type I, we may label only one component.

Remark 3.9. If a hyperbolic trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ is realizable by a real trigonal curve C in Σ_n, then the curve C is of type I because the projection $\pi_n\colon \Sigma_n \to \mathbb{C}P^1$ (see Subsection 3.1) gives a totally real morphism on $\mathbb{C}P^1$.

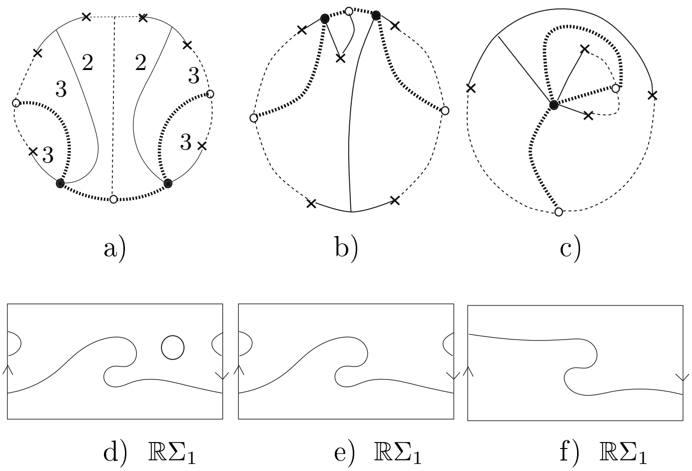


Figure 10. Cubic trigonal graphs.

3.3.1. *Gluing real trigonal graphs.* We call *cubic trigonal graph of type* I (respectively, *type* II) a real trigonal graph of degree 1 and type I (respectively, type II). The graph in Figure 10 a) is a cubic trigonal graph of type I, it has a unique type I labeling and associated trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_1$ as depicted in Figure 10 d). At the same time, the graphs depicted in Figure 10 b) and c) are of type II and have associated trigonal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_1$ as depicted, respectively, in Figure 10 e) and f).

Let Γ_1 (respectively, Γ_2) be a real trigonal graph. Denote by D_1 (respectively, D_2) the disk on which one of the two symmetric halves of Γ_1 (respectively, Γ_2) lies. Consider the disjoint union $\Gamma_1 \sqcup \Gamma_2 \subset D_1 \sqcup D_2$. Let $I_i \subset D_i$, $i = 1, 2$, be a segment in $\mathbb{R}P^1$ whose endpoints are not vertices of Γ_i and such that I_i contains a single $\circ$-vertex or a monochrome dashed vertex ¯╷¯. Let $\phi : I_1 \to I_2$ be an isomorphism preserving orientation, i.e., $\Gamma_1 \cap I_1 \to \Gamma_2 \cap I_2$ is an isomorphism preserving the types of vertices and edges, and preserving orientation. Consider the quotient $D_1 \sqcup_\phi D_2 = D_1 \sqcup D_2/(x \sim \phi(x))$ and $\Gamma_\phi \subset D_1 \sqcup_\phi D_2$ the quotient of the image of $\Gamma_1 \sqcup \Gamma_2$. We call such operation *gluing*. The gluing of real trigonal graphs is still a real trigonal graph (see [DIK08, Subsection 5.6] for details). Up to $\mathcal{L}$-isotopy, there is a finite number of pos-

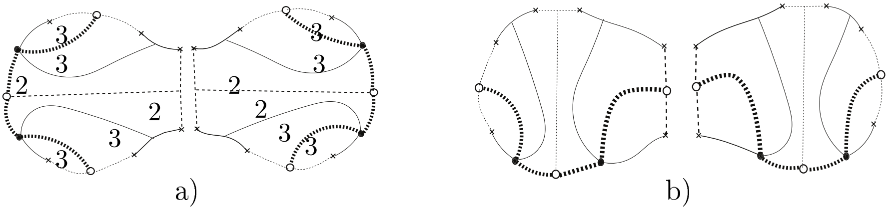


Figure 11. How to glue two cubic trigonal graphs.

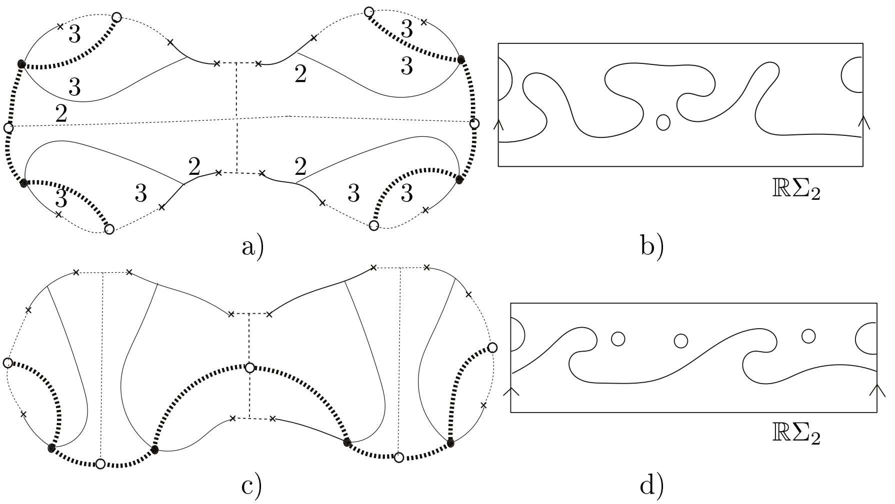


Figure 12. Gluing of two cubic trigonal graphs and associated trigonal $\mathcal{L}$-scheme.

sibilities to glue n cubic trigonal graphs into a real trigonal graph of degree n. We do not know whether any given trigonal graph of degree n is equivalent to the result of gluing of a union of n cubic trigonal graphs (see [Ore03, DIK08]).

Gluing type I real trigonal graphs that are glued to each other along vertices whose neighboring connected components have the same labels, we get a type I real trigonal graph. As example, look at the gluing of two cubic trigonal graphs of type I in a), respectively, b), of Figure 11 and a), respectively, c) of Figure 12. The obtained graphs are real trigonal graphs of degree 2 and type I. The respective associated trigonal $\mathcal{L}$-schemes are depicted in b) and d) of Figure 12.

§4. Construction

4.1. Trigonal construction.

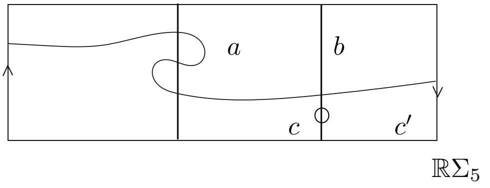


Figure 13. The union of a trigonal $\mathcal{L}$-scheme and two fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_5$: $\eta_{a,b,c,c'}$.

In this subsection we give some intermediate constructions of real algebraic curves that we will need later on.

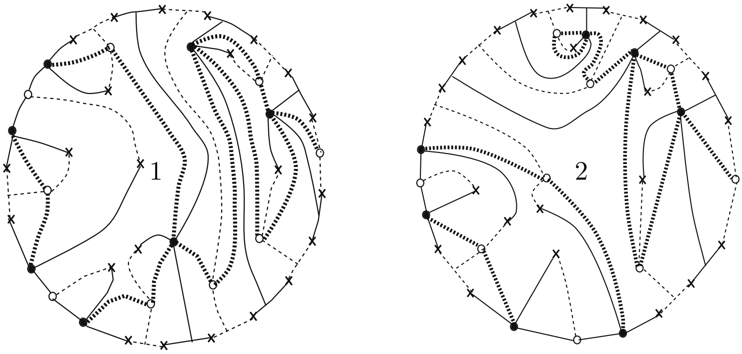
Proposition 4.1. *Let $\eta_{a,b,c,c'}$ be, up to isotopy of $\mathbb{R}\Sigma_5$, the union of a trigonal $\mathcal{L}$-scheme with two fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_5$ as depicted in Figure 13, where the letters a, b, c, c' denote numbers of ovals. Let h, j, and t be nonnegative integers. Then, there exist real algebraic trigonal curves in Σ_5 realizing the real schemes $\eta_{a,b,c,c'}$ for all a, b, c, c' such that $0 \leqslant c + c' \leqslant h, 0 \leqslant a \leqslant j$, and $0 \leqslant b \leqslant t$, where h, j, and t are as follows:*

(1) $j + h + t = 12$ *with*
- $h = 1$ *and* $j \in \{0, 1, 4, 7, 10, 11\}$,
- $h = 5$ *and* $j \in \{0, 1, 2, 3, 4, 5, 6, 7\}$,
- $h = 9$ *and* $j \in \{0, 1, 2, 3\}$.

(2) $j + h + t = 10$ *with*
- $h = 0$ *and* $j \in \{4, 6, 8\}$,
- $h = 2$ *and* $j \in \{0, 1, 2, 4, 5, 6, 8\}$,
- $h = 4$ *and* $j \in \{0, 1, 2, 4, 5, 6\}$,
- $h = 6$ *and* $j \in \{0, 1, 2, 4\}$,
- $h = 8$ *and* $j \in \{0, 1, 2\}$.

(3) $j + h + t = 8$ *with*
- $h = 1$ *and* $j = 3$,
- $h = 3$ *and* $j \in \{1, 2, 5\}$.

Furthermore, the real trigonal curves with $c + c' = h, a = j$, and $b = t$, are of type I. *Also, there exist real trigonal curves of type* I *in Σ_5 realizing $\eta_{a,b,c,c'}$ for $(a, b, c, c') = (1, 5, 0, 0)$ and $(3, 3, 0, 0)$.*

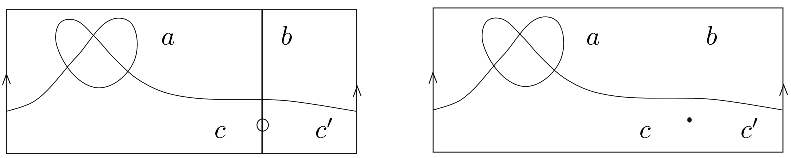


a) b)

Figure 14. Real trigonal graph of degree 5 and type I.

Proof. Thanks to Theorems 3.5 and 3.7, if the real graphs associated with $\eta_{a,b,c,c'}$ are completable in degree 5 to a real trigonal graph of type I (respectively, type II), then there exist real algebraic trigonal curves of type I (respectively, type II) realizing $\eta_{a,b,c,c'}$.

We can glue 5 cubic trigonal graphs (see Subsection 3.3) in such a way that we obtain type I (respectively, type II) real trigonal graphs of degree 5, which complete the real graph associated with $\eta_{a,b,c,c'}$, where a, b, c, and c' are such that $c + c' = h$, $a = j$, and $b = t$ (respectively, $0 \leqslant c + c' < h$, $0 \leqslant a < j$, and $0 \leqslant b < t$), for h, j, and t as listed in 1–3 above. Finally, a type I completion (see Remark 3.8) of the real graph associated with $\eta_{a,b,c,c'}$ for $(a, b, c, c') = (3, 3, 0, 0)$, respectively, $(a, b, c, c') = (1, 5, 0, 0)$, is depicted in Figure 14 a), respectively, b). □



a) $\mathbb{R}\Sigma_6$ b) $\mathbb{R}\Sigma_6$

Figure 15. a) The union of a trigonal $\mathcal{L}$-scheme and a fiber of $\mathcal{L}$ on $\mathbb{R}\Sigma_6$. b) A nodal trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$: $\widetilde{\eta}_{a,b,c,c'}$.

Proposition 4.2. *Let $\widetilde{\eta}_{a,b,c,c'}$ be, up to isotopy of $\mathbb{R}\Sigma_6$, a trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$ as depicted in Figure* 15 b), *where the letters a, b, c, c' denote numbers of ovals. Then, there exist real algebraic trigonal curves in Σ_6 realizing the real schemes $\widetilde{\eta}_{a,b,c,c'}$ for all a, b, c, c' as listed in Proposition* 4.1.

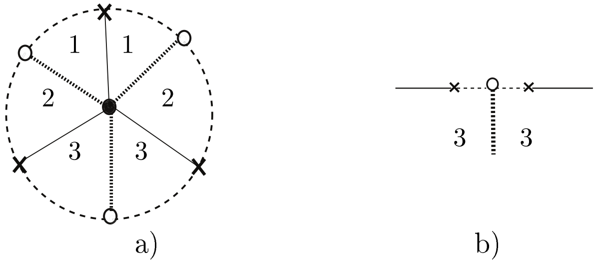


Figure 16. a) A real trigonal graph of degree 1 and type I: ξ. b) A local type I labeling.

Proof. Thanks to Theorems 3.5, 3.7, if the real graphs associated with $\widetilde{\eta}_{a,b,c,c'}$ are completable in degree 6 to a real trigonal graph of type I (respectively, type II), then there exist real algebraic trigonal curves of type I (respectively, type II) realizing $\widetilde{\eta}_{a,b,c,c'}$.

For a, b, c, c' as listed in Proposition 4.1, the existence of real trigonal graphs of degree 6 and type I (respectively, type II) completing the real graph associated with $\widetilde{\eta}_{a,b,c,c'}$ is equivalent to the existence of those of type I (respectively, type II) associated with the $\mathcal{L}$-scheme depicted in Figure 15 a), see [Ore03].

Let ξ be the cubic trigonal graph of type I depicted in Figure 16 a). Take any real trigonal graph Γ of degree 5 constructed in the proof of Proposition 4.1 realizing a trigonal $\mathcal{L}$-scheme $\eta_{a,b,c,c'}$. In a neighborhood of $\Gamma \cap \mathbb{R}P^1$, we denote by δ the subgraph of Γ that is as depicted in Figure 16 b) and whose associated $\mathcal{L}$-scheme is the part of $\eta_{a,b,c,c'}$ such that one fixed fiber of $\mathcal{L}$ passes through it (see Figure 13). Glue Γ along the $\circ$-vertex of δ to the $\circ$-vertex with same labeling if Γ is of type I of the cubic trigonal graph ξ. The gluing is a real trigonal graph of degree 6 that completes the real graph associated with the union of a trigonal $\mathcal{L}$-scheme with one fiber of $\mathcal{L}$ as depicted in Figure 15 a) on $\mathbb{R}\Sigma_6$, for all a, b, c, c' as in Proposition 4.1. Furthemore, the gluing is of type I (respectively, of type II) for all a, b, c, c' for which Γ is of type I (respectively, type II). □

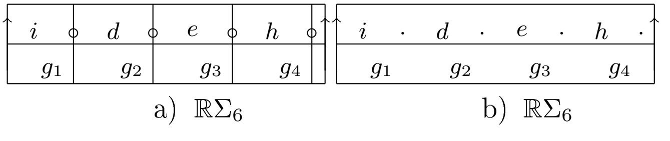


Figure 17. a) The union of a trigonal $\mathcal{L}$-scheme and four fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_6$: $\widetilde{\eta}_{i,d,e,h,g}$. b) A nodal trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$: $\eta_{i,d,e,h,g}$.

Proposition 4.3. *Let $\eta_{i,d,e,h,g}$ be, up to isotopy of $\mathbb{R}\Sigma_6$, the trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$ depicted in Figure 17 b), where the letters i, d, e, h, g_j for $j = 1, 2, 3, 4$ denote numbers of ovals. Let g be $\sum_{j=1}^{4} g_j$ and let s, k be nonnegative integers. Then, there exist real algebraic trigonal curves in Σ_6 realizing the real schemes $\eta_{i,d,e,h,g}$ for all i, d, e, h, g such that $0 \leqslant g \leqslant s$, $0 \leqslant i+d+e+h \leqslant k$, where s, k are as follows*:

(1) $s + k = 12$ *with* $s \in \{6, 10\}$,
(2) $s + k = 10$ *with* $s \in \{5, 9\}$,
(3) $s + k = 8$ *with* $s \in \{0, 4, 6, 8\}$.

Furthermore, the real trigonal curves with $g = s$ and $i + d + e + h = k$, are of type I. *Also, there exist real algebraic trigonal curves of type* I *in Σ_6 realizing $\eta_{i,d,e,h,g}$ for*

(4) $i + d + e + h + g = 8$ *with* $g = 0$;
(5) $i + d + e + h + g = 6$ *with* $g \in \{1, 3, 5\}$;
(6) $i + d + e + h + g = 4$ *with* $g \in \{2, 4\}$.

Proof. Thanks to Theorems 3.5, 3.7, if the real graphs associated with $\eta_{i,d,e,h,g}$ are completable in degree 6 to a real trigonal graph of type I (respectively, II), then there exist real algebraic trigonal curves of type I (respectively, type II) realizing $\eta_{i,d,e,h,g}$.

Let $\widetilde{\eta}_{i,d,e,h,g}$ be, up to isotopy of $\mathbb{R}\Sigma_6$, the union of a trigonal $\mathcal{L}$-scheme with four fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_6$ as depicted in Figure 17 a). Remark that, for i, d, e, h, g as listed in items 1–6 above, the existence of real trigonal graphs of degree 6 and type I (respectively, type II) completing the real graph associated with $\eta_{i,d,e,h,g}$ is equivalent to the existence of those of type I (respectively, type II) associated with $\widetilde{\eta}_{i,d,e,h,g}$ (see [Ore03]).

We can glue 6 cubic trigonal graphs in such a way that we obtain real trigonal graphs of degree 6, which complete the real graph associated with $\widetilde{\eta}_{i,d,e,h,g}$ where i, d, e, h, g are such that $0 \leqslant g \leqslant s$ and $0 \leqslant i + d + e + h \leqslant k$, for s, k as listed in items 1–3 above. Type I completions of the real graph associated with $\widetilde{\eta}_{i,d,e,h,g}$, for values listed in items 4–6 above, are pictured in Figure 18. □

Proposition 4.4. *There exist real algebraic trigonal curves of type* I *in Σ_6 realizing the trigonal $\mathcal{L}$-schemes, respectively, depicted in Figure* 19 a) *and* b). *Moreover, there exists a real algebraic trigonal curve in Σ_6 realizing the hyperbolic* (*see Remark* 3.9) *$\mathcal{L}$-scheme depicted in Figure* 19 c).

Proof. Thanks to Theorems 3.5, 3.7, if the real graphs associated with the real schemes in the statement are completable in degree 6 to a real trigonal

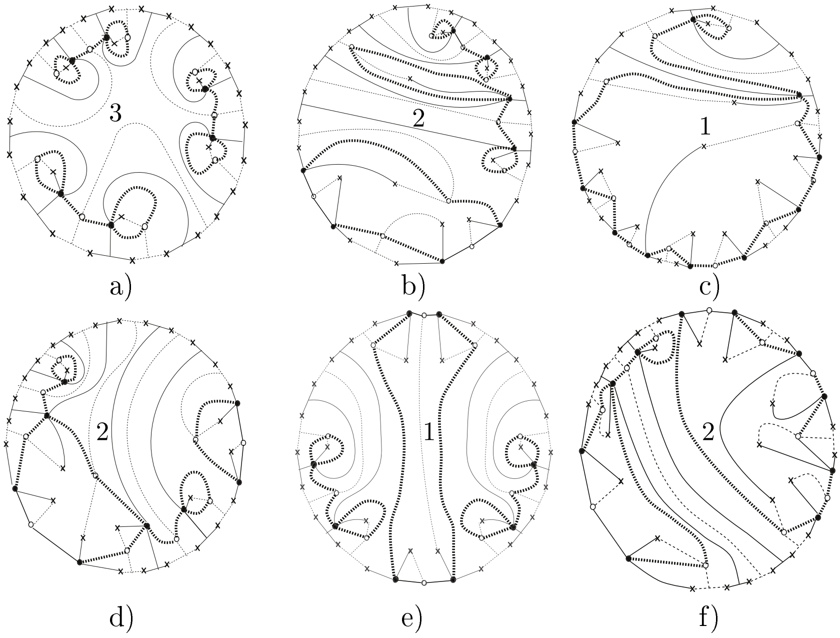


Figure 18. Real trigonal graphs of degree 6 and type I.

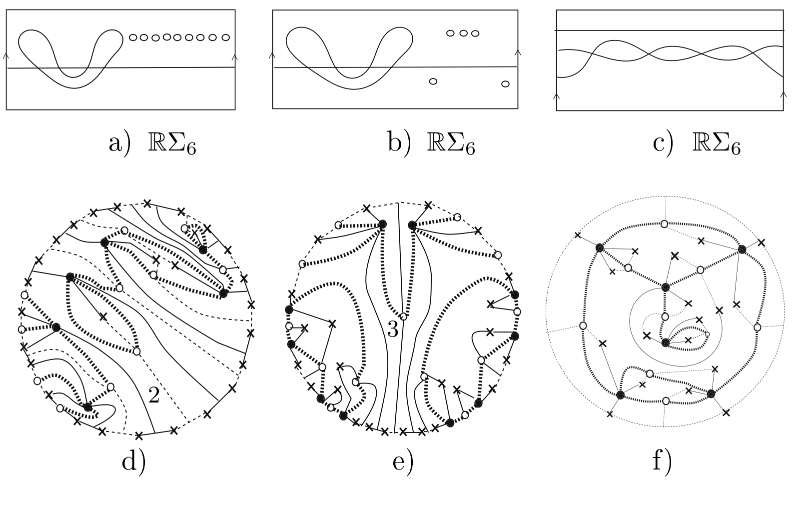


Figure 19. Trigonal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_6$ and the completion of their real graphs in degree 6.

graph of type I (respectively, II), then there exist real algebraic trigonal curves of type I (respectively, type II) realizing them.

Respective completions in degree 6 of the real graphs associated with the $\mathcal{L}$-schemes in Figure 19 a), b), and c) are pictured in Figure 19 d), e), and f).

Furthermore, the trigonal graphs depicted in Figure 19 d) and e) are of type I. □

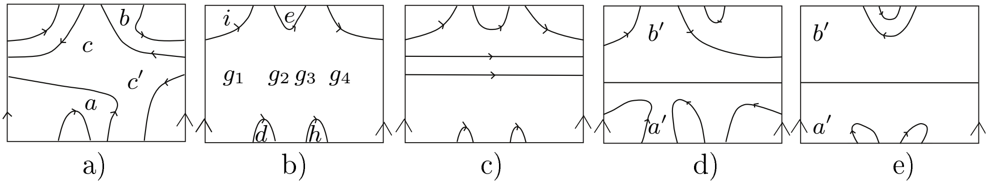


Figure 20. $\mathcal{L}$-schemes $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ on $\mathbb{R}\Sigma_2$.

Proposition 4.5. *Let η_1, respectively, η_2, η_3, and η_4, be a trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_2$, up to isotopy of $\mathbb{R}\Sigma_2$, as depicted in Figure* 20 a), *respectively,* b), c) *and* d), *where a, b, c, c', respectively, i, d, e, h, g, respectively, a', b' denote numbers of ovals. Then, such an $\mathcal{L}$-scheme is realizable by a nonsingular real algebraic curve C_1 (respectively, C_2, C_3, and C_4) of bidegree $(3,4)$ on Σ_2 for a, b, c, c' as listed in Proposition* 4.1 (*respectively, i, d, e, h, g as listed in Proposition* 4.3, *respectively, $(a', b') = (3, 2)$ and $(9, 0)$).*

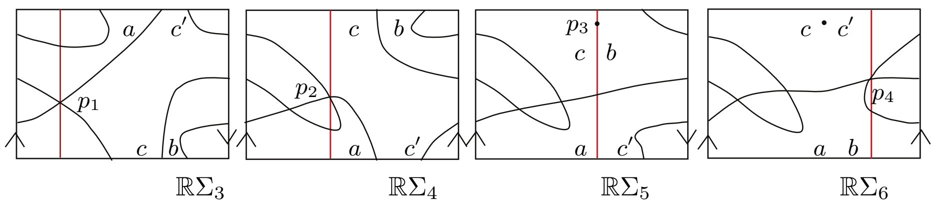


Figure 21. Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\widetilde{C_1})$, from right to left.

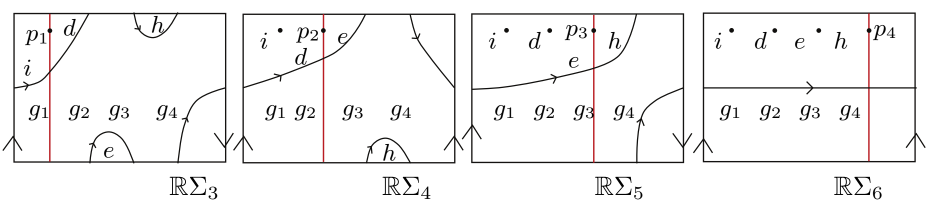


Figure 22. Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\widetilde{C_2})$, from right to left.

Proof. Denote by $\widetilde{C}_1$ (respectively, $\widetilde{C}_2$, $\widetilde{C}_3$, and $\widetilde{C}_4$) any real algebraic trigonal curves in Σ_6 constructed in Propositions 4.2 (respectively, Proposition 4.3,

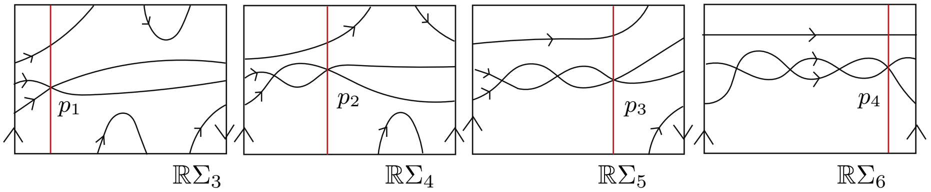


Figure 23. Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\widetilde{C}_3)$, from right to left.

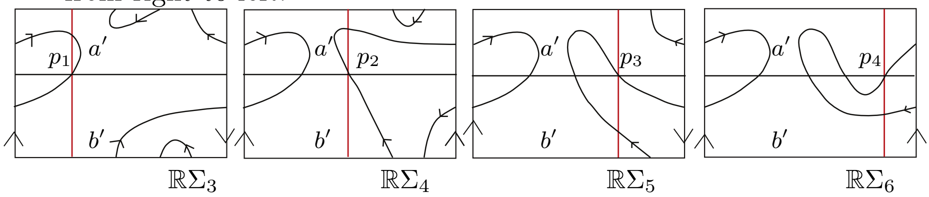


Figure 24. Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\widetilde{C_4})$, from right to left.

respectively, Proposition 4.4). As defined in Subsection 3.1, for each curve $\widetilde{C}_j$ we consider the birational transformation

$$\Xi_j =: \beta_{p_1}^{-1}\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1} : (\Sigma_6, \widetilde{C}_j) \to (\Sigma_2, C_j),$$

where the points p_k, $k = 1, 2, 3, 4$, are real double points such that p_4 belongs to $\mathbb{R}\widetilde{C}_j$ and p_k, $k = 3, 2, 1$, belong to the image of $\mathbb{R}\widetilde{C}_j$ via $\beta_{p_{k+1}}^{-1}$. In Figures 21, 22, 23, and 24 we also depict the fiber of $\mathbb{R}\Sigma_{k+2}$ intersecting the point p_k.

The birational transformation $\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1}((\mathbb{R}\Sigma_6, \mathbb{R}\widetilde{C}_j))$ is depicted in Figure 21, respectively, in Figure 22, Figure 23, and Figure 24 from right to left and $\Xi_j((\mathbb{R}\Sigma_6, \mathbb{R}\widetilde{C}_j))$ in Figure 20 a) for $j = 1$, respectively, in Figure 20 b), c), and d) for $j = 2, 3, 4$. □

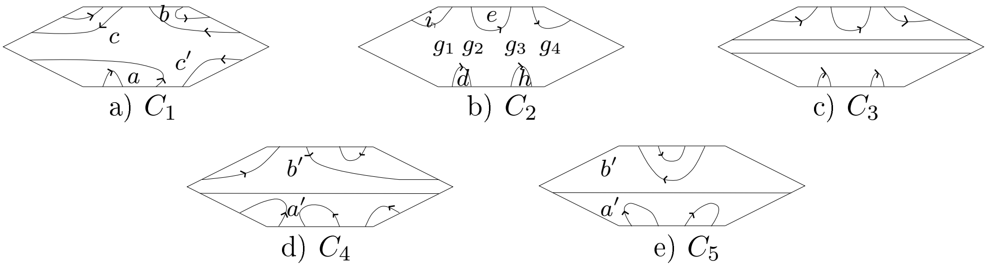


Figure 25. Charts of real curves C_1, C_2, C_3, C_4, C_5 of bidegree $(3, 4)$ and type I in Σ_2.

Since we will apply Viro's patchworking method in Subsection 4.2 to construct real algebraic curves of bidegree $(5, 0)$ on Σ_2 (see [Vir84a, Vir84b, Vir89, Vir06]), in Figure 25 a), b), c), and d) we depict the charts of nonsingular real algebraic curves of bidigree $(3,4)$ in Σ_2 constructed in Proposition 4.5. Moreover, performing a coordinate transformation to a curve C_4 with chart as depicted in Figure 25 d), we obtain a type I real algebraic curve C_5 of bidigree $(3,4)$ in Σ_2 with chart, respectively, real $\mathcal{L}$-scheme, as depicted in Figure 25 e), respectively, as depicted in Figure 20 e), where a', b' still denote numbers of ovals and $(a', b') = (3,2)$ or $(9,0)$.

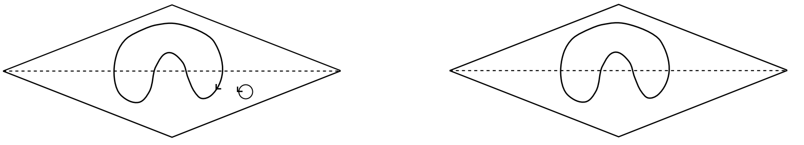
4.2. Final constructions and patchworking. In this subsection we end the proof of Theorems 1.9, 1.10, 1.11, and 1.12. We need Viro's patchworking method. Most of all, we use Viro's original patchworking method which is a tool for constructing nonsingular real algebraic hypersurfaces with prescribed topology in real toric varieties ([Vir84a, Vir84b, Vir89, Vir06]). Finally, for a particular construction, we use a variant of patchworking developed by Shustin ([Shu98, Shu02, Shu05, Shu06]), which exploits the deformation pattern technique and allows one to glue charts of polynomials presenting a tangency point with the boundary of the chart.

Remark 4.6. We want to construct bidegree $(5,5)$ nonsingular real algebraic curves on the quadric ellipsoid realizing all real schemes listed in Theorems 1.9, 1.10, 1.11, and 1.12. We can reduce the problem of construction of such bidegree $(5,5)$ real curves to the construction of bidegree $(5,0)$ real curves in Σ_2 with topology prescribed as follows. Let B be one of the real schemes listed in Theorems 1.9, 1.10, 1.11, and 1.12. We can choose a real scheme η in $\mathbb{R}\Sigma_2$ such that from the arrangement of the triplet $(\mathbb{R}\Sigma_2, \mathbb{R}E_2, \eta)$ one recovers the arrangement B on a 2-sphere as explained in Subsection 3.2. Thanks to Proposition 3.2, if we construct a bidegree $(5,0)$ real curve realizing η in Σ_2, we also construct a bidegree $(5,5)$ real curve realizing B on the quadric ellipsoid.

Notation 4.7. In Propositions 4.8, 4.9, and 4.10 the real schemes marked with the symbol $^\circ$ (respectively, *) are realized by a real algebraic curve of type I (respectively, type II).

Proposition 4.8. *All the real schemes in the following list are realizable by nonsingular real algebraic curves of bidegree* $(5,5)$ *on the quadric ellipsoid*:

(1) *all real schemes listed in Theorem* 1.9 *but the real schemes* $1 \sqcup \langle 4 \rangle \sqcup \langle 10 \rangle$ *and* $1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$;
(2) *all real schemes listed in Theorem* 1.10 *but the real schemes* $\langle 4 \rangle \sqcup \langle 10 \rangle$ *and* $\langle 7 \rangle \sqcup \langle 7 \rangle$;
(3) *all real schemes listed in Theorem* 1.11 *but the real scheme* $\langle 4 \rangle \sqcup \langle 9 \rangle^\circ$;
(4) *all real schemes listed in Theorem* 1.12 *but the real schemes* 13°, $1 \sqcup \langle 1 \rangle \sqcup \langle 9 \rangle^\circ$, $1 \sqcup \langle 3 \rangle \sqcup \langle 7 \rangle^\circ$, $\langle 1 \rangle \sqcup \langle 4 \rangle^\circ$, 1 *and* 0.



a) D_1 b) D_2

Figure 26. Charts and arrangements with respect to the coordinate axis $\{y = 0\}$ of real algebraic curves of bidegree $(2, 0)$ on Σ_2.

Proof. Thanks to Remark 4.6, to prove the statement we only need to construct bidegree $(5, 0)$ real curves in Σ_2 whose charts are the patchworking of one of the charts depicted in Figure 25, with one of those depicted in Figure 26. First, we construct real algebraic curves D_1, D_2 of bidegree $(2, 0)$ in Σ_2 intersecting a given real curve of bidegree $(1, 0)$ at four fixed real points and with charts, respectively, as depicted in a) and b) of Figure 26. Let $\widetilde{H}$ be a bidegree $(1, 0)$ real algebraic curve in Σ_2. For any two fixed distinct real points on $\widetilde{H}$, there exists a bidegree $(1, 0)$ real algebraic curve H passing through them. Let $P_0(x, y)P_1(x, y) = 0$ be a polynomial equation defining the union of $\widetilde{H}$ and H. For any four fixed distinct real points on a connected component $\mathcal{F}$ of $\mathbb{R}\widetilde{H} \setminus \mathbb{R}H$, there exist two real curves H_1 and H_2 of bidegree $(1, 0)$ such that $H_1 \cup H_2$ passes through the fixed four points. Replace the left side of the equation $P_0(x, y)P_1(x, y) = 0$ with $P_0(x, y)P_1(x, y) + \varepsilon f_1(x, y)f_2(x, y)$, where $f_i(x, y) = 0$ is an equation for H_i and ε is a sufficiently small real number. Up to a choice of the sign of ε, one constructs a small perturbation D_1 of $\widetilde{H} \cup H$, where D_1 is a bidegree $(2, 0)$ nonsingular real curve such that $\bigcup_{i=1}^{2} H_i \cap \widetilde{H} = D_1 \cap \widetilde{H}$ and whose chart is as depicted in a) of Figure 26, where $\widetilde{H}$, up to a change of coordinates, is $\{y = 0\}$ (in dashed). Analogously, it is easy to construct a bidegree $(2, 0)$ real curve D_2 whose chart, for any four fixed real points on the coordinates axis $\{y = 0\}$, is as depicted in b) of Figure 26 and intersects $\{y = 0\}$ at the four fixed points. Thanks to Viro's patchworking method, we realize every real scheme listed in items 1–4 by gluing the polynomial and the chart of a real algebraic curve C_i, $i = 1, 2, 3, 4, 5$, constructed in Proposition 4.5 and at the end of Subsection 4.1, with those of a real algebraic curve D_j, $j = 1, 2$. In case we patchwork real curves of type I, we get type I bidegree $(5, 0)$ real algebraic curves on Σ_2 only if, fixing a complex orientation on the C_i's, up to a choice of a complex orientation on D_1, we can patchwork such curves in a compatible way with their complex orientations. For example, consider the curves C_i of type I with charts as depicted in Figure 25 (the fixed complex orientation is represented by the arrows). Now, up to

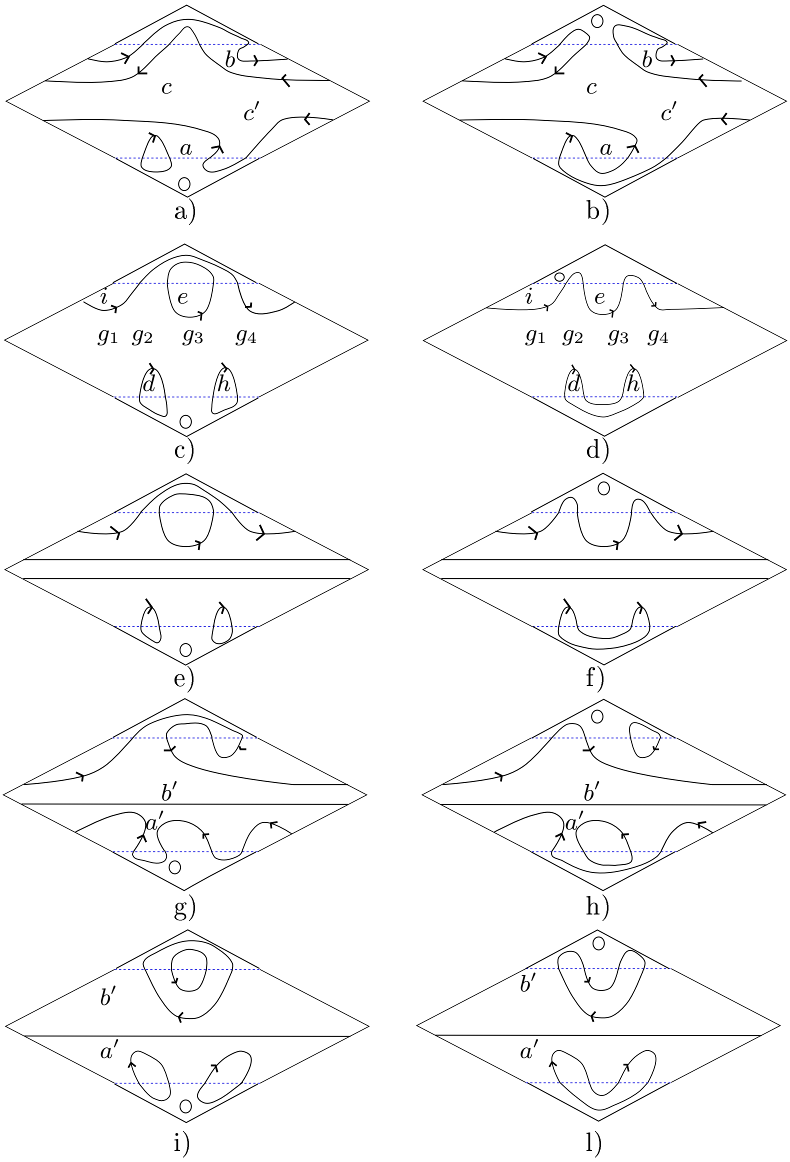


Figure 27. Charts of curves of bidegree $(5,0)$ in Σ_2 obtained by patchworking the charts of the real algebraic curves C_i from Proposition 4.5 and D_1.

a choice of a complex orientation of the real curve D_1, one can glue the charts of the C_i's with those of D_1 in a compatible way with respect to the arrows and obtain charts of bidegree $(5,0)$ real curves of type I in Σ_2 as depicted in Figure 27. □

Proposition 4.9. *The real schemes* $1 \sqcup \langle 4 \rangle \sqcup \langle 10 \rangle$, $\langle 4 \rangle \sqcup \langle 10 \rangle$, $\langle 4 \rangle \sqcup \langle 9 \rangle^\circ$, 13°, $1 \sqcup \langle 1 \rangle \sqcup \langle 9 \rangle^\circ$, *and* $1 \sqcup \langle 3 \rangle \sqcup \langle 7 \rangle^\circ$ *are realizable by nonsingular real algebraic curves of bidegree* $(5,5)$ *on the quadric ellipsoid.*

Proof. Thanks to Remark 4.6, in order to prove the statement we only need to construct bidegree $(5,0)$ real curves of type I (respectively, of type II) in Σ_2 whose charts are as depicted in a) of Figure 28 with

(1) $(\alpha,\beta,\gamma) = (5,0,2)$ and $(t,s) = (2,1)$;
(2) $(\alpha,\beta,\gamma) = (8,1,0)$ and $(t,s) = (0,3)$;
(3) $(\alpha,\beta,\gamma) = (4,1,0)$ and $(t,s) = (3,0)$;
(4) $(\alpha,\beta,\gamma) = (4,1,0)$ and $(t,s) = (1,2)$;

and in b) of Figure 28 with $(\alpha,\beta,\gamma) = (5,0,2)$ (respectively, in a) of Figure 28 with $(\alpha,\beta,\gamma) = (7,0,1)$ and $(t,s) = (1,2)$), where α,β,γ and t,s denote numbers of ovals.

The strategy is to construct real algebraic curves C of bidegree $(4,0)$ on Σ_2 (respectively, C' of bidegree $(2,4)$ in Σ_1 intersecting $\mathbb{R}E_1$ at four fixed real points) with chart as depicted in a) of Figure 29 (respectively, b) and c) of Figure 29). Thanks to [OS16], there exist real algebraic curves C of bidegree $(4,0)$ and type I, respectively, type II, in Σ_2 whose charts, up to a coordinate change, and arrangements with respect to the coordinates axis $\{x=0\}$ are as depicted in a) of Figure 29 with $(\alpha,\beta,\gamma) = (5,0,2), (8,1,0)$ and $(4,1,0)$, respectively, $(\alpha,\beta,\gamma) = (7,0,1)$.

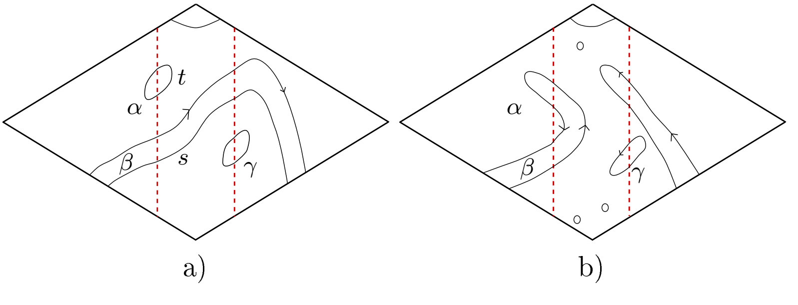


Figure 28. Charts of real algebraic curves of bidegree $(5,0)$ on Σ_2.

Let us construct real curves C' in Σ_1 with the properties described above. Let $\widetilde{H}$ be a bidegree $(1,0)$ real curve in Σ_5, and let us fix ten real points on $\widetilde{H}$. As in the proof of Proposition 4.8, via small perturbations method it is possible to construct a bidegree $(2,0)$ real curve $\widetilde{C}$ intersecting $\widetilde{H}$ at the ten fixed points and such that the arrangement of $(\mathbb{R}\Sigma_1, \mathbb{R}\widetilde{H}, \mathbb{R}\widetilde{C})$ is as depicted in a) of

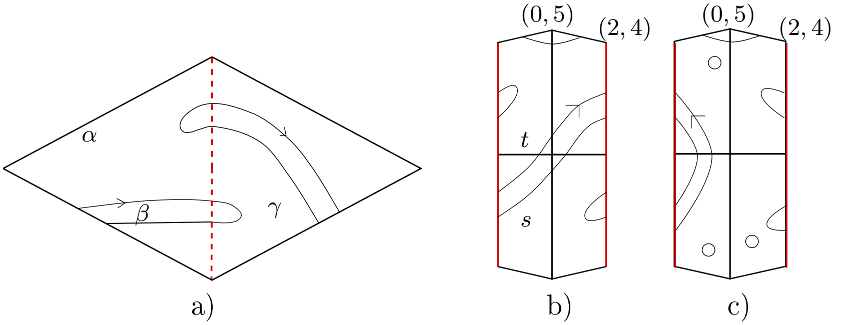


Figure 29. *a*) Charts of real algebraic curves C of bidegree $(4,0)$ on Σ_2. *b*)–*c*) Charts of real algebraic curves C' of bidegree $(2,4)$ on Σ_1.

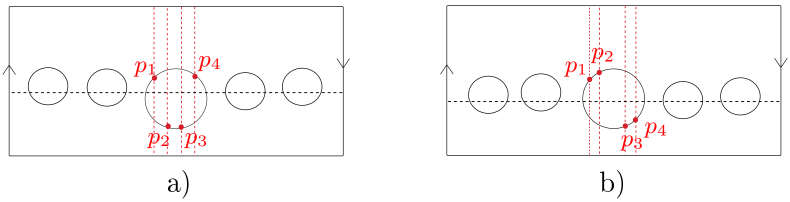


Figure 30. $\mathcal{L}$-schemes of real algebraic maximal curves of bidegree $(2,0)$ on $\mathbb{R}\Sigma_5$.

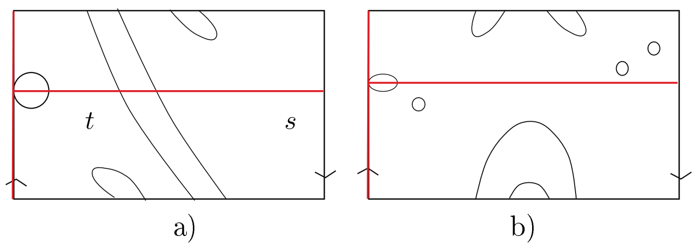


Figure 31. $\mathcal{L}$-schemes of real algebraic curves of bidegree $(2,4)$ on Σ_1.

Figure 30, respectively, b) of Figure 30. For any fixed connected component $\mathcal{O}$ of $\mathbb{R}\widetilde{C}$, we can pick four real points, p_1, p_2, p_3, p_4 on it as depicted in a) of Figure 30, respectively, b) of Figure 30. Then, consider the birational transformation $\beta_{p_1}^{-1}\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1} : (\Sigma_5, \widetilde{C}) \to (\Sigma_1, C')$, as defined in Subsection 3.1, where we call p_i also the image of p_i via $\beta_{p_j}^{-1}$, $j > i$, $i = 1,2,3$. Choose the coordinate axes in $\mathbb{R}\Sigma_1$ such that $\mathbb{R}C'$ has an arrangement as depicted in b) of Figure 31, respectively, a) of Figure 31, where $t + s = 3$. The charts of C' are as depicted in c) of Figure 29, respectively, b) of Figure 29.

Applying Viro's patchworking method, gluing the polynomials and the charts of the curves C with those of the curves C' (the patchworking of their charts is depicted in Figure 28), one constructs the required bidegree $(5,0)$ real curves in Σ_2. $\square$

Proposition 4.10. *The real schemes* $1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$, $\langle 7 \rangle \sqcup \langle 7 \rangle$, $\langle 1 \rangle \sqcup \langle 4 \rangle^\circ$, 1, *and* 0 *are realizable by nonsingular real algebraic curves of bidegree* $(5,5)$ *on the quadric ellipsoid* Q.

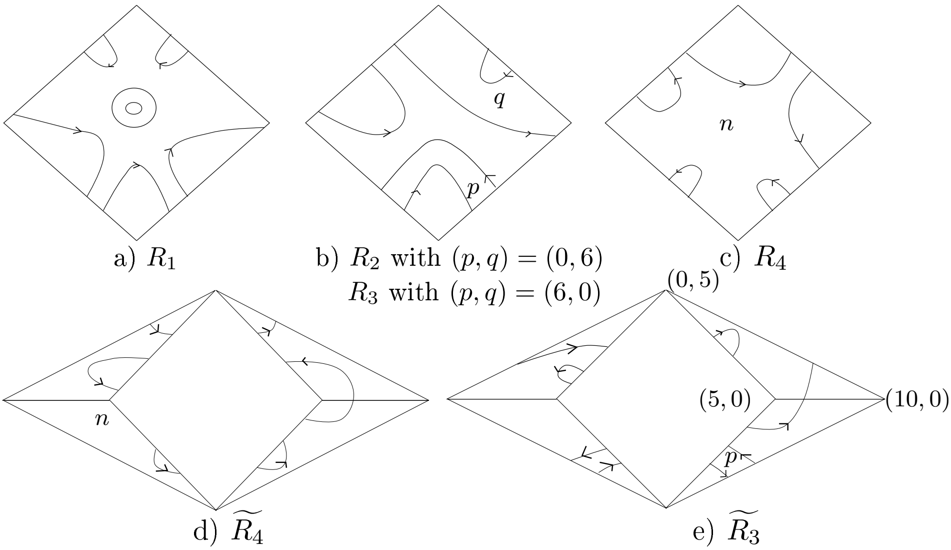


Figure 32. Charts of real algebraic curves.

Proof. First of all, in order to realize the real scheme 0 (respectively, 1), we merely perturb the union of five real hyperplane sections in $\mathbb{C}P^3$ not intersecting $\mathbb{R}X$ (respectively, only one of which intersects $\mathbb{R}X$) to a smooth surface of degree 5. For the realization of the real schemes $\langle 1 \rangle \sqcup \langle 4 \rangle^\circ$ and $1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$ we give some intermediate constructions, then we apply Viro's patchworking method. The strategy is to construct bidegree $(5,0)$ real curves in Σ_2 whose charts are given by the patchworking of the charts depicted in a) of Figure 32 (respectively, b) of Figure 32 with $(p,q)=(0,6)$) with the charts depicted in d) of Figure 32 with $n=4$ (respectively, e) of Figure 32 with $p=6$), where n, p, and q denote numbers of ovals. Then, Remark 4.6 implies that there exist real algebraic curves of bidegree $(5,5)$ in the quadric ellipsoid realizing the real schemes $\langle 1 \rangle \sqcup \langle 4 \rangle^\circ$ and $1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$.

Let us start constructing

- real algebraic plane quintics with charts as depicted in a) and b), with $(p, q) = (0, 6)$, of Figure 32 and intersecting a given real line at five fixed real points;
- real algebraic plane quintics with charts as depicted in b), with $(p, q) = (6, 0)$, and c), with $n = 4$, of Figure 32.

Thanks to [Vir89, Propositions 4.3B, 4.3C] and [Pol77], there exist nonsingular real affine quintics, with five real asymptotes pointing in five different directions, arranged in the real plane as depicted in Figure 33, where $(p, q) = (0, 6)$, $(6, 0)$ and $n = 4$. Moreover, due to [Shu83], for any fixed five directions for the asymptotes, there exist real affine quintics with arrangements in the real plane as depicted in Figure 33. It follows that, for a fixed real line L in the complex projective plane and fixed five distinct real points on L, there exist nonsingular real plane quintics R_1 and R_2 passing through the five given points such that their charts are, respectively, as depicted in a) and b) of Figure 32, where $(p, q) = (0, 6)$. Furthermore, there exist real plane quintics R_3 and R_4 with charts, respectively, as depicted in b) and c) of Figure 32, with $(p, q) = (6, 0)$ and $n = 4$.

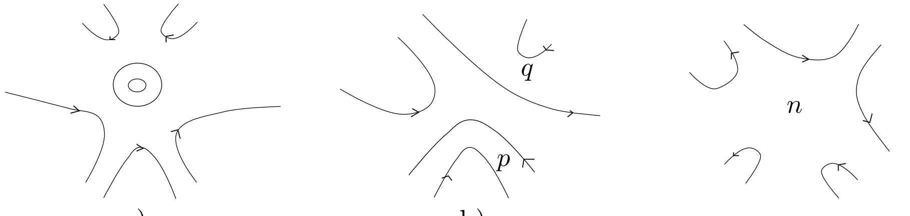


Figure 33. Nonsingular real affine quintics.

Now, we construct real algebraic curves with charts as depicted in d) and e) of Figure 32, respectively, with $n = 4$ and $p = 6$. Let

$$P_h(x, y, z) = \sum_{i+j \leqslant 5} a_{i,j} x^i y^j z^{5-i-j}$$

be a polynomial of a quintic R_h, with $h = 3, 4$. Applying the transformation $T : P_h(x, y, z) \mapsto P_h(xz, yz, x^2)$ to the polynomials P_h, we construct real algebraic plane curves $\widetilde{R}_h$ whose polynomials are

$$\widetilde{P}_h(x, y, z) = \sum_{i+j \leqslant 5} a_{i,j} x^{10-i-2j} y^j z^{i+j}$$

and whose charts are as depicted in d) and e) of Figure 32, respectively, with $n = 4$ and $p = 6$. In order to realize the real scheme $\langle 1 \rangle \sqcup \langle 4 \rangle^{\circ}$, respectively,

$1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$, we apply Viro's patchworking method gluing the polynomials and the charts of a real plane quintic R_1, respectively, R_2 with $(p, q) = (0, 6)$, with those of $\widetilde{R}_4$ with $n = 4$, respectively, $\widetilde{R}_3$ with $p = 6$.

The construction of a real algebraic curve of bidegree $(5, 5)$ on the quadric ellipsoid realizing the real scheme $\langle 7 \rangle \sqcup \langle 7 \rangle$ mimics the construction realizing $1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$, but it requires a variant of the patchworking theorem due to Shustin (see [Shu05, Theorem 5]). The interested reader can find the results of the following paragraph in all their generalities in [Shu05]. Here, we only present an application of [Shu05, Theorem 5] in a particular case.

Let Δ be a convex polygon in $\mathbb{R}^2_+$ and let $\mathrm{Tor}(\Delta)$ be its associated toric variety. Let $S : \Delta = \bigcup_{i=1}^{k} \Delta_i$ be a convex subdivision of Δ. Let

$$f_i = \sum_{(j,h) \in \Delta_i \cap \mathbb{Z}^2} a_{j,h} x^j y^h$$

be real polynomials, with $a_{j,h} \in \mathbb{R}$ and such that $C_i = \{f_i = 0\} \subset \mathrm{Tor}(\Delta_i)$ are nonsingular real algebraic curves. Suppose that there exist some faces $\Gamma_{st} \subset \Delta_s \cap \Delta_t$ such that $\Gamma_{st} \not\subset \partial\Delta$ and $z_{st} \in \mathrm{Tor}(\Gamma_{st}) \cap C_j$ is a real tangency point of C_j with $\mathrm{Tor}(\Gamma_{st})$, for $j = s, t$, and locally as depicted in Figure 34 on the right. Furthermore, suppose that, out of the tangency points z_{ts}, each curve C_i crosses $\mathrm{Tor}(\Gamma')$ transversely for any face $\Gamma' \subset \Delta_i$, with $i = 1, \ldots, k$. Gluing the charts of the C_i's, one does not obtain a chart of a polynomial. Consider the following topological construction: replace a neighborhood of the tangency points and of their symmetric points in $\mathbb{R}\,\mathrm{Tor}(\Delta)$ (Figure 34) with a *deformation pattern*, i.e., two disks as depicted in a) or b) of Figure 35. Then, Shustin (see [Shu05]) proved that such a topological construction is realizable algebraically and it produces a chart of a nonsingular real polynomial in $\mathrm{Tor}(\Delta)$. We want to

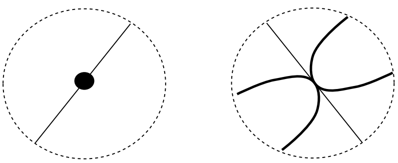


Figure 34

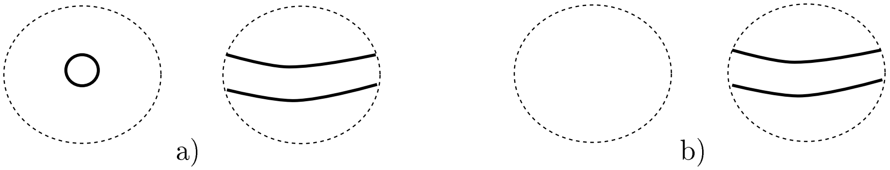
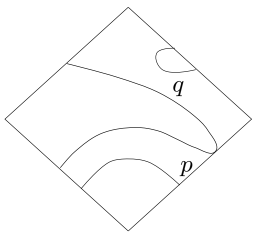


Figure 35

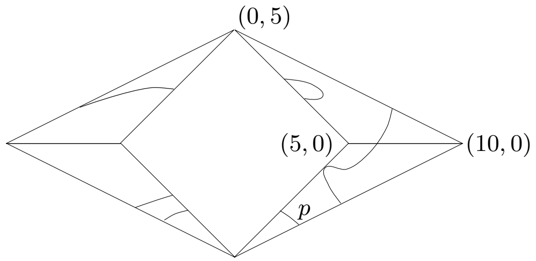
construct a bidegree $(5,0)$ real algebraic curve in Σ_2 with chart as depicted in b) of Figure 37.

First of all, we construct two nonsingular plane quintics C_1, C_2 such that

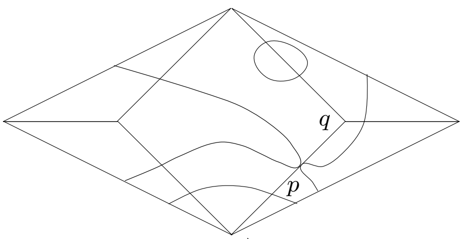
- C_1 passes through three fixed real points on a given real line L, it is tangent to L at another fixed real point and it has chart as depicted in a) of Figure 36, with $(p,q)=(0,6)$;
- C_2 has chart as depicted in a) of Figure 36, with $(p,q)=(6,0)$.



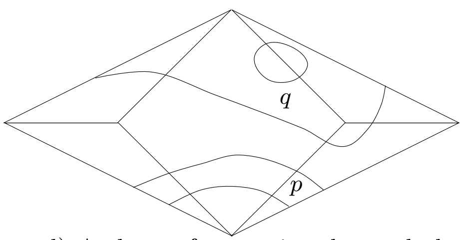
a) A chart of a real algebraic plane quintic.



b) A chart of a real algebraic plane curve.

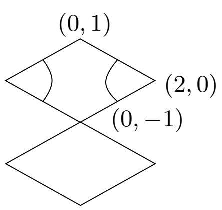


c)

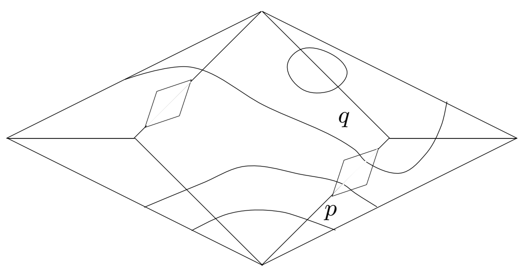


d) A chart of a nonsingular real algebraic curve of bidegree $(5,0)$ on Σ_2.

Figure 36



a) A deformation pattern.



b) A chart of a nonsingular real algebraic curve of bidegree $(5,0)$ on Σ_2.

Figure 37

In order to construct C_1, we fix four distinct real points p_1, p_2, p_3, p_4 on a given real line L. In some local chart one has $L = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ and $p_i = (x_i, 0)$, with $i = 1,2,3,4$. For any fixed small real number $\varepsilon \neq 0$, there exists a nonsingular real plane quintic C_0 passing through p_1, p_2, p_3, $(x_4+\varepsilon, 0)$, $(x_4-\varepsilon, 0)$ and whose chart is as depicted in b) of Figure 32, where $(p,q) = (0,6)$. The quintic C_0 is locally given by

$$\Big\{(x,y) \in \mathbb{R}^2 : \sum_{i+j\leqslant 5, j\neq 0} a_{i,j}x^iy^j + (x-x_1)(x-x_2)(x-x_3)(x-x_4+\varepsilon)(x-x_4-\varepsilon) = 0\Big\}.$$

Consider the real one-parameter family $\mathcal{H}_t$, with $t \in [0,1]$, of nonsingular real plane quintics

$$C_t := \Big\{ \sum_{i+j\leqslant 5,\, j\neq 0} a_{i,j}x^iy^j + (x-x_1)(x-x_2)(x-x_3) \times (x-x_4+(1-t)\varepsilon)(x-x_4-(1-t)\varepsilon) = 0\Big\}.$$

The real quintic C_1 of the family is the required real curve: it passes through p_1, p_2, p_3, it is tangent to L at p_4, and it has chart as depicted in a) of Figure 36, with $(p,q) = (0,6)$.

Since there exists a real plane quintic with chart as depicted in b) of Figure 32 with $(p,q) = (6,0)$, it is easy to see that there exists a real algebraic plane quintic C_2 whose chart is as depicted in a) of Figure 36, with $(p,q) = (6,0)$. Applying the transformation T to the real quintic C_2, one constructs a nonsingular real plane curve $\widetilde{C}_2$ with chart as depicted in b) of Figure 36, where $p = 6$.

Unlike in the construction of a bidegree $(5,5)$ real curve in the quadric ellipsoid realizing the real scheme $1 \sqcup \langle 7\rangle \sqcup \langle 7\rangle$, one cannot glue the charts of C_1 and $\widetilde{C}_2$ as depicted in c) of Figure 36 (where p and q are both equal to 6), because we do not obtain a chart of a polynomial. But, the variant of the patchworking method developed by Shustin allows us to replace a neighborhood of the tangency point and its symmetric point with a *deformation pattern*; see a) and b) of Figure 37. At the end, we obtain a chart of a real algebraic curve of bidegree $(5,0)$ in Σ_2 as depicted in d) of Figure 36, with $p = 6$ and $q = 6$. □

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University of Oslo, UiO
Postboks 1053 Blindern,
0316 OSLO, Norway
E-mail: manzarom@math.uio.no

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