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Greedy Bases in L^p Spaces[1]

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We consider a weighted L^p space $L^p(w)$ with a weight function w. It is known that the Haar system $\mathcal{H}_p$ normalized in L^p is a greedy basis of L^p, $1 < p < \infty$. We study a question of when the Haar system $\mathcal{H}_p^w$ normalized in $L^p(w)$ is a greedy basis of $L^p(w)$, $1 < p < \infty$. We prove that if w is such that $\mathcal{H}_p^w$ is a Schauder basis of $L^p(w)$, then $\mathcal{H}_p^w$ is also a greedy basis of $L^p(w)$, $1 < p < \infty$. Moreover, we prove that a subsystem of the Haar system obtained by discarding finitely many elements from it is a Schauder basis in a weighted norm space $L^p(w)$; then it is a greedy basis.

1. INTRODUCTION

We discuss greedy bases in different Banach spaces. In Sections 2 and 3 we consider weighted L^p spaces $L^p(w)$. We prove in Section 2 that for some weights w the Haar system and its biorthogonal system form a bidemocratic pair for $L^p(w)$. This result, combined with known results, implies that the Haar system is a greedy basis of $L^p(w)$. In Section 3 we consider some subsystems of the Haar system and prove that if those subsystems are Schauder bases in a space $L^p(w)$, then they are greedy bases. We begin with a general introduction to the subject.

Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_k\}_{k=1}^{\infty}$ be a normalized basis for X ($\|\psi_k\| = 1$, $k \in \mathbb{N}$). For a given $f \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(f) := \sigma_m(f,\Psi)_X := \inf_{b_k,\Lambda} \left\| f - \sum_{k\in\Lambda} b_k \psi_k \right\|_X,$$

where the infimum is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \dots$, of the positive integers *decreasing* and write $\rho \in D(f)$ if

$$|c_{k_1}(f)| \ge |c_{k_2}(f)| \ge \dots.$$

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In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the mth greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by the formula

$$G_m(f) := G_m(f,\Psi) := G_m(f,\Psi,\rho) := \sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}.$$

It is a simple algorithm which describes a theoretical scheme for m-term approximation of an element f. This algorithm is known in the theory of nonlinear approximation under the name of Thresholding Greedy Algorithm (TGA). The best we can achieve with the algorithm G_m is

$$\|f - G_m(f)\|_X = \sigma_m(f,\Psi)_X,$$

or a little weaker

$$\|f - G_m(f)\|_X \le C\sigma_m(f,\Psi)_X$$

for all $f \in X$ with a constant C independent of f and m. The following concept of a greedy basis was introduced in [4].

Definition 1.1. We call a basis Ψ a *greedy basis* if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that

$$\|f - G_m(f,\Psi,\rho)\|_X \le C\sigma_m(f,\Psi)_X$$

with a constant C independent of f and m.

The reader can find a discussion of greedy bases in [5; 10; 11; 12, Ch. 1].

In Section 2 we study a question of when the Haar system is a greedy basis for a weighted space $L^p(w)$. Afterwards in Section 3 we study a similar question for some subsystems of the Haar system. Let w be a nonnegative integrable function on $[0,1]$. Define for $1 \le p < \infty$

$$\|f\|_{p,w}^p := \int_0^1 |f(x)|^p w(x)\,dx, \qquad \langle f,g\rangle := \int_0^1 f(x)g(x)\,dx, \qquad \langle f,g\rangle_w := \int_0^1 f(x)g(x)w(x)\,dx.$$

Let Δ be a collection of all dyadic intervals of $[0,1]$,

$$\Delta := \bigl\{I \subset [0,1]\colon\ I = [2^{-n}(l-1), 2^{-n}l),\ n = 0,1,\dots,\ l = 1,\dots,2^n\bigr\} \cup \{[0,1]\}.$$

For an interval $I = [2^{-n}(l-1), 2^{-n}l)$ define

$$I_+ = \bigl[2^{-n}(l-1), 2^{-n}(l-1/2)\bigr), \qquad I_- = \bigl[2^{-n}(l-1/2), 2^{-n}l\bigr).$$

Denote

$$w(I) := \int_I w(x)\,dx.$$

We now discuss weights from Muckenhoupt's dyadic class $A_p^{(d)}$. For $1 < p < \infty$ define $w_p(x) := w(x)^{-1/(p-1)}$. We say that $w \in A_p^{(d)}$ if there exists a constant $B(p) > 0$ such that for any dyadic interval I we have

$$w(I)w_p(I)^{p-1} \le B(p)|I|^p. \tag{1.1}$$

Let $\mathcal{H} = \{h_k\}_{k=0}^{\infty}$ be the Haar system in its natural enumeration. In some cases we are interested in identifying Haar functions by their support, except for the function $h_0(x) \equiv 1$ if $x \in [0,1]$. For any

$I \in \Delta$ denote by h_I, $h_I(x) = 1$ if $x \in I_+$, $h_I(x) = -1$ if $x \in I_-$ and $h_I(x) = 0$ if $x \in I^c = [0,1] \setminus I$, the corresponding Haar function normalized in L^∞. Haar functions normalized in the space $L^p(w)$ are denoted as follows:

$$H^w_{I,p} := w(I)^{-1/p} h_I \qquad \text{and} \qquad H^w_{0,p} := w(I)^{-1/p} h_0.$$

A.S. Krantsberg [6] proved that the condition $w \in A_p^{(d)}$ is a necessary and sufficient condition for the Haar system $\mathcal{H}$ to be a basis of $L^p(w)$, $1 < p < \infty$. In [2] it was pointed out that the condition $w \in A_p^{(d)}$ guarantees that $\mathcal{H}$ is an unconditional basis of $L^p(w)$, $1 < p < \infty$. In this paper we prove that the condition $w \in A_p^{(d)}$ implies that $\mathcal{H}$ is a greedy basis of $L^p(w)$, $1 < p < \infty$.

It was proved in [9] that the Haar basis $\mathcal{H}_p$ normalized in L^p is a greedy basis of L^p, $1 < p < \infty$. The following characterization theorem of greedy bases is from [4].

Theorem 1.1. *A basis is greedy if and only if it is unconditional and democratic.*

We now give the definitions of unconditional and democratic bases.

Definition 1.2. A basis $\Psi = \{\psi_k\}_{k=1}^\infty$ of a Banach space X is said to be *unconditional* if, for every choice of signs $\theta = \{\theta_k\}_{k=1}^\infty$, $\theta_k = 1$ or -1, $k = 1, 2, \dots$, the linear operators $M_{\theta,N}$, $N \in \mathbb{N}$, defined by

$$M_{\theta,N}\left(\sum_{k=1}^\infty a_k \psi_k\right) = \sum_{k=1}^N a_k \theta_k \psi_k$$

are uniformly bounded from X into X.

Definition 1.3. We say that a basis (system) $\Psi = \{\psi_k\}_{k=1}^\infty$ is a *democratic basis* (*system*) *for* X if there exists a constant $D := D(X, \Psi) > 1$ such that, for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$, we have

$$\left\| \sum_{k \in P} \frac{\psi_k}{\|\psi_k\|_X} \right\| \le D \left\| \sum_{k \in Q} \frac{\psi_k}{\|\psi_k\|_X} \right\|. \tag{1.2}$$

In [1] bidemocratic bases have been studied (see Section 2 below for a discussion). Observe that if w is a weight function and w_p, $1 < p < \infty$, is integrable, then the Haar system $\mathcal{H}$ is a complete minimal system in $L^p(w)$ and its biorthogonal system $\mathcal{H}^*_w = \{h^*_k\}_{k=0}^\infty$ is defined by the following equations:

$$h^*_k(x) = \frac{1}{w(x)} h_k(x), \qquad k = 0, 1, \dots.$$

We will use also the notation

$$h^*_I(x) = \frac{1}{|I| w(x)} h_I(x), \qquad I \in \Delta.$$

In Section 2 we prove that the pair of biorthogonal systems $(\mathcal{H}, \mathcal{H}^*_w)$ is bidemocratic for $L^p(w)$ if and only if $w \in A_p^{(d)}$. Having in mind that for $w \in A_p^{(d)}$ the Haar system is an unconditional basis for $L^p(w)$ by Theorem 1.1, we get that $\mathcal{H}$ is a greedy basis of $L^p(w)$, $1 < p < \infty$. We can use the above results to construct for any $1 \le a < b < \infty$ a system of functions which is a greedy basis of L^p if and only if $p \in (a, b)$. Note that for $1 < p < \infty$ a system $\Psi = \{\psi_k\}_{k=1}^\infty$ is a basis of $L^p(w)$ if and only if $\Psi^w := \{\psi_k w^{1/p}\}_{k=1}^\infty$ is a basis of L^p.

For $1 < r < \infty$ let $M_{\eta,r}(x) = x^{\eta/r}$ for $x \in [0,1]$ and $\eta = \pm 1$. It is easy to check that $v_p(x) := M^p_{-1,r}(x) \in A_p^{(d)}$ if $p \in [1, r)$ and $v_p \notin A_r^{(d)}$ if $p \ge r$ and $u_p(x) := M^p_{1,r}(x) \in A_p^{(d)}$ if and only if $p \in \left(\frac{r}{r-1}, \infty\right)$. Define $M^{a,b}(x) = M_{-1,b}(x)(2x)$ if $x \in [0,1/2]$ and $M^{a,b}(x) = M_{1,\frac{a}{a-1}}(2x-1)$ if $x \in (1/2,1]$. Then the following result holds.

Proposition 1.1. *For any $r > 1$ the system $\{M_{-1,r}h_k\}_{k=0}^{\infty}$ is a greedy basis in L^p if and only if $p \in (1,r)$. For any $r' > 1$ the system $\{M_{1,r}h_k\}_{k=0}^{\infty}$, $r = \frac{r'}{r'-1}$, is a greedy basis in L^p if and only if $p \in (r',\infty)$. For $1 < a < b < \infty$ the system $\{M^{a,b}h_k\}_{k=0}^{\infty}$ is a greedy basis in L^p if and only if $p \in (a,b)$.*

2. THE HAAR SYSTEM IN $L^p(w)$

We will say that the weight function $w(x)$ *satisfies condition* (D_p) if there exists $C(p) > 0$ such that for any sequence of dyadic intervals $\mathcal{J} = \{J_k\}_{k=0}^{\infty} \subset \Delta$, $J_0 \supset J_1 \supset \dots$, $|J_k| = 2^{-k}$, and for any $m \in \mathbb{N}$

$$\sum_{k=0}^{m} \left(\frac{w(J_m)}{w(J_k)}\right)^{1/p} \le C(p). \tag{2.1}$$

Assume that the weight function $w(x)$ satisfies the following condition: For any $I \in \Delta$ we have

$$w(I_-) \le q w(I), \qquad w(I_+) \le q w(I) \tag{2.2}$$

with some $q \in (0,1)$ independent of I; then evidently $w(x)$ satisfies condition (D_p) for any $p > 0$.

By the Hölder inequality we get that for any interval J

$$w(J) w_p(J)^{p-1} \ge |J|^p. \tag{2.3}$$

Inequality (1.1) implies that for any $I \in \Delta$

$$\frac{w(I_+)}{w(I_-)} \le 2^p B(p), \qquad \frac{w(I_-)}{w(I_+)} \le 2^p B(p). \tag{2.4}$$

Indeed, by (1.1) we have

$$w(I_+) w_p(I_-)^{p-1} \le B(p)|I|^p, \qquad w(I_-) w_p(I_+)^{p-1} \le B(p)|I|^p. \tag{2.5}$$

Combining (2.5) and (2.3) with $J = I_+$ and $J = I_-$, we obtain (2.4).

It is easy to see that inequalities (2.4) imply (2.2). Thus any $w \in A_p^{(d)}$ satisfies (2.2). It is easily checked that if $w \in A_p^{(d)}$, then $w_p \in A_{p'}^{(d)}$.

Following [1] we denote

$$\varphi(N) = \sup_{|P|=N} \left\| \sum_{k \in P} \frac{\psi_k}{\|\psi_k\|_X} \right\|_X, \qquad \varphi^*(N) = \left\| \sum_{k \in P} \|\psi_k\|_X \, \psi_k^* \right\|_{X^*}$$

and will say that a pair (Ψ, Ψ^*) of biorthogonal systems $\Psi = \{\psi_k\}_{k=1}^{\infty} \subset X$, $\Psi^* = \{\psi_k^*\}_{k=1}^{\infty} \subset X^*$ is *bidemocratic for* X if there exists $C > 0$ such that for any $N \in \mathbb{N}$

$$\varphi(N)\varphi^*(N) \le CN. \tag{2.6}$$

Evidently we have

Remark 2.1. If a pair (Ψ, Ψ^*), where $\Psi \subset X$ and $\Psi^* \subset X^*$, of biorthogonal systems is bidemocratic for X, then

$$\limsup_k \|\psi_k\|_X \|\psi_k^*\|_{X^*} \le C.$$

It is proved in [1] that a bidemocratic basis is a democratic basis. The above definition of bidemocratic system is given for minimal systems which are not necessarily bases. However, the proof given in [1] for bases also works in our case of minimal systems. Hence, we skip the proof of the following

Proposition 2.1. *Let* (Ψ, Ψ^*), *where* $\Psi \subset X$ *and* $\Psi^* \subset X^*$, *be a pair of biorthogonal systems that is bidemocratic for* X. *Then the system* Ψ *is democratic for* X.

We prove the following result.

Theorem 2.1. *Let* $1 < p < \infty$ *and let* w *be a weight function so that* $w_p \in L^1$. *Then the pair of biorthogonal systems* $(\mathcal{H}, \mathcal{H}_w^*)$ *is bidemocratic for* $L^p(w)$ *if and only if* $w \in A_p^{(d)}$.

Proof. Suppose that $(\mathcal{H}, \mathcal{H}_w^*)$ is bidemocratic for $L^p(w)$. Then, writing conditions (2.6) for $N = 1$, we find that for any $I \in \Delta$

$$\left(\int\limits_I |I|^{-p/2} w(x)\, dx\right)^{1/p} \left(\int\limits_I |I|^{-p'/2} w(x)^{-p'} w(x)\, dx\right)^{1/p'} \le C,$$

where $1/p + 1/p' = 1$. It is easily checked that the above inequality is the same as (1.1) with $B(p) = C^p$. Hence the necessity is proved. The proof of sufficiency begins with the following lemma.

Lemma 2.1. *Let* $1 < p < \infty$, w *satisfy condition* (D_p) *and*

$$f := \sum_{I \in P^*} H_{I,p}^w, \qquad |P^*| = N. \tag{2.7}$$

Then

$$\|f\|_{p,w} \le C_1(p) N^{1/p}.$$

Proof. We prove this lemma by induction. For $N = 1$ it follows from normalization of the system $\{H_{I,p}^w\}$ in $L^p(w)$. Suppose that the statement of Lemma 2.1 holds for $|P^*| = N - 1$. We will prove it for $|P^*| = N$. We have

$$\|f\|_{p,w}^p = \int\limits_0^1 \left|\sum_{I \in P^*} H_{I,p}^w(x)\right|^p w(x)\, dx \le \int\limits_0^1 \left(\sum_{I \in P^*} w(I)^{-1/p} |h_I(x)|\right)^p w(x)\, dx.$$

Let $I_0 \in P^*$ be one of the intervals of minimal length. Denote $P_1^* := P^* \setminus \{I_0\}$. If $I_0 \cap J = \varnothing$ for all $J \in P_1^*$, then

$$\|f\|_{p,w}^p = 1 + \left\|\sum_{J \in P_1^*} H_{I,p}^w\right\|_{p,w}^p \le 1 + C_1(p)^p (N-1) \le C_1(p)^p N$$

provided $C_1(p) \ge 1$.

Assume now that there are intervals $J \in P_1^*$ such that $I_0 \cap J \ne \varnothing$. Our assumption that I_0 has the minimal length implies that $I_0 \subset J$. Let $I_k \in P_1^*$, $k = 1, \dots, s$, be such that $I_0 \subset I_k$ and $|I_0| < |I_1| < \ldots < |I_s|$. Then for $x \in I_0$ we have

$$|f(x)| \le \sum_{k=0}^s w(I_k)^{-1/p} \le w(I_0)^{-1/p} \sum_{k=0}^s \left(\frac{w(I_0)}{w(I_k)}\right)^{1/p} \le C(p) w(I_0)^{-1/p}. \tag{2.8}$$

Let $f_1 := f - H^w_{I_0,p}$. Inequality (2.8) implies

$$\int_0^1 |f(x)|^p w(x)\,dx = \int_{I_0} |f(x)|^p w(x)\,dx + \int_{[0,1]\setminus I_0} |f(x)|^p w(x)\,dx \le C(p)^p + \int_0^1 |f_1(x)|^p w(x)\,dx$$

$$\le C(p)^p + C_1(p)^p(N-1) \le C_1(p)^p N$$

provided $C_1(p) \ge C(p)$. $\square$

We have not included the first Haar function in the sum f in order to avoid complicated notation. Evidently Lemma 2.1 is true if in the sum f we have the term $H^w_{0,p}$.

Remark 2.2. If condition (2.1) holds for any sequence of dyadic intervals except for a fixed sequence of dyadic intervals $\mathcal{I} = \{I_k\}_{k=0}^\infty \subset \Delta$, $I_0 \supset I_1 \supset \dots$, $|I_k| = 2^{-k}$, then the assertion of Lemma 2.1 holds if $P^* \cap \mathcal{I} = \varnothing$.

By Lemma 2.1 we have

$$\Biggl\| \sum_{I\in P^*} H^{w_p}_{I,p'} \Biggr\|_{p',w_p} \le C_1(p')N^{1/p'}. \tag{2.9}$$

In order to prove inequalities (2.6) for the Haar system in the space $L^p(w)$, we write

$$\Biggl\| \sum_{k\in P} \|h_k\|_{p,w}\, h_k^* \Biggr\|_{p',w} = \Biggl\| \sum_{I\in P^*} w(I)^{1/p}|I|^{-1}h_I \Biggr\|_{p',w_p} = \Biggl\| \sum_{I\in P^*} |I|^{-1}w(I)^{1/p}w_p(I)^{1/p'} H^{w_p}_{I,p'} \Biggr\|_{p',w_p}$$

$$\le B(p)^{1/p}C_2(p')\Biggl\| \sum_{I\in P^*} H^{w_p}_{I,p'} \Biggr\|_{p',w_p},$$

where the last inequality follows by (1.1) from the well-known fact about the unconditional bases (see, e.g., [7, p. 19]) and from the above cited result (see [2]) that the Haar system is an unconditional basis for the space $L^{p'}(w_p)$ with unconditional basis constant $C_2(p')$. By (2.9)

$$\Biggl\| \sum_{k\in P} \|h_k\|_{p,w}\, h_k^* \Biggr\|_{p',w} \le B(p)^{1/p}C_1(p')N^{1/p'}.$$

Hence, by Lemma 2.1 we finish the proof. $\square$

Theorem 2.1 and Proposition 2.1 imply

Corollary 2.1. *Let $1 < p < \infty$ and let $w \in A_p^{(d)}$. Then the Haar system $\mathcal{H}$ normalized in $L^p(w)$ is a democratic system of $L^p(w)$.*

Democratic bases in weighted norm spaces with the weight functions satisfying the A_p condition have been studied by other authors (see, e.g., [8]).

In the introduction we have commented that if $w \in A_p^{(d)}$, then the Haar system $\mathcal{H}$ is an unconditional basis for $L^p(w)$. Hence, by Corollary 2.1 and Theorem 1.1 we obtain the following result.

Theorem 2.2. *Assume that $w \in A_p^{(d)}$, $1 < p < \infty$. Then $\mathcal{H}_p^w$ is a greedy basis of $L^p(w)$.*

We formulate one more theorem that is a simple corollary of Theorem 2.2 and Krantsberg's result [6].

Theorem 2.3. *Assume that a weight w is such that $\mathcal{H}_p^w$ is a basis of $L^p(w)$, $1 < p < \infty$. Then $\mathcal{H}_p^w$ is a greedy basis of $L^p(w)$.*

3. SUBSYSTEMS OF THE HAAR SYSTEM IN $L^p(w)$

Let us consider the Haar system without the first function. We put $\overline{\mathcal{H}} := \{H^w_{I,p},\ I \in \Delta\}$. In [2] the class of weight functions w such that the system $\overline{\mathcal{H}}$ is a basis and unconditional basis in $L^p(w)$, $1 < p < \infty$, was described.

Theorem 3.1. *The system $\overline{\mathcal{H}}$ is a basis in $L^p(w)$ for $1 < p < \infty$ if and only if there exists a sequence of dyadic intervals $\mathcal{I} = \{I_k\}_{k=0}^{\infty} \subset \Delta$, $I_0 \supset I_1 \supset \dots$, $|I_k| = 2^{-k}$, such that*

$$w_p(I_k^{\mathrm{c}}) < +\infty, \qquad w_p(I_k) = +\infty \qquad \text{for all} \quad k \in \mathbb{N}, \tag{3.1}$$

there exists $C(p) > 0$ such that

$$w(I_k) w_p(I_k^{\mathrm{c}})^{p-1} \le C(p) |I_k|^p \qquad \text{for all} \quad k \in \mathbb{N}, \tag{3.2}$$

and for any $I \in \Delta \setminus \mathcal{I}$ inequality (1.1) *holds.*

The following lemma was proved in [3].

Lemma 3.1. *The system $\overline{\mathcal{H}}$ is a complete minimal system in $L^p(w)$ for $1 < p < \infty$ if and only if there exists a sequence of dyadic intervals $\mathcal{I} = \{I_k\}_{k=0}^{\infty} \subset \Delta$, $I_0 \supset I_1 \supset \dots$, $|I_k| = 2^{-k}$, such that conditions* (3.1) *hold.*

Let $G := \{g_I\}$ be the system dual to $\overline{\mathcal{H}}$. Let $\{i_k\}_{k=0}^{\infty}$ be a sequence defined according to the following rule: $i_k = 1$ if I_{k+1} is the left half of the interval I_k and $i_k = 2$ otherwise. As it was shown in [2],

$$g_{I_k}(x) = w(x)^{-1} \bigl[2^k w(I_k)^{1/p} h_{I_k}(x) + (-1)^{i_k} \cdot 2^k w(I_k)^{1/p}\bigr], \qquad k = 0, 1, 2, \dots,$$

and for any $I \in \Delta$, $I \notin \mathcal{I}$,

$$g_I(x) = w(x)^{-1} w(I)^{1/p} |I|^{-1} h_I(x).$$

We prove the following

Theorem 3.2. *Let $1 < p < \infty$ and let w be a weight function so that there exists a sequence of dyadic intervals $\mathcal{I} = \{I_k\}_{k=0}^{\infty} \subset \Delta$, $I_0 \supset I_1 \supset \dots$, $|I_k| = 2^{-k}$, such that conditions* (3.1) *hold. Then the pair of biorthogonal systems $(\overline{\mathcal{H}}, G)$ is bidemocratic for $L^p(w)$ if and only if there exists $C(p) > 0$ such that conditions* (3.2) *hold.*

Proof. Suppose that $(\overline{\mathcal{H}}, G)$ is bidemocratic for $L^p(w)$. Then, writing conditions (2.6) for $N = 1$, for any $I_k \in \mathcal{I}$ we obtain

$$w(I_k)^{1/p} |I_k|^{-1} w_p(I_k^{\mathrm{c}})^{1/p'} \le C.$$

If $I \in \Delta$, $I \notin \mathcal{I}$, then we find that (1.1) holds with $B(p) = C^p$ as in the proof of Theorem 2.1. For the proof of sufficiency we denote by Λ_1 the subspace of $L^p(w)$ spanned by the system $\{h_{I_j}\}_{j=0}^{\infty} \subset \overline{\mathcal{H}}$. Having in mind the definition of the system G, by Theorem 2.1 and Remark 2.2 we see that the theorem will be proved if we prove the following

Lemma 3.2. *The pair of biorthogonal systems $\bigl(\{h_{I_j}\}_{j=0}^{\infty}, \{g_{I_j}\}_{j=0}^{\infty}\bigr)$ is bidemocratic for Λ_1.*

For the proof of Lemma 3.2 we have to prove some lemmas.

Lemma 3.3. *If w is a weight function which satisfies the conditions of Theorem* 3.1, *then there exists $S > 1$ such that for any $1 \le j < k$*

$$\frac{w(I_k)}{w(I_j)} \le C(p) \cdot 2^{p(j+1-k)} S^{p(j+1-k)} \tag{3.3}$$

where $C(p) > 0$ is the constant from (3.2).

Proof. In [2, (43)] it was proved that there exists $S>1$ such that for any $l\in\mathbb{N}$

$$\frac{w_p(I_l^{\mathrm{c}})^{p-1}}{w_p(I_{l-1}^{\mathrm{c}})^{p-1}}\ge S^p. \tag{3.4}$$

We have

$$\frac{w(I_k)}{w(I_j)}\le\frac{C(p)|I_k|^p\,w_p(I_k^{\mathrm{c}})^{1-p}}{w(I_j\setminus I_{j+1})}\le\frac{C(p)|I_k|^p\,w_p(I_k^{\mathrm{c}})^{1-p}}{|I_j\setminus I_{j+1}|^p\,w_p(I_j\setminus I_{j+1})^{1-p}}\le C(p)\cdot 2^{p(j+1-k)}\frac{w_p(I_{j+1}^{\mathrm{c}})^{p-1}}{w_p(I_k^{\mathrm{c}})^{p-1}}.$$

By (3.4) we finish the proof. $\square$

Lemma 3.3 yields the following assertion.

Proposition 3.1. *If $w\ge 0$ satisfies the conditions of Theorem 3.1, then $w\in(D_p)$.*

Let $0\le k_1<k_2<\ldots<k_N$ and let

$$\psi(x)=\sum_{j=1}^N\frac{1}{w(I_{k_j})^{1/p}}h_{I_{k_j}}(x). \tag{3.5}$$

By Lemma 2.1 and Proposition 3.1 we have

$$\|\psi\|_{p,w}\le C_1(p)N^{1/p}. \tag{3.6}$$

By a well-known result from the general theory of bases, for some $C>1$ we have

$$C^{-1}\le\|g_I\|_{p',w}\le C.$$

Let $\{\alpha_k\}_{k=0}^\infty$ be a sequence of positive numbers such that $\|\alpha_k g_{I_k}\|_{p',w}=1$, $k=0,1,\ldots$. Evidently, $C^{-1}\le\alpha_k\le C$ for all k.

Lemma 3.4. *Let $1<p<\infty$ and let $w\ge0$ satisfy the conditions of Theorem 3.1. Then for the function ϕ defined as*

$$\phi(x)=\sum_{j=1}^N g_{I_{k_j}}(x) \tag{3.7}$$

the following inequality holds:

$$\|\phi\|_{p',w}\le C_3(p)N^{1/p'}. \tag{3.8}$$

Proof. By induction one can easily check that

$$\phi(x)=[w(x)]^{-1}\left[\sum_{j=1}^N(-1)^{i_{k_j}}w(I_{k_j})^{1/p}\cdot2^{k_j}\right]\qquad\text{if}\quad x\in I_{k_1}^{\mathrm{c}},$$

$$\phi(x)=[w(x)]^{-1}\left[(-1)^{i_{k_1}}w(I_{k_1})^{1/p}\cdot2^{k_1+1}+\sum_{j=2}^N(-1)^{i_{k_j}}w(I_{k_j})^{1/p}\cdot2^{k_j}\right]\qquad\text{if}\quad x\in I_{k_1}\setminus I_{k_1+1},$$

for any $1<m\le N$

$$\phi(x)=[w(x)]^{-1}\left[\sum_{j=m}^N(-1)^{i_{k_j}}w(I_{k_j})^{1/p}\cdot2^{k_j}\right]\qquad\text{if}\quad x\in I_{k_{m-1}+1}\setminus I_{k_m},$$

$$\phi(x)=[w(x)]^{-1}\left[(-1)^{i_{k_m}}w(I_{k_m})^{1/p}\cdot2^{k_m+1}+\sum_{j=m+1}^N(-1)^{i_{k_j}}w(I_{k_j})^{1/p}\cdot2^{k_j}\right]\qquad\text{if}\quad x\in I_{k_m}\setminus I_{k_m+1}$$

and $\phi(x)=0$ if $x\in I_{k_N+1}$.

Then we have

$$
\begin{aligned}
\|\phi\|_{p',w}^{p'} \le {}& \left|\sum_{j=1}^{N} w(I_{k_j})^{1/p}\cdot 2^{k_j}\right|^{p'} w_p(I_{k_1}^{\mathrm{c}}) \\
&+ \sum_{m=2}^{N}\left|2w(I_{k_{m-1}})^{1/p}\cdot 2^{k_{m-1}} + \sum_{j=m}^{N} w(I_{k_j})^{1/p}\cdot 2^{k_j}\right|^{p'} w_p\bigl(I_{k_{m-1}}\setminus I_{k_{m-1}+1}\bigr) \\
&+ |2w(I_{k_N})^{1/p}\cdot 2^{k_N}|^{p'} w_p(I_{k_N}\setminus I_{k_N+1}) + \sum_{m=2}^{N}\left|\sum_{j=m}^{N} w(I_{k_j})^{1/p}\cdot 2^{k_j}\right|^{p'} w_p\bigl(I_{k_m-1}\setminus I_{k_{m-1}}\bigr) \\
\le {}& C(p)^{1/(p-1)}\sum_{m=1}^{N}\left|1+\sum_{j=m+1}^{N}\frac{w(I_{k_j})^{1/p}}{w(I_{k_m})^{1/p}}\cdot 2^{k_j-k_m}\right|^{p'} \\
&+ \sum_{m=2}^{N} C(p)^{1/(p-1)}\left|1+\sum_{j=m}^{N}\frac{w(I_{k_j})^{1/p}}{w(I_{k_{m-1}})^{1/p}}\cdot 2^{k_j-k_{m-1}-1}\right|^{p'} \frac{w(I_{k_{m-1}})^{1/(p-1)}}{w\bigl(I_{k_{m-1}}\setminus I_{k_{m-1}+1}\bigr)^{1/(p-1)}} \\
&+ C(p)^{1/(p-1)}\frac{w(I_{k_{N-1}})^{1/(p-1)}}{w\bigl(I_{k_{N-1}}\setminus I_{k_{N-1}+1}\bigr)^{1/(p-1)}}.
\end{aligned}
$$

By Lemma 3.3 we have

$$
\sum_{j=m+1}^{N}\frac{w(I_{k_j})^{1/p}}{w(I_{k_m})^{1/p}}\cdot 2^{k_j-k_m} \le C(p)^{1/p}\sum_{j=m+1}^{N} S^{k_m-k_j} \le C(p)^{1/p}\frac{S}{S-1}.
$$

We easily estimate

$$
\begin{aligned}
\frac{w(I_{k_{m-1}})}{w\bigl(I_{k_{m-1}}\setminus I_{k_{m-1}+1}\bigr)} &= 1+\frac{w(I_{k_{m-1}+1})}{w\bigl(I_{k_{m-1}}\setminus I_{k_{m-1}+1}\bigr)} \\
&\le 1+2^{(k_{m-1}+1)p} w(I_{k_{m-1}+1}) w_p\bigl(I_{k_{m-1}}\setminus I_{k_{m-1}+1}\bigr)^{p-1} \le 1+C(p).
\end{aligned}
$$

Putting together the obtained inequalities, we obtain (3.8). □

Lemma 3.5. *Let $1<p<\infty$ and let the weight function $w\ge 0$ satisfy the conditions of Theorem* 3.1. *Then for the function ψ defined by* (3.5) *the following inequality holds*:

$$
\|\psi\|_{p,w} \ge C_4(p)N^{1/p}. \tag{3.9}
$$

Proof. We have

$$
N = |\langle\phi,\psi\rangle_w| \le \|\psi\|_{p,w}\|\phi\|_{p',w} \le C_3(p)N^{1/p'}\|\psi\|_{p,w},
$$

which yields inequality (3.9). □

By Lemmas 3.4 and 3.5 we finish the proof of Lemma 3.2. □

By Theorem 2.1, as in [2], we obtain the following theorem.

Theorem 3.3. *Let $1<p<\infty$ and let $\overline{\mathcal{H}}_N$ be the subsystem of the Haar system obtained by deleting its first N elements. Then $\overline{\mathcal{H}}_N$ is a greedy basis in any space $L^p(w)$ where it is a Schauder basis.*

Let $\Omega(N)$ be any set of N distinct nonnegative integers and let $\overline{\mathcal{H}}_{\Omega(N)}$ be the subsystem of the Haar system obtained by deleting Haar functions with the indices $\Omega(N)$. In [3] necessary and sufficient conditions were found on the weight function w for which the system $\overline{\mathcal{H}}_{\Omega(N)}$ is complete minimal in $L^p(w)$. Those conditions can be interpreted as follows: for any $\Omega(N)$ there exist N dyadic open intervals which are mutually disjoint and on any of those intervals w satisfies conditions similar to (3.1). The following theorem can be proved in the same way as the above theorem with some technical modifications.

Theorem 3.4. *Let* $1 < p < \infty$ *and let* $\Omega(N)$ *be any set of* N *distinct nonnegative integers. Then* $\overline{\mathcal{H}}_{\Omega(N)}$ *is a greedy basis in any space* $L^p(w)$ *where it is a Schauder basis.*

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