D. R. Yafaev, A new representation of Hankel operators and its spectral consequences,
Algebra i Analiz, 2018, Volume 30, Issue 3, 286–310

Алгебра и анализ
Том 30 (2018), №3

To the memory of Mikhail Zakharovich Solomyak

A NEW REPRESENTATION OF HANKEL OPERATORS AND ITS SPECTRAL CONSEQUENCES

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In the paper, the Hankel operators H are represented as pseudo-differential operators A in the space of functions defined on the whole axis. The amplitudes of such operators A have a very special structure: they are products of functions of a one variable only. This representation has numerous spectral consequences, both for compact Hankel operators and for operators with the continuous spectrum.

§1. Introduction

1.1. This is a short survey based on the talk given by the author at the 9th St.Petersburg Spectral Theory Conference held at the Euler Institute (St.Petersburg, Russia) during 3–6 July, 2017.

Among numerous papers of M. Sh. Birman and M. Z. Solomyak on spectral theory of selfadjoint operators, their study (summarized in [2]) of the Weyl asymptotics of eigenvalues of differential operators plays a distinguished role. The methods developed in their papers on this subject were extended by the authors to pseudo-differential and integral operators in [1] and [3]. We directly use the results of [1, 3] in this paper.

1.2. The Hankel operators can be defined as integral operators

$$(Hu)(t) = \int_0^\infty h(t+s)u(s)ds \tag{1.1}$$

in the space $L^2(\mathbb{R}_+)$ with kernels h that depend on the sum of variables only. If necessary, we write $H = H(h)$ for the operator (1.1). We refer to the books

Key words: Hankel operators, spectral properties, absolutely continuous and discrete spectra, asymptotics of eigenvalues.

Supported by Russian Science Foundation, project № 17-11-01126.

[7, 8] for basic information on Hankel operators. Of course, H is symmetric if $h(t) = \overline{h(t)}$. There are very few cases when Hankel operators can be diagonalized explicitly. The simplest and most important example $h(t) = t^{-1}$ was considered by T. Carleman. The corresponding Hankel operator (1.1) is bounded but not compact; actually, it has absolutely continuous spectrum $[0, \pi]$ of multiplicity 2. It follows that a Hankel operator H is compact if, for example, $h \in L^\infty_{\rm loc}(\mathbb{R}_+)$ and $h(t) = o(t^{-1})$ as $t \to \infty$ and as $t \to 0$. It turns out that the singular values $s_n(H)$ of H (and its eigenvalues in the selfadjoint case) have power asymptotics as $n \to \infty$ if the kernel $h(t)$ is close to t^{-1} in the logarithmic scale (both for large and small t), that is, if $h(t)$ behaves like $\kappa_\infty t^{-1} |\ln t|^{-\alpha}$ with some $\alpha > 0$ as $t \to \infty$ and $\kappa_0 t^{-1} |\ln t|^{-\alpha}$ as $t \to 0$. On the contrary, H is unbounded if $h(t)t \to \infty$ as $t \to \infty$ or as $t \to 0$.

Our first goal is to describe in §2 a procedure suggested in [16, 19] and reducing an arbitrary Hankel operator H by an explicit unitary transformation $\mathbf{M} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ (essentially, by the Mellin transform) to a special integral, or pseudo-differential, operator A in the space $L^2(\mathbb{R})$:

$$H = \mathbf{M}^{-1} A \mathbf{M}. \tag{1.2}$$

In many cases, the spectral properties of the operators A are easier to study than those of the original Hankel operators H. We emphasize that identity (1.2) does not require that the operators H be symmetric, but in our spectral applications all H are selfadjoint (except in §6).

The operator A can be defined as follows. Put

$$(Xu)(x) = xu(x), \quad (Du)(x) = -iu'(x).$$

Then

$$A = v(X)s(D)v(X) \tag{1.3}$$

where the standard function

$$v(x) = \frac{\sqrt{\pi}}{\sqrt{\cosh(\pi x)}} \tag{1.4}$$

is quite explicit; it is one and the same for all Hankel operators. The function $s(\xi)$ depends of course on $h(t)$, and it can be constructed in the following way.

Let us formally define the so-called *sigma-function* by the equation

$$h(t) = \int_0^\infty e^{-t\lambda} \sigma(\lambda) \, d\lambda; \tag{1.5}$$

in general, $\sigma(\lambda)$ is a distribution. The function $s(\xi)$ (in [16] it was called the *sign-function* of a Hankel operator H) differs from $\sigma(\lambda)$ by a change of variables only:

$$s(\xi) = \sigma(e^{-\xi}). \tag{1.6}$$

Thus, the operator A can be viewed in the space $L^2(\mathbb{R})$ either as a ΨDO (pseudo-differential operator) with the amplitude

$$a(x, y; \xi) = v(x)s(\xi)v(y), \quad x, y, \xi \in \mathbb{R},$$

or as an integral operator with the kernel

$$(2\pi)^{-1/2} v(x)(\Phi^* s)(x-y)v(y)$$

where Φ,

$$(\Phi f)(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx,$$

is the Fourier transform. Of course, a consistent definition of the operator A requires some assumptions on $h(t)$; this will be discussed later.

1.3. We apply this construction to two essentially different classes of Hankel operators H which lead to two essentially different classes of ΨDO A.

In §3, we consider generalized Carleman operators with kernels

$$h(t) = P(\ln t) t^{-1} \tag{1.7}$$

where

$$P(\xi) = \sum_{m=0}^{n} p_m \xi^m, \quad p_n = 1, \tag{1.8}$$

is an arbitrary real polynomial. Obviously, the kernels (1.7) have two singular points $t = \infty$ and $t = 0$. For $n \geqslant 1$, such Hankel operators are unbounded but are well defined as selfadjoint operators.

For the kernels (1.7), the sign-function $s(\xi)$ is a real polynomial

$$s(\xi) = \sum_{m=0}^{n} (-1)^m q_m \xi^m =: Q(\xi) \tag{1.9}$$

determined by $P(\xi)$. In this case

$$A = v(X)Q(D)v(X) \tag{1.10}$$

is a differential operator. The polynomials $P(\xi)$ and $Q(\xi)$ have the same degree, and their coefficients are linked by an explicit formula (see formula (3.2) below); in particular, $q_n = 1$. If $n = 0$, then

$$Q(\xi) = P(\xi) = 1,$$

so that A is the operator of multiplication by $v(x)^2$. This yields the familiar diagonalization of the Carleman operator.

Observe that the highest order term of the operator A equals $(-1)^n v^2(x) D^n$, where $v^2(x)$ tends to zero (exponentially) as $|x| \to \infty$. Apparently, such differential operators were never studied before, and we are led to fill in this gap. Studying the differential operators (1.10) in §3, we do not make the specific assumption (1.4) and consider sufficiently arbitrary real functions $v(x)$ tending to zero as $|x| \to \infty$. The essential spectrum of the differential operators (1.10) was localized in [15], where it was shown that $\operatorname{spec}_{\rm ess}(A) = \operatorname{spec}(A) = \mathbb{R}$ if n is odd, and $\operatorname{spec}_{\rm ess}(A) = [0, \infty)$ if n is even. The last result should be compared with the fact that $\operatorname{spec}_{\rm ess}(A) = [\min Q(\xi), \infty)$ if $v(x) = 1$. Thus, even in this relatively simple question, the degeneracy of $v(x)$ at infinity significantly changes spectral properties of the differential operators A. The detailed spectral structure, in particular, the absolutely continuous spectrum of the differential operators (1.10) and hence of the Hankel operators with kernels (1.7) was described in [20].

1.4. In §4, we are interested in the compact selfadjoint Hankel operators (1.1) with power-like asymptotics of eigenvalues $\lambda_n^{\pm}(H)$ as $n \to \infty$. Let $\{\lambda_n^{+}(H)\}_{n=1}^{\infty}$ denote the nonincreasing sequence of positive eigenvalues of a compact selfadjoint operator H (with multiplicities taken into account), and set

$$\lambda_n^{-}(H) = \lambda_n^{+}(-H).$$

Sharp estimates of $\lambda_n^{\pm}(H)$ (and, more generally, of the singular values

$$s_n(H) = \lambda_n^{+}(\sqrt{H^* H})$$

in the nonselfadjoint case) are very well known. Thus, V. V. Peller found (see Chapter 6 of his book [8]) necessary and sufficient conditions for the validity of the estimates $s_n(H) = O(n^{-\alpha})$. At the same time, there are practically no results on the asymptotic behavior of eigenvalues of Hankel operators. This state of affairs is in a sharp contrast with the case of differential operators, where the Weyl type asymptotics of eigenvalues is established in a very large variety of situations. Our goal here is to fill this gap by describing classes of compact Hankel operators where the leading term of eigenvalue asymptotics can be found explicitly.

In general, the study of eigenvalue asymptotics for any class of operators involves two steps: construction of an appropriate model problem where the eigenvalue asymptotics can be determined sufficiently explicitly, and using eigenvalue estimates (or variational methods) to extend the asymptotics to a wider class of operators. Apparently, for a given Hankel operator H, there is no natural model operator in the class of Hankel operators. So, the crucial step of our approach is a construction of the model operator ΨDO A_*. The spectral asymptotics of A_* and hence of the corresponding Hankel operators H_* is given by the Weyl law. This result is then extended to the initial operator H.

1.5. Hankel operators can also be realized naturally in the space $\ell^2(\mathbb{Z}_+)$ of sequences. Namely, for sequences $g = \{g(j)\}_{n\in\mathbb{Z}_+}$, the Hankel operators $G = G(g)$ are defined by infinite matrices:

$$(Gu)(j) = \sum_{k=0}^{\infty} g(j+k)u(k). \tag{1.11}$$

As is well known, the operators H and G give two different representations of one and the same object. Indeed, let us introduce a unitary transformation

$$U\colon \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)$$

by the formula

$$(Uu)(t) = \sum_{j=0}^{\infty} \mathcal{L}_j(t)u(j)e^{-t/2}, \quad u = \{u(j)\}_{j\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+),$$

where the $\mathcal{L}_j = \mathcal{L}_j^0$ are the Laguerre polynomials. The Hankel operators H and G are linked by this transformation, i.e.,

$$UGU^* = H, \tag{1.12}$$

provided

$$h(t) = \sum_{j=0}^{\infty} \mathcal{L}_j^1(t)g(j)e^{-t/2}$$

where the $\mathcal{L}_j^1$ are the generalized Laguerre polynomials.

Although the operators H and G are unitarily equivalent, it is convenient to study the eigenvalues asymptotics of G independently of the results on H. This is done in §5.

In the discrete case, the role of the Carleman operator is played by the Hankel operator G (the Hilbert matrix) with the matrix elements $g(j) = (j+1)^{-1}$. This operator is bounded but not compact; actually, it has the simple absolutely continuous spectrum $[0,\pi]$. Note that the asymptotic behavior of a kernel $h(t)$ like $t^{-1}|\ln t|^{-\alpha}$ as $t\to\infty$ (respectively, as $t\to 0$) is essentially equivalent to the asymptotic behavior of the matrix elements of the corresponding Hankel operator G like $(-1)^j j^{-1}(\ln j)^{-\alpha}$ (respectively, like $j^{-1}(\ln j)^{-\alpha}$) as $j\to\infty$. It is often useful to keep in mind that the Hankel operators G and $\widetilde{G}$ are unitarily equivalent provided their matrix elements are linked by the relation $\widetilde{g}(j) = (-1)^j g(j)$.

It is natural to expect that a faster rate of convergence to zero as $j\to\infty$ of the sequence $g(j)$ results in a faster convergence to zero as $n\to\infty$ of the eigenvalues $\lambda_n^{\pm}(G)$. Indeed, there is a deep result of H. Widom who showed in [14] that for $\gamma > 1$ the Hankel operator corresponding to the sequence $g(j) = (j+1)^{-\gamma}$ is nonnegative and its eigenvalues converge to zero as

$$\lambda_n^{+}(G) = \exp(-\pi\sqrt{2\gamma n} + o(\sqrt{n})), \quad n \to \infty.$$

By the way, for the proof of this result, H. Widom also used a reduction of the Hankel operators he considered to the ΨDO $s(D)^{1/2}v(X)^2 s(D)^{1/2}$. Such a reduction is possible if $s(\xi) \geqslant 0$.

Note that both Hankel operators H and G can be realized in Hardy spaces of analytic functions, where these operators are determined by their symbols. We do not discuss the representations of Hankel operators in Hardy spaces but emphasize that their symbols briefly mentioned in §6 and the sigma-functions are completely different objects — see §3 in [19].

1.6. Finally, in §6 we discuss more general results on singular values and eigenvalues of Hankel operators with kernels $h(t)$ oscillating as $t \to \infty$ and with matrix elements $g(j)$ oscillating as $j \to \infty$.

The results on the power-like asymptotics of singular values have direct applications to rational approximations of functions with logarithmic singularities. A result of this type is stated in §6.

§2. Main identity

A detailed presentation of the results of this section can be found in the papers [16, 19].

2.1. For a given Hankel operator H, let the sigma-function $\sigma(\lambda)$ be *formally* defined by equation (1.5), and let Σ be the operator of multiplication by σ, that is,

$$(\Sigma g)(\lambda) = \sigma(\lambda) g(\lambda), \quad \lambda > 0. \tag{2.1}$$

We shall show that

$$H = \mathsf{L}^* \Sigma \mathsf{L} \tag{2.2}$$

where L is the Laplace transform:

$$(\mathsf{L} f)(\lambda) = \int_0^\infty e^{-t\lambda} f(t)\, dt. \tag{2.3}$$

A *formal* proof of identity (2.2) is quite simple. Indeed, the integral kernel of the operator on the right-hand side of (2.2) equals

$$\int_0^\infty e^{-\lambda t}\sigma(\lambda) e^{-\lambda s} d\lambda = h(t+s)$$

if $\sigma(\lambda)$ and $h(t)$ are linked by formula (1.5). Thus, it equals the integral kernel of the operator defined by (1.1).

The precise meaning of formula (2.2) needs of course to be clarified. Observe that, by its definition (1.5), $\sigma(\lambda)$ can be a regular function only for kernels $h(t)$ satisfying some specific analytic assumptions. Without such very restrictive assumptions, σ is necessarily a distribution. Even for very good kernels $h(t)$ (and especially for them), $\sigma(\lambda)$ may be a highly singular distribution. For example, for $h(t) = t^k e^{-\alpha t}$ with $\operatorname{Re}\alpha > 0$ (α may be complex) and $k = 0, 1, \ldots$, the sigma-function

$$\sigma(\lambda) = \delta^{(k)}(\lambda - \alpha)$$

is a derivative of the delta-function. On the contrary, singular kernels $h(t)$ may yield sigma-functions $\sigma(\lambda)$ smooth on $\mathbb{R}_+$. For example, if $h(t) = t^{-q}$ where $q > 0$ may be arbitrarily large, then

$$\sigma(\lambda) = \Gamma(q)^{-1}\lambda^{1-q};$$

here and below $\Gamma(\,\cdot\,)$ is the gamma function.

Thus, we replace (2.2) by the identity

$$(Hf_1, f_2) = (\Sigma \mathsf{L} f_1, \mathsf{L} f_2)$$

for arbitrary test functions $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$. With respect to h we require only that $h \in C_0^\infty(\mathbb{R}_+)'$. Let us introduce the Laplace convolution

$$(\bar{f_1} \star f_2)(t) = \int_0^t \overline{f_1(s)} f_2(t-s)\,ds.$$

Then, formally,

$$(Hf_1, f_2) = \langle h, \bar{f_1} \star f_2 \rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the duality symbol.

By versions of the Paley–Wiener theorem, the Laplace transform L is an isomorphism of $C_0^\infty(\mathbb{R}_+)$ onto the space $\mathcal{Y}$ of analytic functions $g(\lambda)$ of $\lambda \in \mathbb{C}$ exponentially decaying as $\operatorname{Re}\lambda \to +\infty$, exponentially bounded as $\operatorname{Re}\lambda \to -\infty$ and decaying faster than any power of $|\lambda|^{-1}$ as $|\operatorname{Im}\lambda| \to \infty$ (see [19] for the details). By duality, $\mathsf{L}^*\colon \mathcal{Y}' \to C_0^\infty(\mathbb{R}_+)'$ is also an isomorphism, and hence, in accordance with the definition (1.5),

$$\sigma = (\mathsf{L}^*)^{-1} h \in \mathcal{Y}' \tag{2.4}$$

if $h \in C_0^\infty(\mathbb{R}_+)'$. This yields a one-to-one correspondence between the kernels $h \in C_0^\infty(\mathbb{R}_+)'$ of Hankel operators and their sigma-functions $\sigma \in \mathcal{Y}'$ and makes the theory self-consistent. Note that instead of operators, we can work with quadratic forms, which is both more general and more convenient. For $g \in \mathcal{Y}$, we set $g^*(\lambda) = \overline{g(\bar{\lambda})}$.

Now we are in a position to state our main identity (2.2) precisely.

Theorem 2.1. *Let $h \in C_0^\infty(\mathbb{R}_+)'$, and let $\sigma \in \mathcal{Y}'$ be defined by formula* (2.4). *Then the identity*

$$\langle h, \bar{f}_1 \star f_2 \rangle = \langle \sigma, (\mathsf{L} f_1)^* \mathsf{L} f_2 \rangle$$

holds true for arbitrary $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$.

Identity (2.2) does not of course give a diagonalization of Hankel operators because the operator L is not unitary. However it is continuously invertible as a mapping $\mathsf{L}\colon C_0^\infty(\mathbb{R}_+) \to \mathcal{Y}$, so that identity (2.2) plays the same role as Sylvester's inertia theorem, which states that two Hermitian matrices H and Σ related as in (2.2) have the same total numbers of positive and negative eigenvalues. In particular, $\pm H \geqslant 0$ if and only if $\pm\Sigma \geqslant 0$. By Theorem 2.1, the same assertion is true for the Hankel operators H and the operators of multiplication Σ. Now the operators H and Σ are of completely different nature, and Σ (but not H) admits explicit spectral analysis. As an example of this approach, in §4 of [18] we show that if $\sigma(\lambda) > 0$ (or $\sigma(\lambda) < 0$) on a set of positive Lebesgue measure, then the Hankel operator H has infinite positive (or negative) spectrum. On the other hand, the singularities of $\sigma(\lambda)$ at some isolated points produce finite numbers (depending on the order of the singularity) of positive or negative eigenvalues (see [16, §4]). In particular, this approach enables us (see [17]) to give an explicit formula for the total numbers of positive and negative eigenvalues of finite-rank Hankel operators.

2.2. To perform spectral analysis for Hankel operators, we transform identity (2.2) using the factorization of the operator L. We introduce the Mellin transform $M\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$,

$$(Mf)(x) = (2\pi)^{-1/2} \int_0^\infty f(t) t^{-1/2-ix}\, dt,$$

and the reflection operator $\mathcal{J}$, $(\mathcal{J}u)(x) = u(-x)$, and set

$$(\mathbf{\Gamma} u)(x) = \Gamma(1/2 + ix) u(x).$$

We use the following elementary fact.

Lemma 2.2. *For the Laplace transform defined by* (2.3) *we have*

$$(\mathsf{L} f)(\lambda) = (M^{-1} \mathcal{J} \mathbf{\Gamma} M f)(\lambda), \quad \lambda > 0, \tag{2.5}$$

for all $f \in L^2(\mathbb{R}_+)$.

By the way, the factorization (2.5) enables one to invert the Laplace transform:

$$\mathsf{L}^{-1} = M^{-1} \mathbf{\Gamma}^{-1} \mathcal{J} M.$$

We use (2.5) to establish the unitary equivalence of the operators H and A. Observe that

$$|\Gamma(1/2+ix)| = \frac{\sqrt{\pi}}{\sqrt{\cosh(\pi x)}} = v(x)$$

and set

$$(\mathbf{M}f)(x) = \frac{\Gamma(1/2-ix)}{|\Gamma(1/2-ix)|}(Mf)(-x).$$

Then (2.5) can be rewritten as

$$\mathsf{L} = M^{-1}v(X)\mathbf{M}. \tag{2.6}$$

Note also that $M = \Phi^{-1}W$, where the unitary operator

$$W\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$$

is defined by $(Wf)(\xi) = e^{-\xi/2}f(e^{-\xi})$, whence from (1.6), (2.1) it follows that

$$s(D) = M^{-1}\Sigma M.$$

In view of (2.6), this yields the identity

$$H = \mathbf{M}^{-1}v(X)s(D)v(X)\mathbf{M}. \tag{2.7}$$

Now we state a simple sufficient condition for the validity of this representation, which has been derived above rather formally.

Theorem 2.3. *Suppose that $\sigma \in L^\infty(\mathbb{R}_+)$, so that the operator $s(D)$ is bounded. Then identity* (2.7) *is true.*

2.3. The above results on the integral operators (1.1) can be extended to the Hankel operators G defined by formula (1.11) in the space $\ell^2(\mathbb{Z}_+)$. In the discrete case, the role of the sigma-function $\sigma(\lambda)$ of $\lambda \in \mathbb{R}_+$ is played by the function $\eta(\mu)$ defined on the interval $(-1,1)$ and linked to $\sigma(\lambda)$ by the relation

$$\sigma(\lambda) = \eta\left(\frac{2\lambda-1}{2\lambda+1}\right). \tag{2.8}$$

Let

$$g(j) = \int_{-1}^{1} \eta(\mu)\mu^j\,d\mu, \quad j = 0,1,2,\dots, \tag{2.9}$$

be the sequence of the moments of η, and let G be the Hankel operator with the matrix elements $g(j)$. We emphasize that equations (2.9) play the role of (1.5). It can easily be shown that relation (1.12) is satisfied if the kernel $h(t)$ of H is given by (1.5). Therefore, an analog of Theorem 2.3 is stated as follows.

Theorem 2.4 ([19, Theorem 7.7]). *Let $\eta \in L^\infty(-1,1)$. Then the Hankel operator G in $\ell^2(\mathbb{Z}_+)$ with the matrix elements* (2.9) *is unitarily equivalent to the ΨDO* (1.3) *in $L^2(\mathbb{R})$ with the sign-function $s(\xi)$ defined by* (1.6), (2.8).

§3. Generalized Carleman operators

A detailed presentation of the results of this section can be found in the papers [15, 20].

3.1. Our goal here is to study spectral properties of generalized Carleman operators with kernels (1.7) where $P(\xi)$ is an arbitrary real polynomial (1.8). In this case, relation (1.5) is satisfied with

$$\sigma(\lambda) = \sum_{m=0}^{n} q_m \ln^m \lambda \tag{3.1}$$

where the coefficients

$$q_m = \sum_{j=m}^{n} \binom{j}{m} \gamma^{(j-m)}(0) p_j, \quad m = 0, \dots, n, \quad \gamma(z) = \Gamma(1-z)^{-1}. \tag{3.2}$$

For example, $q_n = p_n$ and $q_{n-1} = p_{n-1} + \Gamma'(1)\, n\, p_n$ for all n (recall that $-\Gamma'(1)$ is the Euler constant). Of course formulas (3.2) enable one to recover the coefficients $p_n, p_{n-1}, \dots, p_0$ given the coefficients $q_n, q_{n-1}, \dots, q_0$. It follows from (1.6), (3.1) that $s(D) =: Q(D)$ is a differential operator given by the formula

$$Q(D) = \sum_{m=0}^{n} (-1)^m q_m D^m.$$

For the Hankel operators (1.1) with kernels (1.8), identity (2.7) yields the following result.

Theorem 3.1 ([15, Theorem 3.2]). *Let $Q(\xi)$ be the polynomial* (1.9) *with the coefficients q_m defined by formulas* (3.2), *and let A be the differential operator* (1.10). *Then for all functions $u_j \in L^2(\mathbb{R}_+)$, $j = 1, 2$, such that their Mellin transforms Mu_j belong to $C_0^\infty(\mathbb{R})$, we have*

$$(Hu_1, u_2) = (A\mathbf{M}u_1, \mathbf{M}u_2).$$

A large part of our results on generalized Carleman operators can be summarized in the following assertion. Below we denote by $\langle x \rangle$ the operator of multiplication by the function $(1 + x^2)^{1/2}$.

Theorem 3.2. *Let H be the selfadjoint Hankel operator defined by formula* (1.1), *where $h(t)$ is the function* (1.7) *and $P(\xi)$ is a real polynomial* (1.8) *of degree $n \geqslant 1$. Then the following is true.*

(i) *The spectrum of H is absolutely continuous except for eigenvalues that may accumulate to zero and infinity only.*
(ii) *The absolutely continuous spectrum of H covers $\mathbb{R}$ and is simple for n odd. It coincides with $[0,\infty)$ and has multiplicity 2 for n even.*
(iii) *If n is odd, then the multiplicities of the eigenvalues of H are bounded (from above) by $(n-1)/2$. If n is even, then the multiplicities of the positive eigenvalues are bounded by $n/2-1$, and the multiplicities of the negative eigenvalues are bounded by $n/2$.*
(iv) *For all $\delta > 1/2$, the operator-valued function*

$$\langle \ln t\rangle^{-\delta}(H-z)^{-1}\langle \ln t\rangle^{-\delta}, \quad \operatorname{Im} z \neq 0, \tag{3.3}$$

is Hölder continuous with any exponent $\alpha < \delta - 1/2$ (and $\alpha < 1$) up to the real axis, except for the eigenvalues of the operator H and the point zero.

Clearly, this assertion is similar in spirit to the corresponding results for differential operators of Schrödinger type. Statement (iv) is known as the limiting absorption principle.

3.2. In view of Theorem 3.1, the proof of Theorem 3.2 reduces to the proof of the corresponding results for the differential operator A defined by (1.10). However, the standard results on differential operators are not applicable in this case because of a strong degeneracy of $v(x)$ at infinity. Fortunately, the operators (1.10) can be reduced by an explicit unitary transformation $\mathbf{L}$ (the generalized Liouville transformation) to standard differential operators. Set

$$(\mathbf{L}u)(x) = y'(x)^{1/2}u(y(x)),$$

where the variables x and y are linked by the relation

$$y = y(x) = \int_0^x v(s)^{-2/n}\, ds,$$

so that $y'(x) = v(x)^{-2/n}$. Then $B = \mathbf{L}^{-1}A\mathbf{L}$ is also a differential operator in the space $L^2(\mathbb{R})$, and it is given by the formula

$$B = D^n + \sum_{m=0}^{n-1} b_m(y)D^m, \qquad D = D_y = -id/dy. \tag{3.4}$$

Our crucial observation is that the coefficients $b_m(y)$, $m = 0,1,\dots,n-1$, of the operator B decay at infinity. Moreover, all coefficients $b_0(y),\dots,b_{n-2}(y)$ of the operator B are short-range, that is, they decay faster than $|y|^{-1}$ as

$|y| \to \infty$. The coefficient $b_{n-1}(y)$ can be removed by a gauge transformation Φ defined by

$$(\Phi u)(y) = e^{i\phi(y)}u(y) \quad \text{with} \quad \phi(y) = -\frac{1}{n}\int_0^y b_{n-1}(s)ds.$$

This means that, again, the operator $\widetilde{B} = \Phi^* B \Phi$ has the form (3.4) with $\widetilde{b}_{n-1}(y) = 0$. The coefficients $\widetilde{b}_0(y), \dots, \widetilde{b}_{n-2}(y)$ remain short-range.

Using fairly standard methods of scattering theory, we obtain assertions (i), (ii) and (iii) of Theorem 3.2 for the operator B. Since

$$A = \mathbf{L}B\mathbf{L}^{-1} \quad \text{and} \quad H = \mathbf{M}^{-1}A\mathbf{M}, \tag{3.5}$$

these results remain true for the operators A and H. Recall that for differential operators B, their (generalized) eigenfunctions are defined as the special solutions $\psi(y,k)$, $k \in \mathbb{R}$, of the equation $B\psi = k^n\psi$ satisfying some asymptotic conditions as $y \to \infty$ and $y \to -\infty$ and then the expansion theorem over these eigenfunctions is established. Relation (3.5) allows us to carry these results over to the operators A and H. In accordance with to (3.5), one can define the eigenfunctions of the operator H by the relation

$$\begin{aligned}\theta(t,k) &= (\mathbf{M}^{-1}\mathbf{L}\psi(k))(t)\\ &= (2\pi)^{-1/2}t^{-1/2}\int_{-\infty}^{\infty} e^{-ix(y)\ln t}e^{i\eta(x(y))}x'(y)^{1/2}\psi(y,k)\,dy,\end{aligned}$$

where $\eta(x) = \arg\Gamma(1/2 + ix)$. This integral converges, although not absolutely. By application of the stationary phase method, the asymptotics of the eigenfunctions $\theta(t,k)$ as $t \to 0$ and as $t \to \infty$ can be deduced from this representation. In particular, on the compact subsets of $\mathbb{R} \setminus \{0\}$, away from the eigenvalues of H we opbtain the following estimate uniform in k:

$$|\theta(t,k)| \leqslant Ct^{-1/2},$$

and a similar estimate is true for the differences $\theta(t,k') - \theta(t,k)$. Using these estimates, one can prove the assertion (iv) of Theorem 3.2. This result looks like the limiting absorption principle for differential operators. The difference, however, is that the weight is $\langle \ln t\rangle^{-\delta}$ in (3.3) while it is $\langle x\rangle^{-\delta}$ (also with $\delta > 1/2$) for the resolvents of differential operators. Thus the power scale for differential operators corresponds to the logarithmic scale for Hankel operators.

3.3. Actually, the specific expression (1.4) for the function $v(x)$ in the definition (1.10) of the operator A is inessential. It should be noted that if $v(x)$ tends to zero exponentially as $|x| \to \infty$, then the coefficients $b_0(y), \dots, b_{n-2}(y)$ of the operator B decay faster than $|y|^{-1}$ as $|y| \to \infty$. On the contrary, for a slower decay of $v(x)$, these coefficients decay slower than (or as) $|y|^{-1}$. Thus, somewhat counter-intuitively, a stronger degeneracy of the operator (1.10) yields better properties of the operator $B = \mathbf{L}^{-1} A \mathbf{L}$.

We also note that our approach applies to fairly arbitrary differential operators of order n with a degeneracy of the coefficient of D^n.

§4. Compact operators. Asymptotics of singular values and eigenvalues

4.1. In this subsection we collect necessary auxiliary results. First, we recall the Weyl asymptotics of ΨDO. For $x \in \mathbb{R}$, we use the standard notation $x_\pm = \max\{0, \pm x\}$.

Theorem 4.1. *Let $s \in C^\infty(\mathbb{R})$ be a real-valued function such that*

$$s(\xi) = \begin{cases} s_\infty \xi^{-\alpha}(1 + o(1)) & \text{as} \quad \xi \to \infty, \\ s_{-\infty} |\xi|^{-\alpha}(1 + o(1)) & \text{as} \quad \xi \to -\infty \end{cases} \tag{4.1}$$

for some $\alpha > 0$ and some constants s_∞ and $s_{-\infty}$. Assume that $v(x) = \overline{v(x)}$ and

$$|v(x)| \leqslant C \langle x \rangle^{-\rho}, \quad x \in \mathbb{R}, \tag{4.2}$$

where $\rho > \alpha/2$. Put

$$\mathbf{a}^\pm = (2\pi)^{-\alpha} \big((s_{-\infty})_\pm^{1/\alpha} + (s_\infty)_\pm^{1/\alpha} \big)^\alpha \Big(\int_{-\infty}^{\infty} |v(x)|^{2/\alpha} dx \Big)^\alpha .$$

Then for the ΨDO (1.3) in $L^2(\mathbb{R})$ we have

$$\lambda_n^\pm(A) = \mathbf{a}^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty.$$

For compactly supported v, Theorem 4.1 was proven by M. Sh. Birman and M. Z. Solomyak in [1]. Their result can be easily extended to arbitrary functions v satisfying (4.2) (see, e.g., Appendix in [10]). Actually, in [1], operators (1.3) were considered in the space $L^2(\mathbb{R}^d)$ with an arbitrary dimension d. For general ΨDO (acting in a bounded domain) with amplitudes asymptotically homogeneous at infinity, Weyl type formula for the asymptotics of the spectrum was obtained in [3].

We need also some estimates on the singular values $s_n(H)$ of Hankel operators of the form (1.1). They are stated in the next assertion established in [9].

For $\alpha \geqslant 1/2$, the proof of these estimates relies heavily on deep results by Peller (see Chapter 6 of his book [8]). For an arbitrary $\alpha > 0$, we set

$$N(\alpha) = [\alpha] + 1 \text{ if } \alpha \geqslant 1/2 \quad \text{and} \quad N(\alpha) = 0 \text{ if } \alpha < 1/2. \tag{4.3}$$

Theorem 4.2. *Let $\alpha > 0$, and let $h \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ be a complex-valued function; if $\alpha \geqslant 1/2$, suppose also that $h \in C^{N(\alpha)}(\mathbb{R}_+)$. Assume that h satisfies the conditions*

$$h^{(m)}(t) = o(t^{-1-m}\langle \log t\rangle^{-\alpha}) \qquad \text{as} \quad t \to 0 \text{ and as } \quad t \to \infty$$

for all $m = 0, 1, \dots, N(\alpha)$. Then $s_n(H) = o(n^{-\alpha})$ as $n \to \infty$.

Theorem 4.2 remains true if $o(\,\cdot\,)$ (on both occurences above) is replaced by $O(\,\cdot\,)$.

We need also the following standard result (see, e.g., [4, §11.6]) in the spectral perturbation theory, which asserts the stability of eigenvalue asymptotics.

Lemma 4.3. *Let K_0 and K be compact selfadjoint operators, and let $\alpha > 0$. Suppose that, for both signs "±",*

$$\lambda_n^\pm(K_0) = \mathbf{a}^\pm n^{-\alpha} + o(n^{-\alpha}), \qquad \text{and} \qquad s_n(K) = o(n^{-\alpha}), \qquad n \to \infty.$$

Then

$$\lambda_n^\pm(K_0 + K) = \mathbf{a}^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty.$$

4.2. Our main result on the asymptotics of the eigenvalues of Hankel operators (1.1) can be stated as follows. Set

$$\tau(\alpha) = 2^{-\alpha}\pi^{1-2\alpha}B(\tfrac{1}{2\alpha}, \tfrac{1}{2})^\alpha, \tag{4.4}$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

is the Beta function.

Theorem 4.4. *Let $\alpha > 0$, and let the integer $N(\alpha)$ be given by (4.3). Let h be a real-valued function in $L^\infty_{\text{loc}}(\mathbb{R}_+)$; if $\alpha \geqslant 1/2$, assume also that $h \in C^{N(\alpha)}(\mathbb{R}_+)$. Suppose that*

$$\left(\frac{d}{dt}\right)^m \big(h(t) - \kappa_0 t^{-1}(\log(1/t))^{-\alpha}\big) = o(t^{-1-m}\langle \log t\rangle^{-\alpha}), \quad t \to 0,$$

$$\left(\frac{d}{dt}\right)^m \big(h(t) - \kappa_\infty t^{-1}(\log t)^{-\alpha}\big) = o(t^{-1-m}\langle \log t\rangle^{-\alpha}), \quad t \to \infty,$$

for some $\kappa_0, \kappa_\infty \in \mathbb{R}$ and all $m = 0, \dots, N(\alpha)$. Then the eigenvalues of the corresponding Hankel operator H have the asymptotic behavior

$$\lambda_n^\pm(H) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{4.5}$$

where

$$a^{\pm} = \tau(\alpha)\big((\kappa_0)_{\pm}^{1/\alpha} + (\kappa_\infty)_{\pm}^{1/\alpha}\big)^{\alpha}. \tag{4.6}$$

Our proof of this result relies on the following three ingredients.

(i) Theorem 2.3 allows us to replace the Hankel operators (1.1) by the ΨDO A defined by (1.3).

(ii) Weyl type spectral asymptotics for ΨDO of this type stated in Theorem 4.1.

(iii) Estimates on the singular values of Hankel operators, see Theorem 4.2.

4.3. Let us sketch the proof of Theorem 4.4. The first and most important step is to construct a *model operator*. For that, we introduce an auxiliary explicit function by the formula

$$\sigma_*(\lambda) = \kappa_\infty \, |\log\lambda|^{-\alpha}\chi_0(\lambda) + \kappa_0|\log\lambda|^{-\alpha}\chi_\infty(\lambda), \quad \lambda > 0, \tag{4.7}$$

where the cut-off functions $\chi_0, \chi_\infty \in C^\infty(\mathbb{R}_+)$ satisfy

$$\chi_0(t) = \begin{cases} 1 & \text{if } t \leqslant 1/4, \\ 0 & \text{if } t \geqslant 1/2, \end{cases} \qquad \chi_\infty(t) = \begin{cases} 0 & \text{if } t \leqslant 2, \\ 1 & \text{if } t \geqslant 4. \end{cases} \tag{4.8}$$

It turns out that the functions $h(t)$ and $h_*(t) = (\mathsf{L}\sigma_*)(t)$ have the same asymptotics as $t \to \infty$; a similar relation holds true as $t \to 0$. For the proof, we need an elementary technical result about the Laplace transform of functions with logarithmic singularities at $\lambda = 0$ and $\lambda = \infty$.

Lemma 4.5. *Let $\alpha > 0$, $m \in \mathbb{Z}_+$, and put*

$$I_m^{(\infty)}(t) = \int_0^c (-\log\lambda)^{-\alpha}\lambda^m e^{-\lambda t} d\lambda, \quad c \in (0,1),$$

and

$$I_m^{(0)}(t) = \int_c^\infty (\log\lambda)^{-\alpha}\lambda^m e^{-\lambda t} d\lambda, \quad c > 1.$$

Then

$$I_m^{(\infty)}(t) = m!\, t^{-1-m}|\log t|^{-\alpha}\big(1 + O(|\log t|^{-1})\big) \tag{4.9}$$

as $t \to \infty$. The integral $I_m^{(0)}(t)$ has the same asymptotic behavior (4.9) *as $t \to 0$.*

This result is well known; see, e.g., Lemmas 3 and 4 in [5]. A simple straightforward proof can be found in [10].

Corollary 4.6. *Let the function σ_* be given by* (4.7)*, and let $h_* = \mathsf{L}\sigma_*$ be its Laplace transform. Then*

$$h_*(t) = \kappa_0 h_0(t) + \kappa_\infty h_\infty(t) + \widetilde{h_*}(t),$$

where the model kernels h_0, h_∞ are defined by

$$h_0(t) = t^{-1}|\log t|^{-\alpha}\chi_0(t), \quad h_\infty(t) = t^{-1}|\log t|^{-\alpha}\chi_\infty(t), \tag{4.10}$$

and the error term $\widetilde{h} \in C^\infty(\mathbb{R}_+)$ satisfies the estimates

$$|\widetilde{h_*}^{(m)}(t)| \leqslant C_m t^{-1-m}\langle \log t\rangle^{-\alpha-1}, \quad t > 0,$$

for all integers $m \geqslant 0$.

Our model operator is the Hankel operator $H_* = H(h_*)$ with the kernel $h_* = \mathsf{L}\sigma_*$.

Lemma 4.7. *The eigenvalues of the operator H_* obey the asymptotic relation*

$$\lambda_n^\pm(H_*) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{4.11}$$

where the coefficients $a^\pm$ are given by (4.6)*.*

Indeed, by Theorem 2.3, the Hankel operator H_* is unitarily equivalent to the ΨDO $A_* = v(X)s_*(D)v(X)$ in $L^2(\mathbb{R})$. As usual, $v(x)$ is the standard function (1.4), and from (1.6) and (4.7) it follows that

$$s_*(\xi) = \sigma_*(e^{-\xi}) = \kappa_\infty|\xi|^{-\alpha}\chi_0(e^{-\xi}) + \kappa_0|\xi|^{-\alpha}\chi_\infty(e^{-\xi}), \quad \xi \in \mathbb{R}.$$

This function belongs to $C^\infty(\mathbb{R})$ and has the asymptotic behavior (4.1) with $s_\infty = \kappa_\infty$ and $s_{-\infty} = \kappa_0$. Therefore, Theorem 4.1 (the Weyl spectral asymptotics of ΨDO) applies to the operator A_*. This yields the asymptotic formula

$$\lambda_n^\pm(A_*) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{4.12}$$

where

$$a^\pm = (2\pi)^{-\alpha}\big((\kappa_0)_\pm^{1/\alpha} + (\kappa_\infty)_\pm^{1/\alpha}\big)^\alpha \left(\int_{-\infty}^{\infty} (\pi(\cosh(\pi x))^{-1})^{1/\alpha} dx\right)^\alpha.$$

Using the change of variables $y = (\cosh(\pi x))^2$, it is easy to check that the coefficients $a^\pm$ here and in (4.6) coincide. Theorem 2.4 shows that the operators H_* and A_* are unitarily equivalent, so that relation (4.12) yields (4.11). □

Now we are in a position to conclude the proof of Theorem 4.4. By its hypotheses, we have the representation

$$h(t) = \kappa_0 h_0(t) + \kappa_\infty h_\infty(t) + \widetilde{h}(t),$$

where h_0 and h_∞ are given by (4.10) and $\widetilde{h}$ satisfies the assumptions of Theorem 4.2 (singular value estimates). Therefore, from Corollary 4.6 it follows that the difference

$$h - h_* = \widetilde{h} - \widetilde{h_*}$$

also satisfies the hypothesis of Theorem 4.2, whence

$$s_n(H - H_*) = o(n^{-\alpha}), \quad n \to \infty. \tag{4.13}$$

In view of the abstract Lemma 4.3, the asymptotic formula (4.5) is a direct consequence of (4.11) and (4.13). □

4.4. We emphasize that the asymptotics of the spectrum of the integral Hankel operators (1.1) is determined by the behavior of $h(t)$ as $t \to 0$ and $t \to \infty$ as well as by local singularities of $h(t)$. Following [16], we consider Hankel operators whose integral kernels (or their derivatives) have jumps of continuity at some positive point.

Theorem 4.8. *Suppose $l \in \mathbb{Z}_+$, $t_0 > 0$, and let $h(t) = h_0(t_0 - t)^l$ for $t \leqslant t_0$ and $h(t) = 0$ for $t > t_0$. Then the eigenvalues of the Hankel operator H have the asymptotics*

$$\lambda_n^\pm(H) = |h_0| l! (2\pi)^{-l-1} t_0^{l+1} n^{-l-1}(1 + O(n^{-1})) \tag{4.14}$$

as $n \to \infty$.

Of course, the exact expression for $h(t)$ is inessential. Indeed, if a real function $v(t)$ satisfies the assumptions of Theorem 4.2 with $\alpha = l$, then the singular numbers $s_n(V)$ of the Hankel operator V with kernel $v(t)$ satisfy the bound $s_n(V) = o(n^{-l-1})$. Therefore, by Lemma 4.3, the asymptotics (4.14) remains true for the eigenvalues of the Hankel operator $H + V$; however, in this case the remainder $O(n^{-1})$ in (4.14) should be replaced by $o(1)$.

We emphasize that, in accordance with (4.14), the leading terms of the asymptotics of positive and negative eigenvalues of the Hankel operator H are the same. Of course, if $h(t)$ becomes smoother (l increases), then the eigenvalues of H decrease faster as $n \to \infty$. Observe that for $l = 0$ (when the kernel itself is discontinuous), the Hankel operator H does not belong to the trace class. We finally note that under the assumptions of Theorem 4.8 the asymptotics of the singular values of the operator H was found long ago in [6].

§5. Discrete case

5.1. Let now a Hankel operator G be defined by formula (1.11). In the discrete case, the role of derivatives $h^{(m)}(t)$ of a function $h(t)$ is played by iterated differences $g^{(m)}(j)$ of a sequence $g(j)$. Those are defined iteratively by setting $g^{(0)}(j) = g(j)$ and

$$g^{(m)}(j) = g^{(m-1)}(j+1) - g^{(m-1)}(j), \quad m \geqslant 1.$$

The following result plays the role of Theorem 4.2.

Theorem 5.1 ([9]). *Let $\alpha > 0$, and let g be a sequence of complex numbers such that*

$$g^{(m)}(j) = o\big(j^{-1-m}\langle \log j\rangle^{-\alpha}\big), \quad j \to \infty,$$

for all $m = 0, 1, \dots, N(\alpha)$ with $N(\alpha)$ defined by (4.3). *Then $s_n(G) = o(n^{-\alpha})$ as $n \to \infty$.*

Theorem 5.1 remains true if $o(\,\cdot\,)$ (on both occurrences above) is replaced by $O(\,\cdot\,)$.

Below is our main result in the discrete case.

Theorem 5.2. *Suppose $\alpha > 0$, $\varkappa_1, \varkappa_{-1} \in \mathbb{R}$, and let $g(j)$ be a sequence of real numbers given* (*for $j \geqslant 2$*) *by*

$$g(j) = \big(\varkappa_1 + (-1)^j \varkappa_{-1}\big) j^{-1} (\log j)^{-\alpha} + \widetilde{g}_1(j) + (-1)^j \widetilde{g}_{-1}(j) \tag{5.1}$$

where the error terms $\widetilde{g}_{\pm 1}$ satisfy the conditions of Theorem 5.1. *Then the eigenvalues of the corresponding Hankel operator G have the asymptotic behavior*

$$\lambda_n^{\pm}(G) = b^{\pm} n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,$$

where

$$b^{\pm} = \tau(\alpha)\big((\varkappa_1)_{\pm}^{1/\alpha} + (\varkappa_{-1})_{\pm}^{1/\alpha}\big)^{\alpha} \tag{5.2}$$

with $\tau(\alpha)$ given by (4.4).

5.2. We describe the plan of the proof of Theorem 5.2. We follow the same steps as in §4, but instead of the Laplace transform $h_* = \mathsf{L}\sigma_*$ of the function $\sigma_*(\lambda)$, $\lambda > 0$, we consider the sequence of moments

$$g_*(j) = \int_{-1}^{1} \eta_*(\mu) \mu^j \, d\mu, \quad j \geqslant 0, \tag{5.3}$$

of the function

$$\eta_*(\mu) = \Big|\log \frac{1+\mu}{2(1-\mu)}\Big|^{-\alpha} \left(\varkappa_1 \chi_\infty\Big(\frac{1+\mu}{2(1-\mu)}\Big) + \varkappa_{-1}\chi_0\Big(2\frac{1+\mu}{1-\mu}\Big)\right) \tag{5.4}$$

where the smooth cut-off functions χ_∞ and χ_0 are as in (4.8). Note that the function η_* is of class $C^\infty(-1,1)$ and has the following asymptotic behavior:

$$\eta_*(\mu) = \kappa_1 |\log(1-\mu)|^{-\alpha} + o(|\log(1-\mu)|^{-\alpha}), \quad \mu \to 1,$$
$$\eta_*(\mu) = \kappa_{-1} |\log(1+\mu)|^{-\alpha} + o(|\log(1+\mu)|^{-\alpha}), \quad \mu \to -1.$$

These relations allow us to obtain the asymptotics of the sequence $g_*(j)$ as $j \to \infty$. Again, we use Lemma 4.5, but the continuous parameter t should be replaced with the discrete one j. The following assertion plays the role of Corollary 4.6.

Lemma 5.3. *The sequence $g_*(j)$ defined by* (5.3), (5.4) *has the asymptotics*

$$g_*(j) = (\kappa_1 + (-1)^j \kappa_{-1}) j^{-1} (\log j)^{-\alpha} + \widetilde{g}_1(j) + (-1)^j \widetilde{g}_{-1}(j) \tag{5.5}$$

where the error terms $\widetilde{g}_{\pm 1}(j)$ satisfy the estimates

$$\widetilde{g}_{\pm 1}^{(m)}(j) = O(j^{-1-m} (\log j)^{-\alpha-1}), \quad j \to \infty,$$

for all $m = 0, 1, 2, \dots$.

Let $v(x)$ be the function (1.4), and let

$$s_*(\xi) = \eta_* \left(\frac{2e^{-\xi} - 1}{2e^{-\xi} + 1} \right), \quad \xi \in \mathbb{R}.$$

Theorem 2.4 implies that the Hankel operator G_* with the matrix elements (5.3) is unitarily equivalent to the ΨDO $A_* = v(X) s_*(D) v(X)$ acting in $L^2(\mathbb{R})$.

By the definition (5.4) of $\eta_*(\mu)$, we have

$$s_*(\xi) = |\xi|^{-\alpha} (\kappa_1 \chi_\infty(e^{-\xi}) + \kappa_{-1} \chi_0(4e^{-\xi})), \quad \xi \in \mathbb{R}.$$

Applying Theorem 4.1 to the ΨDO A_*, we see that

$$\lambda_n^\pm(G_*) = \lambda_n^\pm(A_*) = b^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{5.6}$$

where the numbers $b^\pm$ are given by (5.2).

5.3. Now we can conclude the proof of Theorem 5.2. Comparing (5.1) and (5.5), we see that

$$g(j) - g_*(j) = f_1(j) + (-1)^j f_{-1}(j),$$

where the error terms $f_{\pm 1}(j) = g_{\pm 1}(j) - \widetilde{g}_{\pm 1}(j)$ satisfy the condition

$$f_{\pm 1}^{(m)}(j) = o(j^{-1-m} (\log j)^{-\alpha}), \quad j \to \infty,$$

for all $m = 0, 1, \dots, N(\alpha)$. By Theorem 5.1 (singular value estimates), we have $s_n(G(f_{\pm 1})) = o(n^{-\alpha})$ as $n \to \infty$.

Put

$$\widetilde{f}_{-1}(j) = (-1)^j f_{-1}(j).$$

Then

$$s_n(G(\widetilde{f}_{-1})) = s_n(G(f_{-1})),$$

whence

$$s_n(G(f_1 + \widetilde{f}_{-1})) = o(n^{-\alpha}), \quad n \to \infty. \tag{5.7}$$

In view of (5.6) and (5.7), we can apply the abstract Lemma 4.3 to the operators $K_0 = G(g_*)$ and $K = G(f_1 + \widetilde{f}_{-1})$. This yields the eigenvalue asymptotics (5.2) for the operator $G = G(g) = K_0 + K$. □

§6. Generalizations and applications

6.1. In this subsection we state results on the asymptotic behavior of singular values of Hankel operators. Now the operators are not assumed to be selfadjoint. Recall that the numerical coefficient $\tau(\alpha)$ is given by formula (4.4).

It is convenient to start with the discrete case.

Theorem 6.1 ([11, Theorem 3.1]). *Let $\alpha > 0$, let $\zeta_1, \dots, \zeta_L \in \mathbb{T}$ be pairwise distinct numbers, and let $\kappa_1, \dots, \kappa_L \in \mathbb{C}$. Let $g(j)$ be a sequence of complex numbers such that*

$$g(j) = \sum_{\ell=1}^{L} \big(\kappa_\ell j^{-1} (\log j)^{-\alpha} + \widetilde{g}_\ell(j)\big) \zeta_\ell^{-j}, \quad j \geqslant 2, \tag{6.1}$$

where the error terms $\widetilde{g}_\ell$, $\ell = 1, \dots, L$, satisfy the estimates

$$\widetilde{g}_\ell^{(m)}(j) = o\big(j^{-1-m} (\log j)^{-\alpha}\big), \quad j \to \infty, \tag{6.2}$$

for all $m = 0, 1, \dots, N(\alpha)$ with $N(\alpha)$ given by (4.3). Then the singular values of the Hankel operator G defined in $\ell^2(\mathbb{Z}_+)$ by formula (1.11) satisfy the asymptotic relation

$$s_n(G) = b\, n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty, \tag{6.3}$$

where

$$b = \tau(\alpha) \Big(\sum_{\ell=1}^{L} |\kappa_\ell|^{1/\alpha} \Big)^{\alpha}. \tag{6.4}$$

The plan of the proof of Theorem 6.1 is the following. For $L = 1$, Theorem 6.1 is a consequence of Theorem 5.2 for the particular case where $\kappa_{-1} = 0$. To pass to the general case, we need the notion of the symbol $\omega(\mu)$ of a Hankel operator G. The function $\omega(\mu)$ can be defined by the relation

$$g(j) = \int_{\mathbb{T}} \omega(\mu) \mu^{-j} d\mathbf{m}(\mu) \tag{6.5}$$

where $d\mathbf{m}(\mu)$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Of course, the function $\omega(\mu)$ satisfying (6.5) is not unique.

Consider the leading part

$$g_{\rm lead}(j) = \sum_{\ell=1}^{L} \kappa_\ell j^{-1} (\log j)^{-\alpha} \zeta_\ell^{-j}, \quad j \geqslant 2, \tag{6.6}$$

of the sequence (6.1). It can easily be checked that for every $\alpha > 0$ the function

$$\omega_0(\mu) = \sum_{j=2}^{\infty} j^{-1} (\log j)^{-\alpha} (\mu^j - \bar{\mu}^j), \quad \mu \in \mathbb{T},$$

is bounded and $\omega_0 \in C^\infty(\mathbb{T}\backslash\{1\})$. This means that its singular support satisfies $\operatorname{sing\,supp} \omega_0 = \{1\}$. Therefore, the singular support of the symbol $\omega_{\rm lead}(\mu)$ corresponding to $g_{\rm lead}(j)$ consists of the points $\zeta_1, \dots, \zeta_L$. It can be deduced from this property that the singular value counting function $\#\{n : s_n(G_{\rm lead}) > \varepsilon\}$ of the Hankel operator $G_{\rm lead}$ with the matrix elements $g_{\rm lead}(j)$ is asymptotically (as $\varepsilon \to +0$) the sum of such functions for the separate terms in (6.6). In [11], this fact was called the localization principle. In terms of singular values, the result on counting functions is equivalent to relations (6.3), (6.4) for the operator $G_{\rm lead}$. The singular values of the operator $G - G_{\rm lead}$ can be estimated easily with the help of Theorem 5.1. □

In the continuous case, we have the following result.

Theorem 6.2 ([11, Theorem 5.1]). *Let $\alpha > 0$, let $\rho_1, \dots, \rho_L \in \mathbb{R}$ be pairwise distinct numbers, and let $\kappa_0, \kappa_1, \dots, \kappa_L \in \mathbb{C}$. Let the number $N(\alpha)$ be given by (4.3). Suppose that $h \in L^\infty_{\rm loc}(\mathbb{R}_+)$ if $\alpha < 1/2$ and $h \in C^{N(\alpha)}(\mathbb{R}_+)$ if $\alpha \geqslant 1/2$. Assume that*

$$h(t) = \sum_{\ell=1}^{L} \big(\kappa_\ell t^{-1} (\log t)^{-\alpha} + \widetilde{h}_\ell(t)\big) e^{-i\rho_\ell t}, \quad t \geqslant 2,$$

$$h(t) = \kappa_0 t^{-1} \big(\log(1/t)\big)^{-\alpha} + \widetilde{h}_0(t), \quad t \leqslant 1/2,$$

where the error terms $\widetilde{h}_\ell$ and their derivatives $\widetilde{h}_\ell^{(m)}$ satisfy the estimates

$$\widetilde{h}_\ell^{(m)}(t) = o(t^{-1-m} \langle \log t \rangle^{-\alpha}), \quad m = 0, \dots, N(\alpha), \tag{6.7}$$

as $t \to \infty$ for $\ell = 1, \dots, L$ and as $t \to 0$ for $\ell = 0$. Then the singular values of the integral Hankel operator H with the kernel $h(t)$ in $L^2(\mathbb{R}_+)$ satisfy the asymptotic relation

$$s_n(H) = a\, n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,$$

where

$$a = \tau(\alpha)\Big(\sum_{\ell=0}^{L} |\kappa_\ell|^{1/\alpha}\Big)^{\alpha}.$$

The proof of Theorem 6.2 follows the same general outline as that of Theorem 6.1. In the continuous case a symbol $\Omega(x)$ may be defined by the relation

$$h(t) = \int_{-\infty}^{\infty} \Omega(x) e^{-ixt} dx.$$

Compared to the proof of Theorem 6.2, the only essential difference is that the singularity of the kernel $h(t)$ at $t = 0$ must be treated separately. It corresponds to the singularity of the symbol $\Omega(x)$ at infinity.

6.2. Here we find the asymptotics of eigenvalues for selfadjoint Hankel operators. The first result generalizes Theorem 5.2. Now we consider real sequences of the form (6.1).

Theorem 6.3 ([12, Theorem 5.7]). *Let* $\alpha > 0$, $p = 1/\alpha$; *let* $\zeta_1, \dots, \zeta_L \in \mathbb{T}$ *be pairwise distinct points with* $\operatorname{Im} \zeta_\ell > 0$, *and let* $\varkappa_1, \varkappa_{-1} \in \mathbb{R}$, $\kappa_1, \dots, \kappa_L \in \mathbb{C}$. *Let* $g(j)$ *be a sequence of real numbers such that*

$$\begin{aligned} g(j) = \varkappa_1 j^{-1} (\log j)^{-\alpha} + \widetilde{\mathrm{g}}_1(j) + (-1)^j \big(\varkappa_{-1} j^{-1} (\log j)^{-\alpha} + \widetilde{\mathrm{g}}_{-1}(j)\big) \\ + 2 \operatorname{Re} \sum_{\ell=1}^{L} \zeta_\ell^{-j} \big(\kappa_\ell j^{-1} (\log j)^{-\alpha} + \widetilde{g}_\ell(j)\big), \quad j \geqslant 2, \end{aligned} \tag{6.8}$$

where all error terms $\widetilde{\mathrm{g}}_1$, $\widetilde{\mathrm{g}}_{-1}$, $\widetilde{g}_1$, $\dots$, $\widetilde{g}_L$ *obey condition* (6.2) *for*

$$m = 0, 1, \dots, N(\alpha)$$

($N(\alpha)$ *is given by* (4.3)). *Then the eigenvalues of the Hankel operator* G *with the matrix elements* (6.8) *satisfy the asymptotic relation* (5.2) *with the coefficient* $b^{\pm}$ *defined by*

$$b^{\pm} = \tau(\alpha)\big((\varkappa_{-1})_{\pm}^{1/\alpha} + (\varkappa_1)_{\pm}^{1/\alpha} + \sum_{\ell=1}^{L} |\kappa_\ell|^{1/\alpha}\big)^{\alpha}.$$

Compared to the proof of Theorem 6.1, we should use additionally the so-called symmetry principle (see [12]). It states that if the singular support of the symbol of a compact selfadjoint Hankel operator G does not include the points 1 and -1, then the spectrum of G is asymptotically symmetric with respect to the point zero.

In the continuous case, we consider real kernels $h(t)$ that are singular at $t = 0$ and involve several oscillating terms at infinity. The assertion below plays the role of Theorem 6.3, and its proof follows essentially the same lines. As in the proof of Theorem 6.2, the contributions of the points $t=\infty$ and $t=0$ should be treated separately.

Theorem 6.4 ([12, Theorem 6.6]). *Let $\alpha>0$, let $\rho_1,\dots,\rho_L\in\mathbb{R}_+$ be pairwise distinct numbers, and let $\kappa_0,\kappa_\infty\in\mathbb{R}$, $\kappa_1,\dots,\kappa_L\in\mathbb{C}$. Suppose that $h\in L^\infty_{\rm loc}(\mathbb{R}_+)$ if $\alpha<1/2$ and $h\in C^{N(\alpha)}(\mathbb{R}_+)$ if $\alpha\geqslant 1/2$, where the number $N(\alpha)$ is as in* (4.3). *Assume that*

$$h(t)=\kappa_\infty t^{-1}(\log t)^{-\alpha}+\widetilde{h}_\infty(t)+2\operatorname{Re}\sum_{\ell=1}^{L}\bigl(\kappa_\ell t^{-1}(\log t)^{-\alpha}+\widetilde{h}_\ell(t)\bigr)e^{-i\rho_\ell t}$$

for $t\geqslant 2$ and

$$h(t)=\kappa_0 t^{-1}\bigl(\log(1/t)\bigr)^{-\alpha}+\widetilde{h}_0(t),\quad t\leqslant 1/2,$$

where the error terms $\widetilde{h}_\infty$, $\widetilde{h}_1,\dots,\widetilde{h}_L$ obey estimates (6.7) *as $t\to\infty$ and $\widetilde{h}_0$ obeys these estimates as $t\to 0$. Then the eigenvalues of the integral Hankel operator H with the kernel $h(t)$ satisfy the asymptotic relation* (4.5), *where*

$$a^\pm=\tau(\alpha)\Bigl((\kappa_0)_\pm^{1/\alpha}+(\kappa_\infty)_\pm^{1/\alpha}+\sum_{\ell=1}^{L}|\kappa_\ell|^{1/\alpha}\Bigr)^\alpha.$$

6.3. As an application of Theorem 6.1, we state here a result on rational approximations of functions $\omega(z)$ analytic on the unit disc $\mathbb{D}$ and acquiring logarithmic singularities on $\mathbb{T}$. Let BMO be the space of functions of bounded mean oscillation on the unit circle $\mathbb{T}$. We study the asymptotic behavior as $n\to\infty$ of the distance $\operatorname{dist}_{\rm BMO}\{\omega,\mathcal{R}_n\}$ in the BMO-norm between ω and the set $\mathcal{R}_n$ of all rational functions of degree at most n without poles on the closed disk $\overline{\mathbb{D}}$. A short description of relevant results in this vast domain can be found in [13]. In view of the Adamyan–Arov–Kreĭn theorem, the problem in question is equivalent to the study of the asymptotic behavior of singular values for the Hankel operator with the symbol $\omega(z)$.

Let us describe the class of admissible functions $\omega(z)$. Let $u(z)$ be analytic in $\mathbb{D}$, $u\in C^\infty(\overline{\mathbb{D}})$; we fix some $\zeta=e^{i\varphi}\in\mathbb{T}$ and assume that

$$-\log(\zeta-z)+u(z)\neq 0,\quad z\in\overline{\mathbb{D}}. \tag{6.9}$$

Define

$$\omega(z)=\bigl(-\log(\zeta-z)+u(z)\bigr)^{1-\alpha},\quad z\in\mathbb{D},\quad \alpha>0.$$

We have introduced $u(z)$ to avoid irrelevant singularities of $\omega(z)$ inside $\mathbb{D}$. The branch of the analytic function $\log(\zeta-z)$ is fixed by the condition

$$\log(\zeta-z)=\log(1-r)+i\varphi\quad\text{if}\quad z=r\zeta\quad\text{for}\quad r\in(0,1).$$

We fix $\arg\bigl(-\log(\zeta-z)+u(z)\bigr)$ by the condition that it tends to zero as $z = re^{i\varphi}$ and $r \to 1-0$. Obviously, the function $\omega(z)$ is analytic in the unit disk $\mathbb{D}$ and is smooth up to the boundary $\mathbb{T}$, except at the point $z = \zeta$.

Theorem 6.1 allows us to consider $\omega(z)$ as well as finite sums of such functions.

Theorem 6.5 ([13, Theorem 3.8]). *Let $\zeta_1, \zeta_2, \dots, \zeta_L \in \mathbb{T}$ be pairwise distinct points, and let functions v_ℓ, u_ℓ, $\ell = 1, \dots, L$, be analytic in $\mathbb{D}$ and $v_\ell, u_\ell \in C^\infty(\overline{\mathbb{D}})$. Assume that (6.9) is satisfied for all u_ℓ, ζ_ℓ and set*

$$\omega(z) = \sum_{\ell=1}^{L} v_\ell(z)\bigl(-\log(\zeta_\ell - z) + u_\ell(z)\bigr)^{1-\alpha}, \quad \alpha > 0.$$

Then the limit

$$\lim_{n\to\infty} n^\alpha \operatorname{dist}_{\mathrm{BMO}}\{\omega, \mathcal{R}_n\} = |1-\alpha|\tau(\alpha)\Bigl(\sum_{\ell=1}^{L} |v_\ell(\zeta_\ell)|^{1/\alpha}\Bigr)^\alpha$$

exists.

Note that for $\alpha < 1$, the functions $\omega(\zeta)$ are unbounded as $\zeta \in \mathbb{T}$ tends to one of the points ζ_ℓ, so that their approximation in the norm of $C(\mathbb{T})$ is *a priori* impossible.

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Univ Rennes, CNRS,
IRMAR–UMR 6625,
F-35000 Rennes, France;
St.Petersburg State University,
Universitetskaya nab. 7/9,
199034 St.Petersburg, Russia
E-mail: yafaev@univ-rennes1.fr

Поступило 12 декабря, 2017