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VACUUM CURVES AND CLASSICAL INTEGRABLE SYSTEMS IN $2+1$ DISCRETE DIMENSIONS

Vacuum curves and vacuum vectors are algebroo-geometrical objects that have arisen in the theory of the quantum Yang–Baxter equation. They seem to have their origin in R. Baxter's works [1, 2], and their general definition was formulated by I. M. Krichever [3]. Baxter, and also Takhtajan and Faddeev [4] used the vacuum vectors to obtain a generalization of the Bethe ansatz for the XYZ spin model – an integrable one-dimensional quantum field theory (and also for the eight-vertex model of two-dimensional statistical physics). Krichever has applied the vacuum curves to classification of the solutions of the quantuum Yang–Baxter equation in the tensor product of 2-dimensional vector spaces. Then, the author of this paper has found some further applications of the vacuum curves and vacuum vectors. In the paper [5] (see also [6]) new solutions of the quantum Yang–Baxter equation were constructed for the first time. They correspond to what is now known as Chiral Potts model. In [7, 8, 6] the degeneracies of the spectrum of the XYZ quantum chhain hamiltonian were examined by means of the vacuum curves, and in [9, 6] the studying of vacuum vector bundles hhas resulted in the construction of solutions to the tetrahedron equation with commuting spin variables on the links.

Here, I try to demonstrate that the vacuum curves may be useful for studying the classical (not quantum) field theory by models as well. A difference equation on the $2+1$-dimensional cubic lattice is presented, for which the solution to the Cauchy problem is constructed, at least in principle, through a rather simple scheme. The evolution is of hyperbolic nature, i.e., the "perturbations" propagater not faster than fixed speed. This field theory comes as a "reduction" of some "non-local" dynamical system. On the other hand, generalizations of this latter system are constructed, for which I use the discrete analog of Lax pair.

Acknoledgements

I owe the idea of "local reduction" (see Sec. 2) to A. B. Shabat. L. D. Fadeev informed me of the paper [10]. I would like to express my

gratitude to them

§1. The definition of the dynamical system. Gauge invariance

Let

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a block matrix, $A, \ldots, D$ being $n \times n$ matrices consisting of complex numbers. Consider the following two operations: the construction of inverse matrix:

$$L \to L^{-1}$$

and the block transposing

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \to \quad L^t = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

Now let a (birational) mapping f be a composition of these two operations:

$$f(L) = (L^{-1})^t. \tag{1.1}$$

Let us introduce the discrete integer-valued time τ, and let the matrix L depend on τ so that

$$L(\tau + 1) = f(L(\tau)). \tag{1.2}$$

This "dynamical system" has been already mentioned in literature [10]. In the present paper, the integrability of this system is demonstrated, if the "motion" is considered up to a "gauge transformation" (see below).

Let G and H be non-degenerate $n \times n$ matrices. The gauge transformation of the matrix L is the following transformation of all its blocks:

$$A \to GAH, \ \ldots, \ D \to GDH. \tag{1.3}$$

Two matrices L and L' connected by the transformation (1.3) will be called gauge equivalent. It is clear that if $L(\tau)$ and $L'(\tau)$ belong to the same class of gauge equivalence, the matrices $L(\tau+1)$ and $L'(\tau+1)$ also do so. Thus, dynamics (1.2) induces a dynamics of the set of classes of gauge equivalence.

§2. Vacuum curves and vacuum vectors

It turns out that the dynamics (1.2) preserves the so-called vacuum curve Γ of the operator L (the bases being fixed, we make no difference between a linear operator and its matrix). To be exact, Γ remains unchanged under the transformation $f \circ f$, and undergoes a simple transformation under f. The curve Γ together with the class of linear equivalence of the pole divisor of the vacuum vectors (see below) determines the matrix L up to a gauge transformation. The set of those classes of linear equivalence is isomorphic to a complex torus – the Jacobian of the curve Γ. The dynamics (1.2) is linearized on the Jacobian, i.e., the transformation f corresponds to a constant shift on the torus. Now, let us discuss these facts in detail.

The vacuum curve of the operator L is an algebraic curve in the space $\mathbb{C}^2$ of two variables u, v. Here are two equivalent definitions of it [3].

Definition 1. *Consider the relation*

$$L(U \otimes X) = V \otimes Y \tag{2.1}$$

wherein

$$U = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} v \\ 1 \end{pmatrix}$$

are two-dimensional vectors, X and Y are n-dimensional vectors. For a generic matrix L, the non-zero solutions (U, V, X, Y) of the relatio (2.1) are parametrizad, up to a scalar factor in X and Y, by points of an algebraic curve Γ of genus $g = (n-1)^2$ given by an equation of the form

$$P(u, v) = 0, \tag{2.2}$$

$P(u, v)$ being a polynomial of degree n in each variable, i.e.,

$$P(u, v) = \sum_{j,k=1}^{n} a_{jk} u^j v^k. \tag{2.3}$$

Γ is called the vacuum curve of the operator L.

Definition 2. *The vacuum curve of the operator L is the curve Γ in $\mathbb{C}^2$ given by the equation*

$$P(u, v) = \det(V^{\perp} L U) = \det(uA + B - uvC - vD) = 0 \tag{2.4}$$

where

$$V^{\perp} = (1, -v).$$

Let us denote the points of the vacuum curve by the letter $z = (u, v) \in \Gamma$. Then $U = U(z)$ and $V = V(z)$ are meromorphic vectors on Γ with the pole divisors D_U and D_V of degree n, while $X = X(z)$ and $Y = Y(z)$, if normalized by, e.g., the condition that their nth coordinates equal unity, become meromorphic vectors with pole divisors D_X and D_Y of degree $n^2 - n$ [3]. Under thhis normalization, a meromorphic scalar factor $h(z)$ must be added into (2.1):

$$L(U(z) \otimes X(z)) = h(z)V(z) \otimes Y(z). \tag{2.5}$$

The linear equivalence of divisors

$$D_U + D_X \sim D_V + D_Y$$

holds and is provided by the function $h(z)$ in the sense that $h(z)$ has its poles in the points of $D_U + D_X$ and zeros in the of $D_V + D_Y$.

As is shown in the paper [3], the vacuum curve equation $P(u, v) = 0$ and the class of linear equivalence of divisor D_X or D_Y determine a generic matrix L to within a gauuge transformation, and vice versa, the gauge transformations do not change the vacuum cuurve and the classes of linear equivalence of divisors. In other words, the correspondence (class of gauge equivalence of L) $\leftrightarrow$ (Γ, the class of D_X) is a birational isomorphism.

We will call $X(z)$ the vacuum vector and $Y(z)$ the covacuum vector in the point z of the curve Γ. $X(z) = X(u, v)$ generates the (one-dimensional) kernel of the matrix

$$uA + B - uvC - vD. \tag{2.6}$$

The Definition 1 allows one to trace what happens with the vacuum curve and vacuum vectors under the transformation $L \to L^{-1}$, while the Definition 2 allows one to trace what happens under the transformation $L \to L^t$. Namely, it is seen from the relation

$$L^{-1}(V(z) \otimes Y(z)) = h(z)^{-1}U(z) \otimes X(z)$$

that the vacuum curve equation for the matrix L^{-1} is

$$P(u, v) = 0,$$

while its vacuum vector in the point (v, u) coincides with the covacuum vector $Y(u, v)$ of the initial matrix L. As for the block transposing, the vacuum curve equation for the matrix L^t

$$\det (uA + C - uvB - vD) = 0$$

may be rewritten as

$$u^n v^n \det(v^{-1}A - B + u^{-1}v^{-1}C - u^{-1}D) = 0,$$

i.e.,

$$u^n v^n P(-v^{-1}, -u^{-1}) = 0.$$

The vacuum vector of the matrix L^t in the point $(-v^{-1}, -u^{-1})$ of its vacuum curve coincides with the vacuum vector $X(u,v)$ of the matrix L.

Combining these considerations, one finds out that the vacuum curve $\widetilde{\Gamma}$ of the matrix $(L^{-1})^t$ is given by equation

$$u^n v^n P(-u^{-1}, -v^{-1}) = 0,$$

while the vacuum vector $X(-u^{-1}, -v^{-1})$ coincides with the vector $Y(u,v)$ of the matrix L.

Identifying the curves Γ and $\widetilde{\Gamma}$ by the isomorphism

$$(u,v) \leftrightarrow (-u^{-1}, -v^{-1}),$$

one sees that

$$D_{\widetilde{X}} \sim D_Y \sim D_X + D_U - D_V,$$

which means that, in essence, the transformation (1.1) results in adding a fixed element of the Picard group, namely the equivalence class of the divisor $D_U - D_V$, to the pole divisor D_X of the vacuum vectors. It is clear also that after two transformations one returns to the initial curve:

$$\widetilde{\widetilde{\Gamma}} = \Gamma.$$

§3. Reduction to an evolution equation in the 2 + 1-dimensional space-time

The dynamical of the previous sections admitsan intresting reduction, i.e., some special choie of the matrices $A, \dots, D$ that is in agreement with the evolution. It will be convenient in this section to treat the matrices $A, \dots, D$ as linear operators acting from the linear spdace $\mathcal{H}_1$ into the linear space $\mathcal{H}_2$ (of the same finite dimension). This being the situation at the moment τ, the operators act, of course, from $\mathcal{H}_2$ into $\mathcal{H}_1$ at the moment $\tau + 1$, and so on.

Let each of the spaces $\mathcal{H}_1, \mathcal{H}_2$ be a direct sum of $\frac{lm}{2}$ identical subspaces of dimension d, where l, m are even numbers.

Let us imagine these subspaces as situated at the vertices of the square lattice on the torus of the sizes $l \times m$ (which will mean the periodic boundary conditions in both discrete space variables). Let the subspace be arranged in checkerboard fashion, as in Fig. 1, where the unshaded circles correspond to subspaces of the space $\mathcal{H}_1$, while the shaded circles correspond to those of the space $\mathcal{H}_2$.

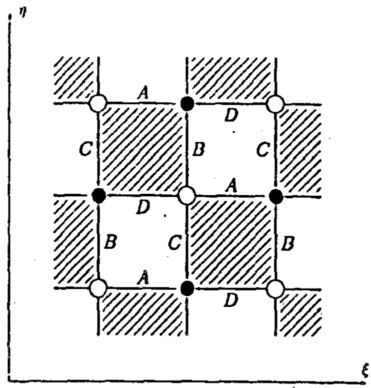

Fig. 1.

Let then the operators $A, \ldots, D$ be such that the image of each of the mentioned d-dimensional subspace of $\mathcal{H}_2$ at which points the arrow marked "A" that links these two subspace (Fig. 1). Analogously, the restrictions on B, C, D are depicted in Fig. 1 (see also formula (3.8) for non-degenerate $A, \ldots, D$). Thus, to each link of the lattice is attached a $d \times d$ matrix that is a block of one of the "large" matrices $A, \ldots, D$. Let us shade half of the squares of the lattice in a checkerboard way, as in Fig. 1. One can verify that the evolution of the system may be described as follows.

At the first step, each of the four $d \times d$ natrices that correspond to the arrows surrounding each non-shaded square is transformed nto a matrix expressed through just these four matrices. This goes according to the foolowing formulae, in which the $d \times d$ blocks are somewhat freely denoted by the same letters $A, \ldots, D$ as the "large" matrices:

$$A \to (A - BD^{-1}C)^{-1}, \tag{3.1a}$$

$$B \to (B - AC^{-1}D)^{-1}, \tag{3.1b}$$

$$C \to (C - DB^{-1}A)^{-1}, \tag{3.1c}$$

$$D \to (D - CA^{-1}B)^{-1}. \tag{3.1d}$$

However, the formulae (3.1) apply equally to the "large" matrices.

After the transformation (3.1), all the arrowa reverse, and at the second step the shaded squares are engaged in the same way according to the same formulae (3.1). Then everything is repeated. Thus, the evolution is of hyperbolic nature: each local perturbation apresds not faster than one unit of length per unit of time.

Let us clarify the symmetries of vacuum curves and divisors D_X in this "reduced" model. Let us introduce two integervalued coordinates ξ, η for the vertices of the lattice, so that ξ increases by 1 in passing from a vertex one step to the right, and η increases by 1 in passing one step upwards. ξ and η are defined modulo l and m respectively. A d-dimesional subspace of $\mathcal{H}_1$ or $\mathcal{H}_2$ will be denoted $\mathcal{H}_{\xi\eta}$ if it corresponds to a vertex with coordinates ξ, η. Consider a linear transformation in spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ consisting in multiplying the vectors of each subspace $\mathcal{H}_{\xi\eta}$ by ω_1, ω_1 being a fixed primitive root of the l-th degree of unity:

$$\omega_1^l = 1.$$

This corresponds to the following transformations of the operators $A, \dots, D$ (form now on we speak of each of these operators "as a whole", not of their blocks):

$$A \to \omega_1 A, \quad B \to \omega_1 B, \quad C \to \omega_1 C, \quad D \to \omega_1 D. \tag{3.2}$$

Consider also another linear transformation in $\mathcal{H}_1$ and $\mathcal{H}_2$, consisting in multiplying the vectors of each subspace by ω_2, ω_2 being a fixed primitive root of the m-th degree of unity:

$$\omega_2^m = 1.$$

This corresponds to the following transformation:

$$A \to A, \quad B \to \omega_2 B, \quad C \to \omega_2^{-1} C, \quad D \to D. \tag{3.3}$$

The vacuum curve of the operator L, which is given be equation (2.4)

$$P(u,v) = \det(uA + B - uvC - vD) = 0,$$

must be invariant under the transformations (3.2), (3.3). This leads to the invariance of the polynomial $P(u,v)$ with respect to the following transformations g_1 and g_2:

$$g_1(u,v) = (\omega_1 u, \omega_1^{-1} v), \tag{3.4}$$

$$g_2(u,v) = (\omega_2^{-1} u, \omega_2 1^{-1} v). \tag{3.5}$$

This invariance, then, leads to the following statement: only those coefficients a_{jk} are non-zero in the vacuum curve equation (see (2.2), (2.3)) for the "reduced" model, for which

$$\begin{aligned} j-k &\equiv 0 \pmod{l}, \\ j+k &\equiv 0 \pmod{m}. \end{aligned} \tag{3.6}$$

As for the divisor D_X, let us recall that it consists of such points in the curve Γ in which vanishes the last coordinate of the vector X (see [3]), the latter being an eigenvector of the matrix (2.6) with zero eigenvalue:

$$(uA + B - uvC - vD)X(u,v) = 0. \tag{3.7}$$

This immediately leads to the conclusion: the divisor D_X is invariant with respect to the transformations (3.4), (3.5).

Under some additional condition, the inverse statement also holds: if the curve Γ and divisor D_X are invariant under the transformations (3.4), (3.5), then the corresponding L-operator comes from a "reduced" model described in this section. For instance, this is true if $\frac{l}{2}$ and $\frac{m}{2}$ are relatively prime numbers. If these numbers are not relatively prime, some conditions are to be imposed on the divisor D_X. To avoid going into details of this latter case, let us not consider it here.

Thus, let an operator $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be given $A, \ldots, D$ being $n \times n$ matrices, $n = \frac{lm}{2}d$, l and m even, and $\frac{l}{2}$ and $\frac{m}{2}$ being relatively prime. Let the vacuum curve Γ of the operator L and the divisor D_X be invariant under action of the group $\mathcal{L}$ generated by its elements g_1, g_2 (3.4), (3.5), ω_1 and ω_2 being primitive roots of degree 1 and m of unity. Then the linear space in which operators $A, \ldots, D$ act is decomposed into a direct sum of $\frac{lm}{2}d$-dimensional subspaces $\mathcal{H}_{\xi\eta}$, ξ and η being integers modulo l and m respectively and such that $\xi + \eta$ is an even number, and the following equalities between the images of these subspaces hold (in a "genetic" case of non-degenerate $A, \ldots, D$):

$$A\mathcal{H}_{\xi-1,\eta+1} = B\mathcal{H}_{\xi\eta} = C\mathcal{H}_{\xi,\eta+2} = D\mathcal{H}_{\xi+1,\eta+1}. \tag{3.8}$$

The equalities (3.8) exactly mean that we are in the situation of Fig. 1.

Let us prove the above statement. First, the natular projection from the curve Γ to its factor $\Gamma/\mathcal{G}$ has no branch points (here the fact that $\frac{l}{2}$ and $\frac{m}{2}$ are relatively prime is used to demonstrate that ramification does not occur when u or v equals zero or infinity). Thus, the n-dimensional linear space of meromorphic functions $x(z) = x(u,v)$ whose pole divisor is D_X decomposes into a direct sum of subspaces of equal dimensions corresponding to the characters of (commutative) group $\mathcal{G}$. Each of these subspaces consists of functions $x(z)$ satisfying relations

$$x(gz) = \chi(g)x(z),$$

the character $\chi_{\xi\eta}$ being a scalar factor

$$\chi_{\xi\eta}(g) = \omega_1^{\xi a}\omega_2^{\eta b}$$

where

$$g = g_1^a g_2^b.$$

The equality $g_1^{l/2} g_2^{m/2} = 1$ means that $\xi + \eta$ must be an even number.

The components of the vector $X(z)$ are exactly the functions $x(z)$. In an appropriate basis, d components correspond to each character $\chi_{\xi\eta}$. Let us denote $\mathcal{H}_{\xi\eta}$ the set of vectors with other components equal to zero. Now, the equalities (3.8) are to be proved to end this section.

Consider the decomposition of vector $X(u,v)$ into a sum

$$X(u,v) = \sum_{\xi,\eta} X_{\xi,\eta}(u,v),$$

where $X_{\xi,\eta} \in \mathcal{H}_{\xi,\eta}$. Then

$$X_{\xi,\eta}(g(u,v)) = \chi_{\xi,\eta}(g)X_{\xi,\eta}(u,v).$$

Consider the sum

$$\sum_{g\in\mathcal{G}} \chi_{\xi,\eta}(g^{-1})g\{(uA + B - uvC - vD)X(u,v)\} = 0 \qquad (3.9)$$

(which is equal to zero because of (3.7)). The action of g upon the braces in (3.9) means that each u and v in the braces is transformed according to (3.4), (3.5), i.e., u changes into $\chi_{1,-1}(g)u$, and v changes into $\chi_{-1,-1}(g)v$. The equality (3.9) gives thus

$$\begin{aligned} uAX_{\xi-1,\eta+1}(u,v) + BX_{\xi\eta}(u,v) - \\ -uvCX_{\xi,\eta+2}(u,v) - vDX_{\xi+1,\eta+1}(u,v) = 0. \end{aligned} \qquad (3.10)$$

Let us set $u = 0$ in (3.10). Then v can take n different values $v)j$ satisfying relation $P(0, v_j) = 0$. To these values v_j correspond d linearly independent vectors $X\xi\eta(0, v_j)$ and also d vectors $X\xi + 1, \eta + 1(0, v_j)$. Thus, the equalities

$$BX_{\xi\eta}(0, v_j) = v_j DX_{\xi+1,\eta+1}(0, v_j)$$

that result from (3.10) give

$$D\mathcal{H}_{\xi\eta} = D\mathcal{H}_{\xi+1,\eta+1}$$

Analogously, one can obtain the rest of equalities (3.8).

§4. The discrete analog of Lax pair and a generalization of the dynamical system

Now let us return from the reduction of the previous section to general matrices $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let us consider the evolution described in Sec. 1 from another viewpoint. Denote

$$(L^{-1})^t = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix}.$$

This means that

$$\begin{pmatrix} \widetilde{A} & \widetilde{C} \\ \widetilde{B} & \widetilde{D} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathbb{O}. \tag{4.1}$$

It follows from the equality (4.1) that

$$\begin{aligned} \widetilde{A}A + \widetilde{C}C &= \widetilde{B}B + \widetilde{D}D, \\ \widetilde{A}B + \widetilde{C}D &= 0, \\ \widetilde{B}A + \widetilde{D}C &= 0. \end{aligned} \tag{4.2}$$

These three equations are equivalent to the fact that the following equality holds for any comlex u:

$$-(\widetilde{A} - u\widetilde{B})^{-1}(\widetilde{C} - u\widetilde{D}) = (uA + B)(uC + D)^{-1}. \tag{4.3}$$

Vice versa, from (4.3) follows

$$\widetilde{L}^t L = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix},$$

F being equal to both sides of (4.2), i.e.,

$$L = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} (L^{-1})^t.$$

It is clear that with any choice of F the matrix L belongs to the same gauge equivalence class. The formula (4.3) defines the same evolution on the space of these classes as it was in Sec. 1, with an agreement that the operators without a tilde correspond to the moment of time τ, while those wiith a tilde correspond to the moment $\tau + 1$.

The formula (4.3) suggests the following generalization. Let, from now on, $A(u)$ and $B(u)$ be matrices depending polynomially on u:

$$A(u) = A_0 + A_1 u + \cdots + A_{m_A} u^{m_A}, \tag{4.4}$$

$$B(u) = B_0 + B_1 u + \cdots + B_{m_B} u^{m_B}, \tag{4.5}$$

We will look for matrices $\widetilde{A}(u)$, $\widetilde{B}(u)$ — the matrix polynomials of the same degrees m_A and m_B in u — that satisfy, for any u, the equation

$$\widetilde{B}(u)^{-1} = A(u)B(u)^{-1}. \tag{4.6}$$

The relation (4.6) provides what is called a discrete analog of the Lax L, A-pair, which means here that the operators $\widetilde{A}(u)\widetilde{B}(u)^{-1}$ and $A(u)B(u)^{-1}$ (which are playing the role of L of the pair) are "isospherical deformations" of one another:

$$\widetilde{A}(u)\widetilde{B}(u)^{-1} = \widetilde{A}(u)A(u)B(u)^{-1}\widetilde{A}(u)^{-1}.$$

Let v be an eigenvalue of both sides of (4.6). Let $Y(u,v)$ be the corresponding eigenvector normalized, as in Sec. 2, so that its last coordinate equals unity, and let $X(u,v)$ be the vector proportional to $B(u)^{-1}Y(u,v)$ and normalized in the same way. One can verify that this may be described by the following formula ($h(u,v)$ being a scalar factor):

$$\begin{pmatrix} A(u) \\ B(u) \end{pmatrix} X(u,v) = h(u,v) \begin{pmatrix} v \\ 1 \end{pmatrix} \otimes Y(u,v), \tag{4.7}$$

which is in obvious analogy to (2.5). The divisor equivalence is

$$mD_u + D_X \sim D_v + D_Y, \tag{4.8}$$

D_u and D_v being pole divisors of the functions u and v, $m = \max(m_A, m_B)$.

For a given u, the eigenvalues v come from the equations

$$P(u,v) = \det\left(A(u) - vB(u)\right) = 0.$$

It defines an algebraic curve Γ – "generalized vacuum curve". Let us calculate the genus g of the curve Γ. First, we need to know the number of branch points of the projection

$$(u,v) \to u \tag{4.9}$$

of the curve Γ onto the complex plain.

Consider $P(u,v)$ as a polynomial in v:

$$P(u,v) = a_0(u) + a_1(u)v + \cdots + a_n(u)v^n. \tag{4.10}$$

One can verify that $a_j(u)$ hhas a degree

$$\deg a_j(u) = (n-j)m_A + jm_B. \tag{4.11}$$

From this one can deduce that the discriminant of $P(u,v)$ considered as a polynomial in v is a polynomial of degree:

$$b=(m_A+m_B)n(n-1)$$

in u. The mapping (4.9) being n-sheeted and the number of branch points equalling b, one obtains from the Riemann–Hurwitz formula:

$$g(n-1)\left(\frac{m_A+m_B}{2}n-1\right). \tag{4.12}$$

So, the following construction has been described. Given two polynomial matrix functions $A(u)$ and $B(u)$, one considers the meromorphic matrix function $A(u)B(u)^{-1}$ (or else $B(u)^{-1}A()$), and from this function the algebro-geometrical oobjects arise: the generalized vacuum curve Γ and the linear equivalence class of the pole divisor D_Y (or, respectively, D_X) of the eigenvectors of the mentioned meromorphic matrix function. Instead of the pair $(A(u),B(u))$, it is sufficient to indicate its equivalence class with respect to gauge transformations

$$A(u)\to GA(u)H,\quad B(u)\to GB(u)H; \tag{4.13}$$

instead of the function $A(u)B(u)^{-1}$, its equivalence class with respect to transformations

$$A(u)B(u)^{-1}\to GA(u)B(u)^{-1}G^{-1}$$

will suffice. Then it turns out that the correspondence between such equivalence classes (either of the pairs $(A(u),B(u))$ or the functions $A(u)B(u)^{-1}$) and the abovementioned algebro-geometrical oobjects is a birational isomorphism, the divisors D_X and D_Y being of degree $g+n-1$, as in Sec. 2.

The easiest way to show this is to start from a given curve Γ defined by the equation

$$P(u,v)=\sum_{j=0}^{n}\sum_{k=0}^{(n-j)m_A+jm_B}a_{jk}v^j u^k=0$$

(compare with (4.10), (4.11)) and a divisor D_X in it of degree $g+n-1$. The number of coefficients a_{jk} minus one common factor equals

$$(n+1)\left(\frac{m_A+m_B}{2}n+1\right)-1. \tag{4.14}$$

The linear equivalence class of divisor D_X is defined, as is known, by g parameters. Adding up the expression (4.14) and (4.12), one gets the total of

$$(m_A + m_B)n^2 + 1 \tag{4.15}$$

parameters.

Then, the gauge equivalence class of the pair $(A(u), B(u))$ is constructed out of relation (4.7). To give more detailes, one must at first choose a divisor D_X satisfying the equivalence (4.8). Then the poles and zeros of the function $h(u, v)$ are determined. For $X(u, v)$ and $Y(u, v)$ one must take columns consisting each of n linearly independent meromorphic functions with corresponding pole divisors. The arbitrariness in these constructions leads exactly to the fact that $A(u)$ and $B(u)$ are determined up to a transformation (4.13).

The pair $(A(u), B(u))$, up to a scalar common factor, is determined by $(m_A + m_B + 2)n^2 - 1$ parameters (see (4.4), (4.5)). In taking the gauge equivalence class, the number of parameters is reduced by $2(n^2 - 1)$. The result is again (4.15). This means that, indeed, to a generic pair $(A(u), B(u))$ corresponds a divisor D_X of degree $g + n - 1$ and the correspondence

$$\begin{matrix}\text{(gauge equivalence class} \\ \text{of the pair } (A(u), B(u)))\end{matrix} \leftrightarrow (\Gamma, \text{ class of } D_X)$$

is a birational isomophism.

Now let us recall that $Y(u, v)$ was defined as an eigenvector of the operator $A(u)B(u)^{-1}$, while $X(u, v)$, as is easily seen, is an eigenvector of $B(u)^{-1}A(u)$. The relation (4.6) means that for the pair $\widetilde{A}(u), \widetilde{B}(u))$ its vector $\widetilde{X}(u, v)$ is nothing else than $Y(u, v)$, i.e., the equivalence holds

$$D_{\widetilde{X}} \sim D_X + (mD_u - D_v) \tag{4.16}$$

Now, assuming that if the quantities without tilde correspond to the moment of time τ than those with tilde correspond to $\tau + 1$, one comes to a conclusion that to the adding of unity to the time corresponds a constant shift (4.16) in the Jacobian of the curve Γ. Thus, the dynamics of the system in the section, as well as in Sec. 2, linearizes.

§5. Discussion

In this paper I study a dynamical system in discrete time, i.e., a mapping and its iterations, acting on some finite set of $n \times n$ matrices. The system appears in several modifications, on which depends the number of matrices as well as the additional conditions that may be imposed on them. The "law of motion" is formulated in a rather simple way, and a large number of "integrals of motion" turn out to exist and be the coefficients of the "vacuum curve" – the object coming from the theoryy of the quantum Yang–Baxter equation. If the motion in the system is considered up to a "gauge transformation", the system is integrable in the sense that there exists a birational isomorphism between the "phase space" and the set of pairs (a vacuum curve, an element of its Picard group), so that in the process of "motion" the vacuum curve doesn't change, while the element of its Picard group depends on the time linearly. Thus, the Cauchy problem is solved through the following scheme: the initial point in the phase space $\to$ the vacuum curve and the element of its Picard group at the initial moment of time $\to$ the same at the moment τ $\to$ the element of the phase space at the moment τ.

There is an interesting problem still unsolved: to describe the evolution of the system in Sec. 1 in full, not up to the gauge equivalence.

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