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This paper is dedicated to the memory of my colleague Mikhail Solomyak.

A FUNCTIONAL MODEL FOR THE FOURIER–PLANCHEREL OPERATOR TRUNCATED TO THE POSITIVE SEMIAXIS

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The truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$,

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} d\xi, \quad t \in \mathbb{R}^+,$$

is studied. The operator $\mathcal{F}_{\mathbb{R}^+}$ is viewed as an operator acting in the space $L^2(\mathbb{R}^+)$. A functional model for the operator $\mathcal{F}_{\mathbb{R}^+}$ is constructed. This functional model is the operator of multiplication by an appropriate (2×2) -matrix function acting in the space $L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$. Using this functional model, the spectrum of the operator $\mathcal{F}_{\mathbb{R}^+}$ is found. The resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$ is estimated near its spectrum.

Notation:

\mathbb{R} is the set of all real numbers.

\mathbb{R}^+ is the set of all positive real numbers.

\mathbb{C} is the set of all complex numbers.

\mathbb{Z} is the set of all integers.

$\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of all natural numbers.

§1. The Fourier–Plancherel operator truncated to the positive semiaxis.

In this paper we study the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$,

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} d\xi, \quad t \in \mathbb{R}^+. \quad (1.1)$$

The operator $\mathcal{F}_{\mathbb{R}^+}$ is viewed as an operator acting in the space $L^2(\mathbb{R}^+)$ of all square integrable complex-valued functions on \mathbb{R}^+ equipped with the inner

Key words: truncated Fourier–Plancherel operator, functional model for a linear operator.

product

$$\langle x, y \rangle_{L^2(\mathbb{R}^+)} = \int_{\mathbb{R}^+} x(t) \overline{y(t)} dt.$$

The operator $\mathcal{F}_{\mathbb{R}^+}^*$ adjoint to $\mathcal{F}_{\mathbb{R}^+}$ with respect to this inner product is

$$(\mathcal{F}_{\mathbb{R}^+}^* x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{-it\xi} d\xi, \quad t \in \mathbb{R}^+. \quad (1.2)$$

The operator $\mathcal{F}_{\mathbb{R}^+}$ has the form

$$\mathcal{F}_{\mathbb{R}^+} = P_{\mathbb{R}^+} \mathcal{F} P_{\mathbb{R}^+}|_{L^2(\mathbb{R}^+)}, \quad (1.3)$$

where \mathcal{F} is the Fourier–Plancherel operator on the whole real axis:

$$(\mathcal{F}x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x(\xi) e^{it\xi} d\xi, \quad t \in \mathbb{R}, \quad (1.4)$$

$$\mathcal{F}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

and $P_{\mathbb{R}^+}$ is the natural orthogonal projector from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$:

$$(P_{\mathbb{R}^+}x)(t) = \mathbf{1}_{\mathbb{R}^+}(t) x(t), \quad x \in L^2(\mathbb{R}), \quad t \in \mathbb{R}^+, \quad (1.5)$$

where $\mathbf{1}_{\mathbb{R}^+}(t)$ is the indicator function of the set \mathbb{R}^+ . For any set E , its indicator function $\mathbf{1}_E$ is

$$\mathbf{1}_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E. \end{cases} \quad (1.6)$$

It should be mentioned that the Fourier operator \mathcal{F} is a unitary operator in $L^2(\mathbb{R})$:

$$\mathcal{F}^* \mathcal{F} = \mathcal{F} \mathcal{F}^* = \mathcal{J}_{L^2(\mathbb{R})}, \quad (1.7)$$

where $\mathcal{J}_{L^2(\mathbb{R})}$ is the identity operator in $L^2(\mathbb{R})$ and \mathcal{F}^* is the operator adjoint to \mathcal{F} with respect to the standard inner product in $L^2(\mathbb{R})$.

From (1.3) and (1.7) it follows that the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$ are contractive: $\|\mathcal{F}_{\mathbb{R}^+}\| \leq 1$, $\|\mathcal{F}_{\mathbb{R}^+}^*\| \leq 1$. Later it will be shown that actually

$$\|\mathcal{F}_{\mathbb{R}^+}\| = 1, \quad \|\mathcal{F}_{\mathbb{R}^+}^*\| = 1. \quad (1.8)$$

Nevertheless, these operators are strictly contractive:

$$\|\mathcal{F}_{\mathbb{R}^+} x\| < \|x\|, \quad \|\mathcal{F}_{\mathbb{R}^+}^* x\| < \|x\| \quad \text{for all } x \in L^2(\mathbb{R}^+), \quad x \neq 0, \quad (1.9)$$

and their spectral radii $r(\mathcal{F}_{\mathbb{R}^+})$ and $r(\mathcal{F}_{\mathbb{R}^+}^*)$ are less than one:

$$r(\mathcal{F}_{\mathbb{R}^+}) = r(\mathcal{F}_{\mathbb{R}^+}^*) = 1/\sqrt{2}. \quad (1.10)$$

In particular, the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$ are contractions of class C_0 in the sense of [1]. (See [1, Chapter 2, §4].)

In [1], the spectral theory of contractions in Hilbert space was developed. The starting point of this theory is the representation of a given contractive operator A acting in a Hilbert space \mathcal{H} in the form

$$A = PUP, \tag{1.11}$$

where U is a unitary operator acting in some *ambient* Hilbert space \mathfrak{H} , $\mathcal{H} \subset \mathfrak{H}$, and P is the orthogonal projector from \mathfrak{H} onto \mathcal{H} . In the construction of [1] it is required that not only (1.11) but also the series of identities

$$A^n = PU^nP, \quad n \in \mathbb{N}, \tag{1.12}$$

be true. The unitary operator U acting in an ambient Hilbert space \mathfrak{H} , $\mathcal{H} \subset \mathfrak{H}$, is called a *unitary dilation of the operator* A , $A : \mathcal{H} \rightarrow \mathcal{H}$, if identities (1.12) are fulfilled. In [1] it was shown that every contractive operator A admits a unitary dilation. By using the unitary dilation, a functional model of the operator A was constructed. This functional model is an operator acting in some Hilbert space of analytic functions. The functional model of the operator A is an operator unitarily equivalent to A . The spectral theory of the original operator A is developed by analyzing its functional model. However, the functional model constructed in [1] is not suitable for the spectral analysis of the truncated Fourier-Plancherel operator $\mathcal{F}_{\mathbb{R}^+}$.

Relation (1.3) is of the form (1.11), where $\mathcal{H} = L^2(\mathbb{R}^+)$, $\mathfrak{H} = L^2(\mathbb{R})$, $U = \mathcal{F}$, $A = \mathcal{F}_{\mathbb{R}^+}$, and $P = P_{\mathbb{R}^+}$ is the orthoprojector from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$, see (1.5). For these objects, identities (1.12) do not hold true for all $n \in \mathbb{N}$, but only for $n = 1$. So, the operator \mathcal{F} is not a unitary dilation of its truncation $\mathcal{F}_{\mathbb{R}^+}$. Nevertheless, we succeeded in constructing a functional model of the operator $\mathcal{F}_{\mathbb{R}^+}$ such that it is easily analyzable. Analyzing this model, we can develop a complete spectral theory of the operator $\mathcal{F}_{\mathbb{R}^+}$.

§2. The model space.

Definition 2.1.

1. The *model space* \mathfrak{M} is the set of all 2×1 columns $\varphi = \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix}$ whose entries φ_+ and φ_- are arbitrary complex-valued functions of class $L^2(\mathbb{R}^+)$.

2. The space \mathfrak{M} is equipped by the natural linear operations.

3. The inner product $\langle \varphi, \psi \rangle_{\mathfrak{M}}$ of columns $\varphi = \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix}$ and $\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$ belonging to this space is defined as

$$\langle \varphi, \psi \rangle_{\mathfrak{M}} = \langle \varphi_+, \psi_+ \rangle_{L^2(\mathbb{R}^+)} + \langle \varphi_-, \psi_- \rangle_{L^2(\mathbb{R}^+)}. \quad (2.1)$$

In particular,

$$\|\varphi\|_{\mathfrak{M}}^2 = \|\varphi_+\|_{L^2(\mathbb{R}^+)}^2 + \|\varphi_-\|_{L^2(\mathbb{R}^+)}^2. \quad (2.2)$$

Remark 2.2. The model space \mathfrak{M} is merely the orthogonal sum of two copies of the space $L^2(\mathbb{R}^+)$. The standard notation $L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$ for such orthogonal sum does not reflect the fact that the elements of \mathfrak{M} are (2×1) -columns.

The notation $\begin{bmatrix} L^2(\mathbb{R}^+) \\ \oplus \\ L^2(\mathbb{R}^+) \end{bmatrix}$ is more logical, but too bulky.

We define a linear mapping U of the space $L^2(\mathbb{R})$ into the model space. For $x \in L^2(\mathbb{R}^+)$, the formal definition is

$$(Ux)(\mu) = \begin{bmatrix} (Ux)_+(\mu) \\ (Ux)_-(\mu) \end{bmatrix}, \quad \mu \in \mathbb{R}^+, \quad (2.3)$$

where

$$(Ux)_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{+i\mu} d\xi, \quad \mu \in \mathbb{R}^+, \quad (2.4a)$$

$$(Ux)_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{-i\mu} d\xi, \quad \mu \in \mathbb{R}^+. \quad (2.4b)$$

Here and in what follows, $\xi^\zeta = e^{\zeta \ln \xi}$, where $\ln \xi \in \mathbb{R}$ for $\xi \in \mathbb{R}^+$.

If $x \in L^2(\mathbb{R}^+)$, the functions $x(\xi) \xi^{-1/2} \xi^{\pm i\mu}$ occurring in (2.4) may fail to be integrable with respect to Lebesgue measure $d\xi$ on \mathbb{R}^+ . Therefore, the integrals in (2.4) may fail to exist as Lebesgue integrals.

Definition 2.3. The set \mathcal{D} is the set of all functions $x \in L^2(\mathbb{R}^+)$ satisfying

$$\int_{\mathbb{R}^+} |x(\xi)| \xi^{-1/2} d\xi < \infty. \quad (2.5)$$

Lemma 2.4. If a function x belongs to $L^2(\mathbb{R}^+)$ and its support $\text{supp } x$ lies strictly inside the positive semiaxis \mathbb{R}^+ , then $x \in \mathcal{D}$.

Proof.

$$\begin{aligned} \int_{\mathbb{R}^+} |x(\xi)| \xi^{-1/2} d\xi &= \int_{\xi \in \text{supp } x} |x(\xi)| \xi^{-1/2} d\xi \\ &\leq \left\{ \int_{\mathbb{R}^+} |x(\xi)|^2 d\xi \right\}^{1/2} \cdot \left\{ \int_{\xi \in \text{supp } x} |\xi|^{-1} d\xi \right\}^{1/2} < \infty. \quad \square \end{aligned}$$

For $x \in \mathcal{D}$, the integrals on the right-hand sides of (2.4a) and (2.4b) exist as Lebesgue integrals for every $\mu \in \mathbb{R}^+$. So, the functions $(Ux)_+(\mu)$ and $(Ux)_-(\mu)$ are well defined for every $\mu \in \mathbb{R}^+$.

Lemma 2.5. *If a function x belongs to \mathcal{D} , then both functions $(Ux)_+$ and $(Ux)_-$ belong to $L^2(\mathbb{R}^+)$. Moreover, we have*

$$\|(Ux)_+\|_{L^2(\mathbb{R}^+)}^2 + \|(Ux)_-\|_{L^2(\mathbb{R}^+)}^2 = \|x\|_{L^2(\mathbb{R}^+)}^2. \quad (2.6)$$

Proof. Changing the variable $\xi = e^\eta$ in (2.4), we obtain

$$(Ux)_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{+i\mu\eta} d\eta, \quad \mu \in \mathbb{R}^+, \quad (2.7a)$$

$$(Ux)_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{-i\mu\eta} d\eta, \quad \mu \in \mathbb{R}^+, \quad (2.7b)$$

where

$$v(\eta) = e^{\eta/2} x(e^\eta). \quad (2.8)$$

We have

$$\int_{\mathbb{R}} |v(\eta)|^2 d\eta = \int_{\mathbb{R}^+} |x(\xi)|^2 d\xi. \quad (2.9)$$

Put

$$u(\nu) = \begin{cases} (Ux)_+(\nu) & \text{if } \nu \in \mathbb{R}^+, \\ (Ux)_-(-\nu) & \text{if } \nu \in \mathbb{R}^-. \end{cases} \quad (2.10)$$

It is clear that

$$\int_{\mathbb{R}} |u(\nu)|^2 d\nu = \int_{\mathbb{R}^+} |(Ux)_+(\mu)|^2 d\mu + \int_{\mathbb{R}^+} |(Ux)_-(\mu)|^2 d\mu. \quad (2.11)$$

From (2.7) and (2.10) it follows that

$$u(\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{i\nu\eta} d\eta, \quad \nu \in \mathbb{R}. \quad (2.12)$$

Thus,

$$u(\nu) = (\mathcal{F}v)(\nu), \quad \nu \in \mathbb{R}, \quad (2.13)$$

where \mathcal{F} is the Fourier–Plancherel operator (1.4). The Parseval identity

$$\int_{\mathbb{R}} |u(\nu)|^2 d\nu = \int_{\mathbb{R}} |v(\eta)|^2 d\eta \quad (2.14)$$

and formulas (2.9) and (2.11) yield (2.6). \square

It is clear that the set \mathcal{D} is a (nonclosed) vector subspace of $L^2(\mathbb{R}^+)$.

Lemma 2.6. *The set \mathcal{D} is dense in $L^2(\mathbb{R}^+)$.*

Proof. Given $x \in L^2(\mathbb{R}^+)$, we define

$$x_n(t) = x(t) \cdot \mathbf{1}_{[1/n, n]}(t), \quad n = 1, 2, \dots, \quad (2.15)$$

where $\mathbf{1}_{[1/n, n]}(t)$ is the indicator function of the interval $[1/n, n]$. (See (1.6).) Clearly,

$$\|x - x_n\|_{L^2(\mathbb{R}^+)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.16)$$

Moreover, $x_n \in \mathcal{D}$ by Lemma 2.4. \square

Definition 2.7. Formula (2.6) means that the operator U defined by (2.3)–(2.4) for $f \in \mathcal{D}$ maps the subspace \mathcal{D} into the model space \mathfrak{M} isometrically. Therefore, the operator U extends from the subspace $\mathcal{D} \subset L^2(\mathbb{R}^+)$ to its closure $L^2(\mathbb{R}^+)$ by continuity:

$$\begin{aligned} \text{if } x \in L^2(\mathbb{R}^+), \quad x_n \in \mathcal{D}, \quad n = 1, 2, \dots, \quad x = \lim_{n \rightarrow \infty} x_n, \\ \text{then } Ux \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} Ux_n. \end{aligned} \quad (2.17)$$

We preserve the notation U for the operator extended in this way.

From now on, we deal with the operator U that is already extended from \mathcal{D} to the whole space $L^2(\mathbb{R}^+)$ in accordance with (2.17). It is clear that U maps $L^2(\mathbb{R}^+)$ onto some closed subspace of the model space \mathfrak{M} .

Theorem 2.8. *The operator U maps $L^2(\mathbb{R}^+)$ onto the whole model space \mathfrak{M} .*

Proof. Let $y = \begin{bmatrix} y_+ \\ y_- \end{bmatrix}$ be an arbitrary element of \mathfrak{M} . Both functions y_+ and y_- belong to $L^2(\mathbb{R}^+)$. We set

$$u(\nu) = \begin{cases} y_+(\nu) & \text{if } \nu > 0, \\ y_-(-\nu) & \text{if } \nu < 0. \end{cases} \quad (2.18)$$

Clearly, $u \in L^2(\mathbb{R})$. As a function of class $L^2(\mathbb{R})$, the function u is representable in the form

$$u(\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{i\nu\eta} d\eta, \quad \nu \in \mathbb{R}, \quad (2.19)$$

where v is a function in $L^2(\mathbb{R})$.

Relation (2.19) can be interpreted as the following pair of formulas:

$$y_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{i\mu\eta} d\eta, \quad \mu \in \mathbb{R}^+, \quad (2.20a)$$

$$y_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{-i\mu\eta} d\eta, \quad \mu \in \mathbb{R}^+, \quad (2.20b)$$

where $v \in L^2(\mathbb{R})$. Changing the variable $\xi = e^\eta$ in (2.20), we reduce (2.20) to the form

$$y_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{i\mu} d\xi, \quad \mu \in \mathbb{R}^+, \quad (2.21a)$$

$$y_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{-i\mu} d\xi, \quad \mu \in \mathbb{R}^+, \quad (2.21b)$$

where

$$v(\eta) = e^{\eta/2} x(e^\eta). \quad (2.22)$$

Moreover, $x \in L^2(\mathbb{R}^+)$:

$$\int_{\mathbb{R}^+} |x(\xi)|^2 d\xi = \int_{\mathbb{R}} |v(\eta)|^2 d\eta. \quad (2.23)$$

By the definition of the operator U , formulas (2.21) mean that

$$y = Ux. \quad (2.24)$$

□

Remark 2.9. The function v in (2.19) may fail to belong to $L^1(\mathbb{R})$. To give a meaning to (2.19), we use the standard approximation procedure. We choose a sequence $\{v_n\}_{n=1,2,\dots}$ such that

$$v_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

for every n and $\|v_n - v\|_{L^2(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$. The sequence $\{u_n\}_{n=1,2,\dots}$, where

$$u_n(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v_n(\xi) e^{it\xi} d\xi,$$

is well defined and converges to u : $\|u_n - u\|_{L^2(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$.

Remark 2.10. The transformation (2.4) is none other than the Mellin transform $\int_{\mathbb{R}^+} x(t) t^{\zeta-1} dt$ restricted to the line $\operatorname{Re} \zeta = \frac{1}{2}$: $\zeta = \frac{1}{2} + i\mu$, $\mu \in \mathbb{R}$. Formula (2.6) is the Parseval identity for the Mellin transform, see [2, §3.17, Theorem 71]. However, we would like to emphasize that we view this Mellin transform not as a single function defined for $\mu \in \mathbb{R}$, but as a pair of functions defined for $\mu \in \mathbb{R}^+$.

§3. The model of the truncated Fourier–Plancherel operator.

In §2 we introduced the operator U that maps the space $L^2(\mathbb{R}^+)$ onto the model space \mathfrak{M} isometrically. In this section we calculate the operator $U\mathcal{F}_{\mathbb{R}^+}U^{-1}$, which serves as a model of the operator $\mathcal{F}_{\mathbb{R}^+}$.

Let $x \in L^2(\mathbb{R}^+)$ and $\begin{bmatrix} y_+ \\ y_- \end{bmatrix} = Ux$, i.e., let (2.4) be true. We would like to express the pair $\begin{bmatrix} z_+ \\ z_- \end{bmatrix} = U\mathcal{F}_{\mathbb{R}^+}x$ in terms of the pair $\begin{bmatrix} y_+ \\ y_- \end{bmatrix}$. Substituting the function

$$(\mathcal{F}_{\mathbb{R}^+}x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} d\xi$$

for x in (2.4), we obtain

$$z_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} d\xi \right) t^{-1/2} t^{i\mu} dt, \quad \mu \in \mathbb{R}^+, \quad (3.1a)$$

$$z_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} d\xi \right) t^{-1/2} t^{-i\mu} dt, \quad \mu \in \mathbb{R}^+. \quad (3.1b)$$

Changing the order of integration in (3.1), we arrive at the formulas

$$z_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} e^{it\xi} t^{-1/2} t^{i\mu} dt \right) d\xi, \quad \mu \in \mathbb{R}^+, \quad (3.2a)$$

$$z_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} e^{it\xi} t^{-1/2} t^{-i\mu} dt \right) d\xi, \quad \mu \in \mathbb{R}^+. \quad (3.2b)$$

Changing $t \rightarrow t/\xi$ in (3.2), we obtain

$$z_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{-i\mu} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} e^{it} t^{-1/2} t^{i\mu} dt \right) d\xi, \quad t \in \mathbb{R}^+, \quad (3.3a)$$

$$z_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{i\mu} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} e^{it} t^{-1/2} t^{-i\mu} dt \right) d\xi, \quad t \in \mathbb{R}^+. \quad (3.3b)$$

The inner integrals in (3.3) do not depend on ξ . Calculating these integrals, we can present (3.3) in the form

$$z_+(\mu) = F_{+-}(\mu) y_-(\mu), \quad \mu \in \mathbb{R}^+, \quad (3.4a)$$

$$z_-(\mu) = F_{-+}(\mu) y_+(\mu), \quad \mu \in \mathbb{R}^+, \quad (3.4b)$$

where

$$F_{+-}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} e^{it} t^{-1/2} t^{i\mu} dt, \quad \mu \in \mathbb{R}^+, \quad (3.5a)$$

$$F_{-+}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} e^{it} t^{-1/2} t^{-i\mu} dt, \quad \mu \in \mathbb{R}^+. \quad (3.5b)$$

The functions F_{+-} and F_{-+} can be expressed in term of the Euler Γ -function:

$$F_{+-}(\mu) = \frac{1}{\sqrt{2\pi}} e^{i\pi/4} e^{-\frac{\pi}{2}\mu} \Gamma\left(\frac{1}{2} + i\mu\right), \quad \mu \in \mathbb{R}^+, \quad (3.6a)$$

$$F_{-+}(\mu) = \frac{1}{\sqrt{2\pi}} e^{i\pi/4} e^{\frac{\pi}{2}\mu} \Gamma\left(\frac{1}{2} - i\mu\right), \quad \mu \in \mathbb{R}^+. \quad (3.6b)$$

We shall not justify the possibility of changing the order of integration in (3.1). The above argument, which leads from (3.1) to (3.4), plays a heuristic role. Actually, we establish formulas (3.4), where the functions $F_{+-}(\mu)$ and $F_{-+}(\mu)$ are of the form (3.6), in a different way.

The pair of identities (3.4) can be presented in the matrix form

$$(U\mathcal{F}_{\mathbb{R}^+x})(\mu) = F(\mu)(Ux)(\mu), \quad \mu \in \mathbb{R}^+, \quad (3.7)$$

where $F(\mu)$ is a (2×2) -matrix:

$$F(\mu) = \begin{bmatrix} 0 & F_{+-}(\mu) \\ F_{-+}(\mu) & 0 \end{bmatrix}, \quad \text{for all } \mu \in \mathbb{R}^+. \quad (3.8)$$

Theorem 3.1. *Let x be an arbitrary function in $L^2(\mathbb{R}^+)$ and $\mathcal{F}_{\mathbb{R}^+x}$ the truncated Fourier-Plancherel transform of x . Then their images Ux and $U\mathcal{F}_{\mathbb{R}^+x}$ under the operator U are related by formulas (3.7), where the entries $F_{+-}(\mu)$ and $F_{-+}(\mu)$ of the matrix $F(\mu)$ are of the form (3.6).*

Proof. It suffices to verify identities (3.4) only for x of the form $x(t) = e_a(t)$, where

$$e_a(t) = e^{-at}, \quad t \in \mathbb{R}^+ \quad (3.9)$$

and a is an arbitrary positive number. It is well known that the linear hull of the family of function $\{e_a(t)\}_{0 < a < \infty}$ is a dense set in $L^2(\mathbb{R}^+)$. The function $(\mathcal{F}_{\mathbb{R}^+} e_a)(t)$ can be calculated explicitly:

$$(\mathcal{F}_{\mathbb{R}^+} e_a)(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a - it}. \quad (3.10)$$

The corresponding elements $\begin{bmatrix} y_+ \\ y_- \end{bmatrix} = Ue_a$ and $\begin{bmatrix} z_+ \\ z_- \end{bmatrix} = (U\mathcal{F}_{\mathbb{R}^+})e_a$ can also be calculated explicitly. By the definition (2.3)–(2.4) of the operator U , we have

$$y_+(\mu) = a^{-\frac{1}{2}-i\mu} \Gamma\left(\frac{1}{2} + i\mu\right), \quad \mu \in \mathbb{R}^+, \quad (3.11a)$$

$$y_-(\mu) = a^{-\frac{1}{2}+i\mu} \Gamma\left(\frac{1}{2} - i\mu\right), \quad \mu \in \mathbb{R}^+, \quad (3.11b)$$

and

$$z_+(\mu) = \sqrt{\frac{\pi}{2}} e^{i\frac{\pi}{4}} a^{-\frac{1}{2}+i\mu} \frac{e^{-\frac{\pi}{2}\mu}}{\cosh \pi\mu}, \quad \mu \in \mathbb{R}^+, \quad (3.12a)$$

$$z_-(\mu) = \sqrt{\frac{\pi}{2}} e^{i\frac{\pi}{4}} a^{-\frac{1}{2}-i\mu} \frac{e^{\frac{\pi}{2}\mu}}{\cosh \pi\mu}, \quad \mu \in \mathbb{R}^+. \quad (3.12b)$$

Relations (3.4) follow from (3.11), (3.12), (3.6), and the identity¹

$$\Gamma(1/2 + i\mu) \Gamma(1/2 - i\mu) = \frac{\pi}{\cosh \pi\mu}. \quad (3.13)$$

□

If $M = \begin{bmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{bmatrix}$ is a (2×2) -matrix with complex entries, then $\|M\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}$ is the norm of the matrix M viewed as an operator in the space \mathbb{C}^2 , where the space \mathbb{C}^2 is equipped with the standard Hermitian norm.

Definition 3.2. Let $M(\mu) = \begin{bmatrix} M_{++}(\mu) & M_{+-}(\mu) \\ M_{-+}(\mu) & M_{--}(\mu) \end{bmatrix}$ be a (2×2) -matrix-valued function of $\mu \in \mathbb{R}^+$ whose entries are complex valued functions defined almost everywhere. The multiplication operator \mathcal{M}_M generated by the matrix function M is defined by the formula

$$(\mathcal{M}_M y)(\mu) = M(\mu)y(\mu), \quad y = \begin{bmatrix} y_+ \\ y_- \end{bmatrix} \in \mathfrak{M}. \quad (3.14)$$

¹This is a special case of the identity $\Gamma(\zeta) \cdot \Gamma(1 - \zeta) = \frac{\pi}{\sin \pi\zeta}$, $\zeta \in \mathbb{C} \setminus \mathbb{Z}$.

Lemma 3.3. *If the matrix function $M(\mu)$ is bounded on \mathbb{R}^+ , i.e.,*

$$\operatorname{ess\,sup}_{\mu \in \mathbb{R}^+} \|M(\mu)\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2} < \infty,$$

then \mathcal{M}_M is a bounded operator in the space \mathfrak{M} , and

$$\|\mathcal{M}_M\|_{\mathfrak{M} \rightarrow \mathfrak{M}} = \operatorname{ess\,sup}_{\mu \in \mathbb{R}^+} \|M(\mu)\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}. \tag{3.15}$$

The expression on the left-hand side of (3.15) means the norm of the multiplication operator \mathcal{M}_M in the space \mathfrak{M} . Concerning the notion of *ess sup* see [3, §2.11, p. 140.]

Remark 3.4. If the matrix function $M(\mu)$ is continuous on \mathbb{R}^+ , then

$$\|\mathcal{M}_M\|_{\mathfrak{M} \rightarrow \mathfrak{M}} = \sup_{\mu \in \mathbb{R}^+} \|M(\mu)\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}. \tag{3.16}$$

Since $x \in L^2(\mathbb{R}^+)$ in (3.7) is arbitrary, we can interpret (3.7) as an identity of operators. The following theorem is the core of the present paper.

Theorem 3.5. *The truncated Fourier-Plancherel operator $\mathcal{F}_{\mathbb{R}^+}$ is unitarily equivalent to the multiplication operator \mathcal{M}_F generated by the matrix function F of the form (3.8)–(3.6) in the space \mathfrak{M} . We have*

$$\mathcal{F}_{\mathbb{R}^+} = U^{-1} \mathcal{M}_F U, \tag{3.17}$$

where U is the unitary operator described in Definition 2.7.

Remark 3.6. The multiplication operator \mathcal{M}_F possesses the same spectral properties as the operator $\mathcal{F}_{\mathbb{R}^+}$. However, to study \mathcal{M}_F is much easier than to study $\mathcal{F}_{\mathbb{R}^+}$. By the *model operator* we mean \mathcal{M}_F .

§4. The spectrum and the resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$.

The unitary equivalence (3.17) allows us to reduce the spectral analysis of the operator $\mathcal{F}_{\mathbb{R}^+} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ to that of the operator $\mathcal{M}_F : \mathfrak{M} \rightarrow \mathfrak{M}$.

To perform the spectral analysis of the operator \mathcal{M}_F acting in the *infinite-dimensional* space \mathfrak{M} , we need to perform the spectral analysis of the (2×2) -matrix $F(\mu)$ acting in the *two-dimensional* space \mathbb{C}^2 . The spectral analysis of the matrix $F(\mu)$ can be done for *each* $\mu \in \mathbb{R}^+$ separately. Then we can *glue* the spectrum $\sigma(\mathcal{M}_F)$ of \mathcal{M}_F from the spectra $\sigma(F(\mu))$ of the matrices $F(\mu)$, and the resolvent of \mathcal{M}_F can be glued from the resolvents of the matrices $F(\mu)$.

For $\mu \in [0, \infty)$, let

$$\zeta(\mu) = e^{i\pi/4} \frac{1}{\sqrt{2 \cosh \pi\mu}}, \tag{4.1}$$

and let

$$\zeta_+(\mu) = \zeta(\mu), \quad \zeta_-(\mu) = -\zeta(\mu). \quad (4.2)$$

It is clear that $\zeta(\mu) \neq 0$, so that $\zeta_+(\mu) \neq \zeta_-(\mu)$ for every $\mu \in [0, \infty)$.

Lemma 4.1. *For $\mu \in [0, \infty)$, the spectrum $\sigma(F(\mu))$ of the matrix $F(\mu)$ given by (3.8) is simple and consists of two different points $\zeta_+(\mu)$ and $\zeta_-(\mu)$, see (4.2) and (4.1):*

$$\sigma(F(\mu)) = \{\zeta_+(\mu), \zeta_-(\mu)\}. \quad (4.3)$$

Proof. Let $D(z, \mu)$ be the determinant of that matrix:

$$D(z, \mu) = \det(zI - F(\mu)). \quad (4.4)$$

The structure (3.8) of $F(\mu)$ shows that

$$D(z, \mu) = z^2 - F_{+-}(\mu) \cdot F_{-+}(\mu). \quad (4.5)$$

The product $F_{+-}(\mu) \cdot F_{-+}(\mu)$ can be calculated by using (3.6) and (3.13):

$$F_{+-}(\mu) \cdot F_{-+}(\mu) = \frac{i}{2 \cosh \pi \mu}.$$

Thus,

$$D(z, \mu) = z^2 - \frac{i}{2 \cosh \pi \mu}. \quad (4.6)$$

□

Definition 4.2. Let a and b be points in \mathbb{C} . By definition, the interval $[a, b]$ is the set $[a, b] = \{(1 - \tau)a + \tau b : \tau \text{ runs over } [0, 1]\}$. The open interval (a, b) as well as half-open intervals are defined similarly.

When μ runs over the interval $[0, \infty)$, the points $\zeta_+(\mu)$ fill the interval $\left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$ and the points $\zeta_-(\mu)$ fill the interval $\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right)$. When μ increases, the points $\zeta_+(\mu)$, $\zeta_-(\mu)$ move monotonically: the point $\zeta_+(\mu)$ moves from $e^{i\pi/4} \frac{1}{\sqrt{2}}$ to 0, the point $\zeta_-(\mu)$ moves from $-e^{i\pi/4} \frac{1}{\sqrt{2}}$ to 0. Thus, the mappings $\mu \rightarrow \zeta_+(\mu)$ is a homeomorphism of $[0, \infty)$ onto $\left(0, e^{i\pi/4} \frac{1}{\sqrt{2}}\right]$ and the mapping $\mu \rightarrow \zeta_-(\mu)$ is a homeomorphism of $[0, \infty)$ onto $\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right)$. Moreover $\zeta_+(\infty) = \zeta_-(\infty) = \{0\}$.

Thus, the interval $\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}}\right]$ is naturally decomposed into the union of three nonintersecting parts:

$$\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}}\right] = \left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right) \cup \{0\} \cup \left(0, e^{i\pi/4} \frac{1}{\sqrt{2}}\right]. \quad (4.7)$$

Theorem 4.3. *The spectrum $\sigma(\mathcal{M}_F)$ of the model operator \mathcal{M}_F looks like this:*

$$\sigma(\mathcal{M}_F) = \left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}} \right]. \tag{4.8}$$

In other words, Theorem 4.3 claims that the spectrum $\sigma(\mathcal{M}_F)$ of \mathcal{M}_F is represented in the form

$$\sigma(\mathcal{M}_F) = \bigcup_{\mu \in [0, \infty]} \sigma(F(\mu)). \tag{4.9}$$

Since the spectra of unitarily equivalent operators coincide, Theorem 4.3 can be reformulated as follows.

Theorem 4.4. *For the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ of the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$ we have*

$$\sigma(\mathcal{F}_{\mathbb{R}^+}) = \left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}} \right]. \tag{4.10}$$

In the present section we prove the description (4.9) of the spectrum $\sigma(\mathcal{M}_F)$ of the model operator \mathcal{M}_F . In passing, we obtain some estimates for the resolvents of the matrices $F(\mu)$. These estimates are not quite evident because the matrices $F(\mu)$ are not selfadjoint. In particular, $F(\infty)$ is a Jordan cell.

Lemma 4.5. *The norm of an arbitrary (2×2) -matrix M ,*

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$

viewed as an operator from \mathbb{C}^2 to \mathbb{C}^2 , admits the estimates

$$\frac{1}{2} \text{trace}(M^*M) \leq \|M\|^2 \leq \text{trace}(M^*M). \tag{4.11}$$

Under the assumption that $\det M \neq 0$, the norm of the inverse matrix M^{-1} can be estimated as follows:

$$\begin{aligned} |(\det M)|^{-2} \text{trace}(M^*M) - \frac{2}{\text{trace}(M^*M)} \\ \leq \|M^{-1}\|^2 \leq |(\det M)|^{-2} \text{trace}(M^*M), \end{aligned} \tag{4.12}$$

where

$$\text{trace } M^*M = |m_{11}|^2 + |m_{12}|^2 + |m_{21}|^2 + |m_{22}|^2. \tag{4.13}$$

Proof. Let s_0 and s_1 be the singular values of the matrix M , i.e.,

$$0 < s_1 \leq s_0 \tag{4.14}$$

and the numbers s_0^2, s_1^2 are eigenvalues of the matrix M^*M . Then

$$\begin{aligned} \|M\| &= s_0, \quad \|M^{-1}\| = s_1^{-1}, \\ \text{trace}(M^*M) &= s_0^2 + s_1^2, \quad |\det(M)|^2 = \det(M^*M) = s_0^2 \cdot s_1^2. \end{aligned}$$

Therefore, inequality (4.11) takes the form

$$\frac{1}{2}(s_0^2 + s_1^2) \leq s_0^2 \leq (s_0^2 + s_1^2),$$

and (4.12) takes the form

$$(s_0 s_1)^{-2}(s_0^2 + s_1^2) - \frac{2}{s_0^2 + s_1^2} \leq s_1^{-2} \leq (s_0 s_1)^{-2}(s_0^2 + s_1^2).$$

The last inequalities are valid for arbitrary numbers s_0, s_1 that satisfy (4.14). \square

Since the numbers $\Gamma(1/2 \pm i\mu)$ are complex conjugate, from (3.13) it follows that

$$|\Gamma(1/2 \pm i\mu)|^2 = \frac{2\pi}{e^{\pi\mu} + e^{-\pi\mu}}, \quad \mu \in \mathbb{R}^+. \quad (4.15)$$

Using (3.6) and (4.15), we calculate the absolute values $|F_{+-}(\mu)|$ and $|F_{-+}(\mu)|$:

$$|F_{+-}(\mu)| = \frac{1}{\sqrt{1 + e^{2\pi\mu}}}, \quad \mu \in \mathbb{R}^+, \quad (4.16a)$$

$$|F_{-+}(\mu)| = \frac{1}{\sqrt{1 + e^{-2\pi\mu}}}, \quad \mu \in \mathbb{R}^+. \quad (4.16b)$$

Note that, in particular,

$$1/\sqrt{2} \leq |F_{-+}(\mu)| < 1, \quad |F_{+-}(\mu)| \leq 1/\sqrt{2}, \quad \mu \in \mathbb{R}^+. \quad (4.17)$$

If μ runs over the interval $[0, \infty)$, then $|F_{-+}(\mu)|$ increases from $2^{-1/2}$ to 1 and $|F_{+-}(\mu)|$ decreases from $2^{-1/2}$ to 0. In particular,

$$\sup_{\mu \in \mathbb{R}^+} |F_{-+}(\mu)| = \text{ess sup}_{\mu \in \mathbb{R}^+} |F_{-+}(\mu)| = 1. \quad (4.18)$$

From (4.16) it follows that

$$|F_{+-}(\mu)|^2 + |F_{-+}(\mu)|^2 = 1. \quad (4.19)$$

For the matrix $F(t)$ defined by (3.8)–(3.6) its norm is

$$\|F(t)\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2} = \frac{1}{\sqrt{1 + e^{-2\pi t}}} \quad \text{for all } t \in \mathbb{R}^+. \quad (4.20)$$

Lemma 4.6. *For every $\mu \in [0, \infty)$ and every $z \in \mathbb{C} \setminus \sigma(F(\mu))$, the matrix $(zI - F(\mu))^{-1}$ admits the estimates*

$$|D(z, \mu)|^{-2} (2|z|^2 + 1) - \frac{2}{2|z|^2 + 1} \tag{4.21}$$

$$\leq \| (zI - F(\mu))^{-1} \|^2 \leq |D(z, \mu)|^{-2} (2|z|^2 + 1),$$

where

$$D(z, \mu) = \det(zI - F(\mu)) \tag{4.22}$$

and $\sigma(F(\mu))$ is the spectrum of the matrix $F(\mu)$.

Proof. We apply estimate (4.12) to the matrix $M = zI - F(\mu)$. We calculate the quantity $\text{trace } M^*M$ with the help of (4.13):

$$\text{trace } (zI - F(\mu))^* (zI - F(\mu)) = 2|z|^2 + |F_{+-}(\mu)|^2 + |F_{-+}(\mu)|^2. \tag{4.23}$$

Using (4.19), we see that

$$\text{trace } (zI - F(\mu))^* (zI - F(\mu)) = 2|z|^2 + 1. \tag{4.24}$$

□

Proof of Theorem 4.3. When μ runs over the interval $[0, \infty)$, the complex numbers $\frac{i}{2 \cosh \pi \mu}$, which occur on the right-hand side of (4.6), fill the interval $(0, i/2]$. Therefore,

$$\inf_{\mu \in (0, \infty)} |D(z, \mu)| = \text{dist}(z^2, [0, i/2]), \tag{4.25}$$

where

$$\text{dist}(z^2, [0, i/2]) = \min_{\zeta \in [0, i/2]} |z^2 - \zeta|. \tag{4.26}$$

In particular,

$$\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)| > 0 \right) \Leftrightarrow (z^2 \notin [0, i/2]),$$

or, in other words,

$$\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)| > 0 \right) \Leftrightarrow \left(z \notin \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \right), \tag{4.27}$$

Inequalities (4.21) imply

$$\frac{2|z|^2 + 1}{\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)| \right)^2} - \frac{2}{2|z|^2 + 1} \tag{4.28}$$

$$\leq \sup_{\mu \in [0, \infty)} \| (zI - F(\mu))^{-1} \|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}^2 \leq \frac{2|z|^2 + 1}{\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)| \right)^2}.$$

From (4.27) and (4.28) it follows that

$$\left(\sup_{\mu \in [0, \infty)} \|(zI - F(\mu))^{-1}\|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2} < \infty \right) \Leftrightarrow \left(z \notin \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \right) \quad (4.29)$$

□

By Lemma 3.3, inequalities (4.28) can be viewed as estimates for the norm of the resolvent of the model operator \mathcal{M}_F , or, in other words, as estimates for the norm of the resolvent of the truncated Fourier–Plancherel operator $\mathcal{F}_{\mathbb{R}^+}$:

$$\begin{aligned} \frac{2|z|^2 + 1}{(\text{dist}(z^2, [0, i/2]))^2} - \frac{2}{2|z|^2 + 1} &\leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)}^2 \\ &\leq \frac{2|z|^2 + 1}{(\text{dist}(z^2, [0, i/2]))^2}. \end{aligned} \quad (4.30)$$

The left inequality can be presented in the form

$$\frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])} \sqrt{1 - \frac{2(\text{dist}(z^2, [0, i/2]))^2}{(2|z|^2 + 1)^2}} \leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)}.$$

Here, the quantity under the square root is positive because

$$\frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2} \leq \frac{2|z|^2}{(2|z|^2 + 1)^2} \leq \frac{1}{2}.$$

Since $(1 - \alpha) \leq \sqrt{1 - \alpha}$ for $0 \leq \alpha \leq 1$, we have

$$1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2} \leq \sqrt{1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2}}.$$

Thus, we get a lower estimate for the norm of the resolvent:

$$\frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])} - \frac{2\text{dist}(z^2, [0, i/2])}{(2|z|^2 + 1)^{3/2}} \leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)}. \quad (4.31a)$$

An upper estimate for the norm of the resolvent is provided by the right inequality in (4.30):

$$\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \leq \frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])}. \quad (4.31b)$$

The smaller is the value $\text{dist}(z^2, [0, i/2])$, the closer are the lower estimate (4.31a) and the upper estimate (4.31b).

However, we would like to estimate the resolvent not in terms of

$$\text{dist}(z^2, [0, i/2]),$$

but rather in terms of $\text{dist}(z, \sigma(\mathcal{F}_{\mathbb{R}^+}))$.

Lemma 4.7. *Let ζ be a point of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$:*

$$\zeta \in \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right], \tag{4.32}$$

and let z lie on the normal to the interval $\left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right]$ at the point ζ :

$$z = \zeta \pm |z - \zeta| e^{i3\pi/4}. \tag{4.33}$$

Then

$$\text{dist}(z^2, [0, i/2]) = \begin{cases} 2|\zeta||z - \zeta| & \text{if } |z - \zeta| \leq |\zeta|, \\ |\zeta|^2 + |z - \zeta|^2 = |z|^2 & \text{if } |z - \zeta| \geq |\zeta|. \end{cases} \tag{4.34}$$

Proof. Condition (4.32) means that $\zeta = \pm|\zeta|e^{i\pi/4}$. Substituting this in (4.33), we obtain

$$z^2 = \pm 2|\zeta||z - \zeta| + i(|\zeta|^2 - |z - \zeta|^2).$$

If $|z - \zeta| \leq |\zeta|$, then the point $i(|\zeta|^2 - |z - \zeta|^2)$ lies on the interval $[0, i/2]$. In this case,

$$\text{dist}(z^2, [0, i/2]) = 2|\zeta||z - \zeta|.$$

If $|z - \zeta| \geq |\zeta|$, then the point $i(|\zeta|^2 - |z - \zeta|^2)$ lies on the half-axis $[0, -i\infty)$. In this case,

$$\text{dist}(z^2, [0, i/2]) = \sqrt{(|\zeta|^2 - |z - \zeta|^2)^2 + 4|\zeta|^2|z - \zeta|^2} = |\zeta|^2 + |z - \zeta|^2 = |z|^2.$$

Since $|\zeta|^2 + |z - \zeta|^2 \geq 2|\zeta||z - \zeta|$, it follows that, in any of the cases above, $|z - \zeta| \leq |\zeta|$, or $|z - \zeta| \geq |\zeta|$, we have

$$\text{dist}(z^2, [0, i/2]) \geq 2|\zeta||z - \zeta|. \tag{4.35}$$

holds. □

Theorem 4.8. *Let ζ be a point of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ of the operator $\mathcal{F}_{\mathbb{R}^+}$, and let z lie on the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$ at the point ζ .*

Then the following statements hold true.

1. *The resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}$ admits the upper estimate*

$$\| (z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1} \|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \leq A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|}, \tag{4.36}$$

where

$$A(z) = \frac{(2|z|^2 + 1)^{1/2}}{2}. \tag{4.37}$$

2. If, moreover, the condition $|z - \zeta| \leq |\zeta|$ is satisfied, then the resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}$ also admits the lower estimate

$$A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|} - B(z) |\zeta| |z - \zeta| \leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)}, \quad (4.38)$$

where $A(z)$ is the same as in (4.37) and

$$B(z) = \frac{4}{(2|z|^2 + 1)^{3/2}}. \quad (4.39)$$

3. For $\zeta = 0$, the resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}$ admits the estimates

$$2A(z) \frac{1}{|z|^2} - B(z) \leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \leq 2A(z) \frac{1}{|z|^2}, \quad (4.40)$$

where $A(z)$ and $B(z)$ are the same as in (4.37), (4.39), and z is an arbitrary point of the normal.

In particular, if $\zeta \neq 0$, and z tends to ζ along the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$, then we have

$$\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} = \frac{A(\zeta)}{|\zeta|} \frac{1}{|z - \zeta|} + O(1). \quad (4.41)$$

If $\zeta = 0$ and z tends to ζ along the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$, then we have

$$\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} = |z|^{-2} + O(1), \quad (4.42)$$

where $O(1)$ is a quantity that remains bounded as z tends to ζ .

Proof. The proof is based on estimates (4.31) for the resolvent and on Lemma 4.7. Combining inequality (4.35) with estimate (4.31b), we obtain estimate (4.36), which is valid for all z lying on the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$ at the point ζ . If, moreover, z is sufficiently close to ζ , namely, the condition $|z - \zeta| \leq |\zeta|$ is satisfied, then equality occurs in (4.35). Combining identity (4.35) with estimate (4.31a), we obtain (4.38).

The asymptotic relation (4.41) is a consequence of inequalities (4.38) and (4.40) because

$$\frac{|A(z) - A(\zeta)|}{|z - \zeta|} = O(1)$$

as z tends to ζ .

The asymptotic relation (4.42) is a consequence of inequalities (4.31) and the relation

$$\text{dist}(z^2, [0, i/2]) = |z|^2,$$

which is valid for all z lying on the normal to the interval $\sigma(\mathcal{M}_F)$ at the point $\zeta = 0$. (See (4.34) for $\zeta = 0$.) \square

Corollary 4.9. *The operator $\mathcal{F}_{\mathbb{R}^+}$ is not similar to any normal operator.*

Should the operator $\mathcal{F}_{\mathbb{R}^+}$ be similar to some normal operator \mathcal{N} , the resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$ would admit the estimate

$$\|(zJ - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \leq C (\text{dist}(z, \sigma(\mathcal{F}_{\mathbb{R}^+}))^{-1},$$

with $C < \infty$ independent of z . However, this estimate is not compatible with the asymptotic relation (4.42).

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