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INVITATION TO BOOLEAN VALUED ANALYSIS

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This is a short invitation to the field of Boolean valued analysis. Model theory evaluates and counts truth and proof. The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from Boolean valued models of set theory. These models stem from the famous works by Paul Cohen on the continuum hypothesis. They belong to logic and yield a profusion of the surprising and unforeseen visualizations of the ingredients of mathematics. Many promising opportunities are open to modeling the powerful habits of reasoning and verification. Boolean valued analysis is a blending of analysis and Boolean valued models. Adaptation of the ideas of Boolean valued models to functional analysis projects among the most important directions of developing the synthetic methods of mathematics. This approach yields the new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now transform matrices into numbers, embed function spaces into a straight line, yet having still uncharted vast territories of new knowledge. The article advertised two books that crown our thought about and research into the field.

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Key words: Boolean valued universe, Boolean truth value, transfer principle, maximum principle, mixing, descending, ascending, Boolean valued reals, Gordon's theorem.

1. Introduction

Humans definitely feel truth but cannot define truth properly. That is what Alfred Tarski explained to us in the 1930s. Mathematics pursues truth by way of proof, as wittily phrased by Saunders Mac Lane. *Boolean valued analysis* is one of the vehicles of the pursuit, resulting from the fusion of analysis and model theory.

Analysis is the technique of differentiation and integration. Differentiation discovers trends, and integration forecasts the future from trends. Analysis opens ways to understanding of the universe.

Model theory evaluates and counts truth and proof. The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from Boolean valued models of set theory. These models stem from the famous works by Paul Cohen on the continuum hypothesis. They belong to logic and yield a profusion of the surprising and unforeseen visualizations of the ingredients of mathematics. Many promising opportunities are open to modeling the powerful habits of reasoning and verification.

Logic organizes and orders our ways of thinking, manumitting us from conservatism in choosing the objects and methods of research. Logic of today is a fine instrument of pursuing truth and an indispensable institution of mathematical freedom. Logic liberates mathematics, providing nonstandard ways of reasoning.

Some model of set theory is *nonstandard* if the membership between the objects of the model differs from that of the originals. In fact, the nonstandard tools of today use a couple of set-theoretic models simultaneously. Boolean valued models reside within the most popular logical tools.

Boolean valued analysis is a blending of analysis and Boolean valued models which originated and distinguishes itself by ascending and descending, mixing, cycling hulls, etc.

2. Invention of Boolean Valued Analysis

Boolean valued analysis signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, we usually take the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Usage of two models for studying a single object is a family feature of the so-called *nonstandard methods of analysis.* For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance. The term *Boolean valued analysis* was coined by G. Takeuti.

Proliferation of Boolean valued analysis stems from the celebrated achievement of P. J. Cohen who proved in the beginning of the 1960s that the negation of the continuum hypothesis, CH, is consistent with the axioms of Zermelo–Fraenkel set theory, ZFC. This result by Cohen, together with consistency of CH with ZFC established earlier by K. Gödel, proves that CH is independent of the conventional axioms of ZFC.

The genuine value of the great step forward by Cohen could be understood better in connection with the serious difficulty explicated by J. Shepherdson and absent from the case settled by Gödel. The crux of the Shepherdson observation lies in impossibility of proving the consistency of (ZFC) + ($\neg$CH) by means of standard models of set theory. Strictly speaking, we can never find a subclass of an arbitrary chosen representation of the von Neumann universe which models (ZFC) + ($\neg$CH) provided that we use the available interpretation of membership. Cohen succeeded in inventing a new powerful method for constructing noninternal, *nonstandard*, models of ZFC. He coined the term *forcing*.. The technique by Cohen invokes the axiom of existence of a standard transitive model of ZFC in company with the forcible and forceful transformation of the latter into an immanently nonstandard model by the method of forcing. His tricks fall in an outright contradiction with the routine mathematical intuition stemming “from our belief into a natural nearly physical model of the mathematical world” as Cohen phrased this himself.

Miraculously, the difficulties in comprehension of Cohen’s results gained a perfect formulation long before they sprang into life. This was done in the famous talk “Real Function Theory: State of the Art” by N. N. Luzin at the All-Russia Congress of Mathematicians in 1927. Then Luzin said: “The first idea that might leap to mind is that the determination of the cardinality of the continuum is a matter of a free axiom like the parallel postulate of geometry. However, when we vary the parallel postulate, keeping intact the rest of the axioms of Euclidean geometry, we in fact change the precise meaning of the words we write or utter, that is, ‘point,’ ‘straight line,’ etc. What words are to change their meaning if we attempt at making the cardinality of the continuum movable along the scale of alephs, while constantly proving consistency of this movement? The cardinality of the continuum, if only we imagine the latter as a set of points, is some unique entity that must reside in the scale of alephs in the place which the cardinality of the continuum belongs to; no matter whether the determination of this place is difficult or even 'impossible for us, the human beings' as J. Hadamard might comment."

P. S. Novikov expressed a very typical attitude to the problem: "...it might be (and it is actually so in my opinion) that the result by Cohen conveys a purely negative message and reveals the termination of the development of 'naive' set theory in the spirit of Cantor."

Intention to obviate obstacles to mastering the technique and results by Cohen led D. Scott and R. Solovay to constructing the so-called Boolean valued models of ZFC which are not only visually attractive from the standpoint of classical mathematicians but also are fully capable of establishing consistency and independence theorems. P. Vopěnka constructed analogous models in the same period of the early 1960s.

The above implies that the Boolean valued models, achieving the same ends as Cohen's forcing, must be nonstandard in some sense and possess some new features that distinguish them from the standard models.

Qualitatively speaking, the *notion of Boolean valued model involves a new conception of modeling* which might be referred to as *modeling by correspondence* or *long-distance modeling.* We explain the particularities of this conception as compared with the routine approach. Encountering two classical models of a single theory, we usually seek for a bijection between the universes of the models. If this bijection exist then we translate predicates and operations from one model to the other and speak about isomorphism between the models. Consequently, this conception of isomorphism implies a direct contact of the models which consists in witnessing to bijection of the universes of discourse.

Imagine that we are physically unable to compare the models pointwise simultaneously. Happily, we take an opportunity to exchange information with the owner of the other model using some means of communication, e.g., by having long-distance calls. While communicating, we easily learn that our interlocutor uses his model to operate on some objects that are the namesakes of ours, i.e., sets, membership, etc. Since we are interested in ZFC, we ask the interlocutor whether or not the axioms of ZFC are valid in his model. Manipulating the model, he returns a positive answer. After checking that he uses the same inference rules as we do, we cannot help but acknowledge his model to be a model of the theory we are all investigating. It is worth noting that this conclusion still leaves unknown for us the objects that make up his universe and the procedures he uses to distinguish between true and false propositions about these objects.

All in all, the *new conception of modeling implies not only refusal from identification of the universes of discourse but also admission of various procedures for verification of propositions.*

This article advertised the books [1] and [2] that crown our thought about and research into the field.

3. Elements of Set Theory

The credo of naive set theory cherishes a dream about the "Cantorian paradise" which is the universe that contains "any many which can be thought of as one, that is, every totality of definite elements which can be united to a whole through a law" or "every collection into a whole M of definite and separate objects m of our perception or our thought.'

The contemporary set theory studies realistic approximations to the ethereal ideal which are appropriate formal systems enabling us to operate on a wide spectrum of particular sets not leaving the comfortable room of soothing logical rigor. The essence of such a formalism lies in constructing a universe that "approximates from below" the world of naive sets so as to achieve the aim of current research. The corresponding axiomatic set theories open up ample opportunities to comprehend and corroborate in full detail the qualitative phenomenological principles that lie behind the standard and nonstandard mathematical models of today. ZFC, Zermelo–Fraenkel set theory, is most popular and elaborate. So, it is no wonder that our exposition proceeds mostly in the realm of ZFC.

We invite the reader to recall the formal technique for constructing universes of sets by some transfinite processes that lead to the so-called cumulative hierarchies. This technique is vital for Boolean valued analysis. Of profound importance is the detailed description of how the von Neumann universe grows from the empty set. So, we thoroughly analyze the status of classes of sets within the formal system stemming from J. von Neumann, K. Gödel, and P. Bernays and serving as a conservative extension of Zermelo–Fraenkel set theory.

4. Elements of the Theory of Boolean Algebras

Obnious is the key role of Boolean algebras in the area of analysisi we discuss. In fact, the influence of Boolean algebras spreads far beyond the theme under presentation. Boolean algebras penetrate into not only every section of mathematics but also practically all chambers of the mental treasure trove of mankind. There are ample grounds to assert that the concept of Boolean algebra reflects something general that is omnipresent in all spheres of human life.

There is a wonderful immanent connection between the "events" of physics and the "sentences" of logic which was revealed by G. Boole (1815–1864) whose name is made immortal by the term "Boolean algebra." Boole algebraized the tribes of events and sentences in a form so terse and lapidary that it has enjoyed everyone from novice to master for more than 150 years.

It is impossible to appraise Boole's contribution to culture better than this was done by his famous compatriot, contemporary, and elder friend A. De Morgan: "Boole's system of logic is but one of many proofs of genius and patience combined.... That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved."

The relevant preliminaries to Boolean algebra include the celebrated Stone Theorem. For the sake of diversity, we demonstrate it by using the Gelfand transform.

5. Elements of Category Theory

Set theory rules in the present-day mathematics. The buffoon's role of "abstract nonsense" is assigned in mathematics to category theory. History and literature demonstrate to us that the relations between the ruler and the jester may be totally intricate and unpredictable. Something very similar transpires in the interrelations of set theory and category theory and the dependency of one of them on the other.

Alongside set theory, the theory of categories serves as a universal language of the modern mathematics. Moreover, it is category theory that one of the most ambitious projects of the twentieth century mathematics was realized within, the project of socializing set theory. This evoked topos theory which provides a profusion of categories of which classical set theory is an ordinary member. It is worth noting that Boolean valued models were extra stimuli in search of a category-theoretic foundation of mathematics.

We sketchi the prerequisites of category theory up to the key concepts of topos and Boolean topos.

6. Boolean Valued Universes

It is the use of various rather unconventional models of set theory that unifies the available nonstandard methods of analysis. In particular, the technique of Boolean valued analysis bases on the properties of a certain cumulative hierarchy $\mathbb{V}^{(B)}$ whose every successive level comprises all functions with domain in the preceding levels and range in a complete Boolean algebra B fixed in advance.

Among our main topics is the construction and study of this hierarchy; i.e., the Boolean valued universe $\mathbb{V}^{(B)}$.

The idea behind the construction of $\mathbb{V}^{(B)}$ is very simple. We first observe that the characteristic function of a set is a good substitute for the set itself. Travelling across the levels of the von Neumann universe and carrying out successive substitutions, we arrive to another representation of the von Neumann universe which consists only of two valued functions. Replacing the two element Boolean algebra $\mathbb{2}$ with an arbitrary Boolean algebra B and repeating the above construction, we arrive at the desired $\mathbb{V}^{(B)}$.

The subtlest aspects, deserving special attention, relate to elaboration of the sense in which we may treat $\mathbb{V}^{(B)}$ as a model of set theory. We expose in full detail the basic technique that lay grounds for Boolean valued analysis; i.e., the transfer, mixing, and maximum principles.

Considerations of logical rigor and expositional independence have requested an ample room for constructing a separated universe and interpreting NGB in $\mathbb{V}^{(B)}$. The reader, interested only in solid applications to analysis, may just cast a casual glance at these rather sophisticated fragments of exposition while getting first acquaintance with the Boolean valued analysis.

7. The Apparatus of Boolean Valued Analysis

The transfer and maximum principles enable us to carry out various constructions of the conventional mathematical practice inside every Boolean valued universe. Therein we encounter the fields of real and complex numbers, Banach spaces, different operators, etc. The objects, representing them, may be perceived to some extent as nonstandard representations of the original mathematical entities.

Therefore, viewing the model $\mathbb{V}^{(B)}$ as a nonstandard presentation of the mathematical universe of discourse and recalling that $\mathbb{V}^{(B)}$ is constructed within the von Neumann universe, we may peek in the Boolean valued world, discovering nonstandard objects in a standard disguise.

Skipping from one B to another, a keen researcher sees many hypostaces of a sole mathematical idea embodied in a set-theoretic formula. Comparing observations is a method for studying a concealed meaning of the formula. The method often shows that essentially different analytical objects are in fact just various appearances of the same concept. This reveals the endoteric reasons of many facts and enables us to clarify the internal reasons for many vague analogies and dim parallelism and also to open new opportunities to study old objects.

This reminds us of the celebrated cave of Plato. If a casual escapee decided to inform his fellow detainees on what he saw at large, he might build a few bonfires in the night. Then each entity will cast several shadows on the wall of the cave (rather than a single shadow suggested by Plato). Now the detainees acquired a possibility of finding the essence of unknown things from analyzing the collection of shadows bearing more information than a sole shadow of an entity.

Comparative analysis with the help of Boolean valued models proceeds usually in two stages which we may agree to call syntactic and semantic.

At the *syntactic stage*, the mathematical fragment under study (a definition, a construction, a property, etc.) is transformed into a formal text of the symbolic language of set theory, or, to be more precise, into a text in a suitable jargon. In this stage we often have to analyze the complexity of the text; in particular, it matters whether the whole text or some of its parts is a bounded formula.

The *semantic stage* consists in interpretation of a formal text inside a Boolean valued universe. In this stage we use the terms of the conventional set theory, i.e. the von Neumann universe $\mathbb{V}$, to interpret (decode or translate) some meaningful texts that contain truth about the objects of the Boolean valued universe $\mathbb{V}^{(B)}$. This is done by using especial operations on the elements and subsets of the von Neumann universe.

We eleborate the basic operations of Boolean valued analysis, i.e., the canonical embedding, descent, ascent, and immersion. The most important properties of these operations are conveniently expressed using the notions of category and functor.

8. Functional Representation of Boolean Valued Universes

Various function spaces reside in functional analysis, and so the intention is natural of replacing an abstract Boolean valued system by some function analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into the class $\mathbb{V}$ of all sets. Truth values in the model $\mathbb{V}^Q$ are various subsets of Q and, in addition, the truth value $[\![\varphi(u_1,\dots,u_n)]\!]$ of $\varphi(t_1,\dots,t_n)$ at functions $u_1,\dots,u_n \in \mathbb{V}^Q$ is calculated as follows:

$$[\![\varphi(u_1,\dots,u_n)]\!] = \big\{q \in Q : \ \varphi\big(u_1(q),\dots,u_n(q)\big)\big\}.$$

We present a solution by A. G. Gutman and G. A. Losenkov to the above problem. To this end, we introduce and study their concept of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, every Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse.

9. Analysis of Algebraic Systems

Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, etc. Applying the descent functor to the established algebraic systems in a Boolean valued model, we distinguish bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

This technique of research, known as *direct Boolean valued interpretation*, allows us to produce new theorems or, to be more exact, to extend the semantical content of the available theorems by means of slavish translation. The information we so acquire might fail to be vital, valuable, or intriguing, in which case the direct Boolean valued interpretation turns out to be a leisurely game.

It thus stands to reason to raise the following questions: What structures significant for mathematical practice are obtainable by the Boolean valued interpretation of the most common algebraic systems? What transfer principles hold true in this process? Clearly, the answers should imply specific objects whose particular features enable us to deal with their Boolean valued representation which, if understood duly, is impossible to implement for arbitrary algebraic systems.

We show that an abstract B-set U embeds in the Boolean valued universe $\mathbb{V}^{(B)}$ so that the Boolean distance between the members of U becomes the Boolean truth-value of the negation of their equality. The corresponding element of $\mathbb{V}^{(B)}$ is, by definition, the *Boolean valued representation* of U. In case the B-set U has some a priori structure we may try to furnish the Boolean valued representation of U with an analogous structure, intending to apply the technique of ascending and descending to studying the original structure of U. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a B-set furnished with some additional structure.

We then analyze the problem for the main objects of general algebra. Located at the center of exposition, the notion of an algebraic B-system refers to a nonempty B-set endowed with a few contractive operations and B-predicates, the latter meaning B-valued contractive mappings.

The Boolean valued representation of an algebraic B-system appears to be a conventional two-valued algebraic system of the same type. This means that an appropriate completion of each algebraic B-system coincides with the descent of some two-valued algebraic system inside $\mathbb{V}^{(B)}$. On the other hand, each two-valued algebraic system may be transformed into an algebraic B-system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a B-system to a two-valued system. In other words, we have to seek here some versions of the transfer principle or the identity preservation principle of long standing in some branches of mathematics.

10. Analysis of Groups, Rings, and Fields

Continuating the previous research, we illustrate the general facts of Boolean valued analysis with particular algebraic systems in which complete Boolean algebras of congruences are connected with the relations of order and disjointness. We restrict exposition mainly to the descents of the systems under study and demonstrate the opportunities that are opened up by Boolean valued analysis. One of the main results (due to E. I. Gordon) reads as follows: *Each rationally complete semiprime commutative ring is an interpretation of a field in an appropriate Boolean valued model.*

11. Analysis of Cardinals

This theme occupies an especial place in the whole book. By now we only considered the Boolean valued universe $\mathbb{V}^{(B)}$ over an arbitrary complete Boolean algebra B. Moreover, we discussed only those properties and constructions that are practically independent of the choice of B. In actuality, many delicate mathematical properties of the members of $\mathbb{V}^{(B)}$ depends essentially on the structure of B. We show here how the choice of a Boolean algebra affects the specific properties of cardinals (and not only cardinals) in the corresponding Boolean valued universe.

It is shown that the canonical embedding of the von Neumann universe $\mathbb{V}$ to $\mathbb{V}^{(B)}$ sends ordinals to Boolean valued ordinals, preserving the order on ordinals. The same happens to cardinals provided that B enjoys the countable chain condition. However, the choice of B is available such that the canonical embedding "glue together" infinite cardinals; i.e., the standard names of two distinct infinite cardinals may have the same cardinality in an appropriate Boolean valued model. This effect is called *cardinal collapsing*. There are various mathematical constructions distorted under the canonical embedding. We discuss a few of them but focus exposition on the classical Gödel–Cohen solution of the continuum problem.

12. Analysis of Vector Lattices

The Boolean valued inverse $\mathbb{V}^{(B)}$ associated with a fixed complete Boolean algebra B is one of the arenas of mathematical events. Indeed, by virtue of the transfer and maximum principles, $\mathbb{V}^{(B)}$ contains numbers and groups as well as the Lebesgue and Riemann integrals, with the Radon–Nikodým and Hahn–Banach theorems available by virtue of the transfer and maximum principles.

The elementary technique of ascending and descending which we become acquainted with when considering algebraic systems shows each of mathematical objects in $\mathbb{V}^{(B)}$ to be a representation of an analogous classical object with an additional structure determined by B. In particular, this is also true in regard to functional-analytical objects. We expose the facts that are associated with Boolean valued representation of the latter objects.

Our main topic is Banach spaces in Boolean valued universes. It turns out that these spaces are inseparable from the concepts of the theory of ordered vector spaces and, above all, with the Dedekind complete vector lattices which were introduced by L. V. Kantorovich at the beginning of the 1930s under the name of K-spaces. They are often referred to as *Kantorovich spaces* nowadays.

The fundamental result of Boolean valued analysis in regard to this aspect is Gordon's Theorem which reads as follows: *Each universally complete Kantorovich space is an interpretation of the reals in an appropriate Boolean valued model.* Conversely, each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals viewed as such over some dense subfield of the reals.

Moreover, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in the original Kantorovich space. Translation of theorems is carried out by appropriate general operations of Boolean valued analysis. We then illustrate the technique of *Boolean valued transfer* by deriving the basic properties of Kantorovich spaces: representation as continuous or spectral functions, the Freudenthal spectral theorem, spectral integration, the functional calculus, etc.

13. Analysis of Lattice Normed Spaces

Also, we consider the structure and properties of a vector space with some norm taking values in a vector lattice. Such a vector space is called a *lattice normed space*. The most important peculiarities of these spaces are connected with decomposability. Use of decomposability allows us in particular to distinguish a complete Boolean algebra of linear projections in a lattice normed space which is isomorphic to the Boolean algebra of band projections of the norm lattice. Most typical in analysis are the lattice normed spaces of continuous or measurable functions.

In much the same way as many structural properties of a Kantorovich space are some properties of the reals in an appropriate Boolean valued model, the basic properties of a lattice normed space presents the Boolean valued interpretations of the relevant properties of normed spaces. The most principal connections are reflected by the three facts:

(1) The internal Banach spaces and external universally complete Banach–Kantorovich spaces are bijective under the bounded descent from a Boolean valued model.

(2) Each lattice normed space is realizable as a dense subspace of a Banach space viewed a vector space over some field, e.g. the rationals, in an appropriate Boolean valued model.

(3) Each Banach space X is a result of the bounded descent of some Banach space in a Boolean valued model if and only if X includes a complete Boolean algebra of norm one projections which possesses the cyclicity property. In other words, X is a Dedekind complete lattice normed space with a mixed norm.

These facts lie behind the approach to involutive algebras.

14. Analysis of Banach Algebras

The theory of Banach algebras is one of the most attractive traditional sections of functional analysis. We presents the basic results of Boolean valued analysis of involutive Banach algebras and Jordan Banach algebras.

The possibility of applying Boolean valued analysis to operator algebras rests on the following observation: If the center of an algebra is properly qualified and perfectly located then it becomes a one dimensional subalgebra after immersion in a suitable Boolean valued universe. This might lead to a simpler algebra. On the other hand, the transfer principle implies that the scope of the formal theory of the initial algebra is the same as that of its Boolean valued representation.

Exposition focuses on the analysis of AW^*-algebras and JB-algebras, i.e. Baer C^*-algebras and Jordan–Banach algebras. These algebras are realized in a Boolean valued model as AW^*-factors and JB-factors. The problem of representation of these objects as operator algebras leas to studying Kaplansky–Hilbert modules.

The dimension of a Hilbert space inside a Boolean valued model is a Boolean valued cardinal which is naturally called the Boolean dimension of the Kaplansky–Hilbert module that is the descent of the original Hilbert space. The cardinal shift reveals itself: some isomorphic Kaplansky–Hilbert modules may fail to have all bases of the same cardinality. This implies that a type I AW^*-algebra may generally split in a direct sum of homogeneous subalgebras in many ways. This was conjectured by I. Kaplansky as far back as in 1953.

Leaning on the results about the Boolean valued immersion of Kaplansky–Hilbert modules, we derive some functional representations of these objects. To put it more precisely, we prove that each AW^*-module is unitarily equivalent to the direct sum of some homogeneous AW^*-modules consisting of continuous vector functions ranging in a Hilbert space. An analogous representation holds for an arbitrary type I AW^*-algebra on replacing continuous vector functions with operator valued functions continuous in the strong operator topology.

We call an AW^*-algebra *embeddable* if it is $*$-isomorphic with the double commutant of some type I AW^*-algebra. Each embeddable AW^*-algebra admits a Boolean valued representation, becoming a von Neumann algebra or factor. We give several characterizations for embeddable AW^*-algebras. In particular, we prove that an AW^*-algebra A is embeddable if and only if the center valued normal states of A separate A. We also consider similar problems for the JB-algebras, a kind of real nonassociative analogs of C^*-algebras.

15. Operator Theory via Boolean Valued Analysis

We also show how Boolean valued analysis transforms the theory of operators in vector lattices, see [2]. We focus on the most recent results not reflected in the monographic literature yet. We start with the Boolean valued interpretations of order bounded operators with the emphasis on lattice homomorphisms and disjointness preserving operators. We provide a complete solution of the Wickstead problem as well as other new results on band preserving operators. Much attention is paid to various applications of order continuous operators to injective Banach lattices, Maharam operators, and related topics.

16. Conclusion

Adaptation of the ideas of Boolean valued models to functional analysis projects among the most important directions of developing the synthetic methods of mathematics. This approach yields the new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now transform matrices into numbers, embed function spaces into a straight line, yet having still uncharted vast territories of new knowledge.

Quite a long time had passed until the classical functional analysis occupied its present position of the language of continuous mathematics. Now the time has come of the new powerful technologies of model theory in mathematical analysis. Not all theoretical and applied mathematicians have already gained the importance of modern tools and learned how to use them. However, there is no backward traffic in science, and the new methods are doomed to reside in the realm of mathematics for ever and they will shortly become as elementary and omnipresent in analysis as Banach spaces and linear operators.

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ПРИГЛАШЕНИЕ В БУЛЕВОЗНАЧНЫЙ АНАЛИЗ

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Это короткое приглашение в область булевозначного анализа. Теория моделей оценивает и исчисляет истинность и доказательства. Поиск истины не только приближает нас к преследуемой цели, но также позволяет постичь многие другие ипостаси истины, к которым мы не стремились и которые мы даже не предвидели в начале предпринятого поиска. Это то, что нам открылось при изучении булевозначных моделей теории множеств. Такие модели проистекают из знаменитых работ П. Дж. Коэна по гипотезе континуума. Они относятся к математической логике и дают обилие непривычных и непредвиденных инкарнаций математических идей. Тем самым открываются новые мощные возможности для моделирования привычных способов умозаключения и верификации. Булевозначный анализ — это синтез анализа и булевозначных моделей. Адаптация идей булевозначного моделирования к функциональному анализу относится к наиболее важным направлениям развития синтетических методов математики. Этот подход дает новые модели чисел, пространств и типов уравнений. Расширяет содержимое всех имеющихся теорем и алгоритмов. Вся методология математического исследования обогащается и обновляется, открывая фантастические возможности. Теперь мы можем трансформировать матрицы в числа, вложить функциональные пространства в вещественную прямую, но при этом остаются неизведанными обширные территории нового знания. Статья представляет собой дайджест двух книг, содержащие итоги наших размышлений и исследований в этой области.

Ключевые слова: булевозначный универсум, булева оценка истинности, принцип переноса, принцип максимума, перемешивание, спуск, подъем, булевозначные числа, теорема Гордона.