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Minimal Sets in Almost Equicontinuous Systems[1]

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Supplying necessary and sufficient conditions such that a transitive system (as a subsystem of the Bebutov system) is uniformly rigid and using the fact that each transitive uniformly rigid system has an almost equicontinuous extension, we construct almost equicontinuous systems containing n ($n \in \mathbb{N}$), countably many, and uncountably many minimal sets, which serve as new examples of almost equicontinuous systems. Our method is quite general as each transitive uniformly rigid system has a factor that is a subsystem of the Bebutov system. Moreover, we explore how the number of connected components in a transitive pointwise recurrent system is related to the connectedness of the minimal sets contained in the system.

1. INTRODUCTION

Let (X,T) denote a given dynamical system on a compact metric space X induced by a continuous surjective map T of X onto X. A point $x \in X$ is *recurrent* provided $T^{n_i}(x) \to x$ for some sequence $n_i \to +\infty$. When each point of X is recurrent, we say that T is *pointwise recurrent* (PR). Recall that (X,T) is *weakly almost periodic* (WAP) if each element in the Ellis semigroup is continuous on X, (X,T) is *almost equicontinuous* (AE) if it is transitive (for an $\mathbb{N}$-action) and each element in the Ellis semigroup is continuous on the set of transitive points of T, and (X,T) is *uniformly rigid* (UR) if there are $n_1 < n_2 < \dots$ such that T^{n_i} tends to *id* uniformly (see [1, 2, 6, 5, 7] for the definitions and the equivalence statements). Note that, under the transitivity setting, a WAP system is AE [2], and an AE system is UR [8].

In this paper, supplying necessary and sufficient conditions such that a transitive system (as a subsystem of the Bebutov system) is UR and using the fact that each transitive UR system has an AE extension, we construct AE systems containing n ($n \in \mathbb{N}$), countably many, and uncountably many minimal sets, which serve as new examples of AE systems. Our method is quite general as each transitive UR system has a factor that is a subsystem of the Bebutov system. Moreover, we explore how the number of connected components in a transitive PR system is related to the connectedness of the minimal sets contained in the system.

After we had finished the paper, Downarowicz informed us about a forthcoming paper by Glasner and Weiss [9], where they defined a new class of AE systems, which is called *locally equicontinuous* (LE) ((X,T) is LE if each transitive subsystem of (X,T) is AE), and showed that WAP $\subsetneqq$ LE $\subsetneqq$ AE. In fact, they constructed an example that is LE and contains uncountably many minimal sets. As each transitive WAP system has a unique minimal set [6] and any transitive PR system on a zero-dimensional space is minimal [10], it will be interesting to know if there is a transitive LE system containing n minimal sets (for a given $n \geq 2$) or countably many minimal sets.

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2. A TRANSITIVE UR SYSTEM CONTAINING UNCOUNTABLY MANY MINIMAL SETS

To warm up, we first construct a transitive UR system containing uncountably many minimal sets as a subsystem of the so-called Bebutov system, which is defined by the shift map S on $I^{\mathbb{N}}$, where $I=[0,1]$, the closed unit interval, $I^{\mathbb{N}}$ is provided with the product topology, and the shift map is defined by $S(\alpha)_n=\alpha_{n+1}$ for $\alpha\in I^{\mathbb{N}}$ and $n\in\mathbb{N}$. A metric that generates the product topology is given by

$$d(\alpha,\beta)=\sup_{n\in\mathbb{N}}\frac{|\alpha_n-\beta_n|}{2^n}.$$

The following lemmas will help us to understand the idea of the construction of our system and to locate the minimal sets in a transitive system. Recall that, for a transitive system (X,T), Tran_T is the set of points whose ω-limit sets are X. For a dynamical system (X,T) with $x\in X$ and $U\subset X$, $N(x,U)=\{n\in\mathbb{N}\colon T^n(x)\in U\}$. A subset S of $\mathbb{N}$ is *syndetic* if there exists $l\in\mathbb{N}$ such that $\{i,\dots,i+l\}\cap S\neq\varnothing$ for each $i\in\mathbb{N}$.

Lemma 2.1. *Assume that (Y,S) is a transitive subsystem of $(I^{\mathbb{N}},S)$ and $y\in\mathrm{Tran}_S$. Then (Y,S) is UR if and only if there are $n_1<n_2<\dots$ such that, for each $\epsilon>0$, there is $i_0\in\mathbb{N}$ with $|y_{n_i+j}-y_j|<\epsilon$ for $i\geq i_0$ and $j\in\mathbb{N}$.*

Proof. Assume that (Y,S) is UR. Then, there are $n_1<n_2\dots$ such that S^{n_i} tends to id uniformly. Thus, for each $\epsilon>0$, there is i_0 such that, for $i\geq i_0$, $d(S^{n_i}(z),z)<\epsilon/2$ for each $z\in Y$. Take $z=S^j(y)$ for $j\in\mathbb{N}$; we have $|y_{n_i+j}-y_j|<\epsilon$ for $i\geq i_0$ and $j\in\mathbb{N}$.

The converse is easily obtained by the assumption that $y\in\mathrm{Tran}_S$. □

Lemma 2.2. *Let (X,T) be a transitive system and $x\in\mathrm{Tran}_T$. If A is a closed invariant subset of X such that $N(x,U)$ is syndetic for each neighborhood of A, then each minimal set of T is contained in A.*

Proof. Assume that M is a minimal set of T that is not contained in A. Then $A\cap M=\varnothing$. Hence, there are a neighborhood U of A and a neighborhood V of M such that $U\cap V=\varnothing$. For any $m\in\mathbb{N}$, as $T(M)=M$, there is a neighborhood $V_1=V_1(m)$ of M such that $T^j(V_1)\subset V$ for $0\leq j\leq m$. As $x\in\mathrm{Tran}_T$, $T^n(x)\in V_1$ for some n. Thus, $N(x,U)$ is not syndetic, a contradiction. This proves that each minimal set of T is contained in A. □

By Lemma 2.1, to construct the mentioned system, it is sufficient to construct a point $y=(y_1,y_2,\dots)\in I^{\mathbb{N}}$ with the following properties:

1. There are $n_1<n_2<\dots$ and $\epsilon_j\to 0$ such that $|y_{n_j+i}-y_i|<\epsilon_j$ for $i,j\in\mathbb{N}$.
2. For each $0\leq u\leq 1$, $\theta_u=(u,u,\dots)$ is a fixed point.

For convenience, we introduce the following notation. By B_n and $A_n(u)$, we denote $(y_1,\dots,y_n)$ and $(\underbrace{u,\dots,u}_{n})$, respectively, where $0\leq u\leq 1$. For $B=(b_1,\dots,b_n)\in I^n$ and $C=(c_1,\dots,c_m)\in I^m$, let $BC=(b_1,\dots,b_n,c_1,\dots,c_m)$ and $(B)_l^k=(b_l,\dots,b_k)$ for $1\leq l\leq k\leq n$.

Now we start the construction. First, we take $\epsilon_n=\frac{1}{2^n}$, $n\in\mathbb{N}$,

$n_1=1$, $B_{n_1}=(0)$ and

$n_2=9$ with $B_{n_2}=(0,\frac14,\frac12,\frac34,1,\frac34,\frac12,\frac14,0)$.

We will construct n_k, B_{n_k} with $k\geq 2$ inductively, which satisfy the following three conditions.

$(1)_k$ $n_{k-1}|n_k$ and B_{n_k} contains $A_{n_{k-1}}(0)$ and $A_{n_{k-1}}(1)$.

$(2)_k$ For $1\leq j\leq k-1$, $B_{n_j}=(B_{n_k})_1^{n_j}=(B_{n_k})_{n_k-n_j+1}^{n_k}$; i.e., $(y_1,\dots,y_{n_j})=(y_{n_k-n_j+1},\dots,y_{n_k})$.

$(3)_k$ $\xi_j^k=\max_{1\leq i\leq n_k}|y_{n_j+i\ (\mathrm{mod}\ n_k)}-y_i|<\epsilon_j$ for each $1\leq j\leq k-1$.

For $k=2$, $(1)_2$–$(3)_2$ are satisfied.

Now assume that, for $2 \le k \le l$, we have constructed n_k, B_{n_k} that satisfy $(1)_k$–$(3)_k$. To construct n_{l+1}, $B_{n_{l+1}}$, take

$$m_l = 2\left(\left[\frac{1}{\min\left\{\frac{\epsilon_1-\xi_1^l}{2}, \frac{\epsilon_2-\xi_2^l}{2}, \dots, \frac{\epsilon_{l-1}-\xi_{l-1}^l}{2}, \epsilon_l\right\}}\right]+1\right)$$

and set

$$B_{n_l}^j(u) = \left(1-\frac{j}{m_l}\right)B_{n_l} + \frac{j}{m_l}A_{n_l}(u)$$

for $0 \le j \le m_l$ and $0 \le u \le 1$. Note that, for $B=(b_1,\dots,b_n), C=(c_1,\dots,c_n) \in I^n$ and $b,c \in \mathbb{R}$, $bB+cC=(bb_1+cc_1,\dots,bb_n+cc_n)$. We observe that if $(B_{n_l})_i = u$, then $(B_{n_l}^j(u))_i = u$. Let

$$\begin{aligned} B_{n_{l+1}} = B_{n_l}B_{n_l}^1(0)\dots B_{n_l}^{m_l-1}(0)A_{n_l}(0)B_{n_l}^{m_l-1}(0)\dots B_{n_l}^1(0)B_{n_l} \\ B_{n_l}^1(1)\dots B_{n_l}^{m_l-1}(1)A_{n_l}(1)B_{n_l}^{m_l-1}(1)\dots B_{n_l}^1(1)B_{n_l}. \end{aligned}$$

Now we verify that n_{l+1}, $B_{n_{l+1}}$ satisfy $(1)_{l+1}$–$(3)_{l+1}$.

$(1)_{l+1}$ and $(2)_{l+1}$ are satisfied by the construction.

For $B=(b_1,\dots,b_n)$, let $\|B\| = \max_{1\le i\le n}|b_i|$. Then,

$$\|B_{n_l}^j(u) - B_{n_l}^{j+1}(u)\| \le \frac{1}{m_l}(\|B_{n_l}\| + \|A_{n_l}(u)\|) \le \frac{2}{m_l} < \epsilon_l$$

for $j=0,1,\dots,m_l-1$ and $0 \le u \le 1$. As $B_{n_{l+1}}$ begins and ends with B_{n_l}, we have

$$\xi_l^{l+1} \le \frac{2}{m_l} < \epsilon_l.$$

For $1 \le j \le l-1$, we have

$$\begin{aligned} \|(B_{n_l}^s(u))_1^{n_j} - (B_{n_l}^{s+1}(u))_{n_l-n_j+1}^{n_l}\| &\le \|(B_{n_l}^s(u))_1^{n_j} - (B_{n_l}^{s+1}(u))_1^{n_j}\| \\ &\quad + \|(B_{n_l}^{s+1}(u))_{n_l-n_j+1}^{n_l} - (B_{n_l}^{s+1}(u))_1^{n_j}\| \\ &\le \frac{2}{m_l} + \xi_j^l \le \frac{\epsilon_j - \xi_j^l}{2} + \xi_j^l = \frac{\epsilon_j + \xi_j^l}{2} < \epsilon_j \end{aligned}$$

for $s=0,\dots,m_l-1$ and $0 \le u \le 1$. By the same reasoning, we have

$$\|(B_{n_l}^s(u))_{n_l-n_j+1}^{n_l} - (B_{n_l}^{s+1}(u))_1^{n_j}\| \le \frac{\epsilon_j+\xi_j^l}{2} < \epsilon_j.$$

Also, we have

$$\max_{1\le i\le n_l}\left|(B_{n_l}^s(u))_{n_j+i \pmod{n_l}} - (B_{n_l}^s(u))_i\right| \le \left(1-\frac{s}{m_l}\right)\xi_j^l \le \xi_j^l \le \frac{\epsilon_j+\xi_j^l}{2} < \epsilon_j$$

for $s=0,\dots,m_l$ and $0 \le u \le 1$.

Finally, since $B_{n_{l+1}}$ satisfies $(2)_{l+1}$, we have

$$\|(B_{n_{l+1}})_1^{n_j} - (B_{n_{l+1}})_{n_{l+1}-n_j+1}^{n_{l+1}}\| = 0$$

for $j=1,\dots,l$.

To sum up, we have proved that $\xi_j^{l+1} < \epsilon_j$ for $j=1,\dots,l$. Thus, $(3)_{l+1}$ is satisfied.

Now, let $y = \lim_{k\to\infty} B_{n_k}$ and Y be the orbit closure of y under S.

Theorem 2.3. *(Y,S) is transitive, uniformly rigid, and contains uncountably many fixed points θ_u, $0 \le u \le 1$.*

Proof. Since $d((S^{n_j}y)_i, y_i) < \epsilon_j$ for $i,j \in \mathbb{N}$, (Y,S) is UR and transitive. It is obvious that, for each $0 \le u \le 1$, $\theta_u \in Y$ is a fixed point of S. Using Lemma 2.2, it is easy to see that they are the only minimal sets of S. $\square$

3. TRANSITIVE UR SYSTEMS CONTAINING n OR COUNTABLY MANY MINIMAL SETS

There are examples of transitive UR systems containing only one minimal set [10, 1]. In fact, if we define

$$B_{n_{l+1}} = B_{n_l} B^1_{n_l}(0) \dots B^{m_l-1}_{n_l}(0) A_{n_l}(0) B^{m_l-1}_{n_l}(0) \dots B^1_{n_l}(0) B_{n_l}$$

in the construction of Section 2, then the resulting system is such a system. Now it is natural to ask if there is a transitive UR system containing exactly n, for a given $n \in \mathbb{N}$, or countably many minimal sets. In fact, such a system exists. In this section, we will construct a transitive UR system containing only two minimal sets and explain how to modify it to get other examples. We remark that the construction here is more delicate than the construction of Section 2 because, for any $\epsilon > 0$, $\{n \in \mathbb{N}\colon \epsilon < y_n < 1-\epsilon\}$ should not be thick if θ_0 and θ_1 are the only two minimal sets in the orbit closure of $y = (y_1, y_2, \dots)$. To obtain the example, one follows Lemmas 2.1 and 2.2. That is, one constructs $y = (y_1, y_2, \dots) \in I^{\mathbb{N}}$ with the following properties:

(1) There are $n_1 < n_2 < \dots$ and $\epsilon_j \to 0$ such that $|y_{n_j+i} - y_i| < \epsilon_j$ for $i,j \in \mathbb{N}$.

(2) For any $M \in \mathbb{N}$, $\{n \in \mathbb{N}\colon y_{n+i} = 0$ for $1 \le i \le M$ or $y_{n+i} = 1$ for $1 \le i \le M\}$ is syndetic, and θ_0 and θ_1 are fixed points in the orbit closure of y.

Firstly, we construct a point $y' = (y'_1, y'_2, \dots) \in (-\infty, 1]^{\mathbb{N}}$ with the following properties:

(1′) There are $n_1 < n_2 < \dots$ and $\epsilon_j \to 0$ such that $|y'_{n_j+i} - y'_i| < \epsilon_j$ for $i,j \in \mathbb{N}$.

(2′) For any $M \in \mathbb{N}$, $\{n \in \mathbb{N}\colon y'_{n+i} \le 0$ for $1 \le i \le M$ or $y'_{n+i} = 1$ for $1 \le i \le M\}$ is syndetic, and there are $m_1, m_2 \in \mathbb{N}$ with $y'_{m_1+i} = 1$ and $y'_{m_2+i} \le 0$ for $1 \le i \le M$.

Secondly, we take $y = (y_1, y_2, \dots) \in I^{\mathbb{N}}$ with $y_i = \max\{y'_i, 0\}$. As $|y'_i - y'_j| \ge |y_i - y_j|$ for any $i,j \in \mathbb{N}$, it is clear that y satisfies (1) and (2).

For convenience, we introduce the following notation. By D_n, we denote $(y'_1, \dots, y'_n)$. For $E = (e_1, \dots, e_n) \in (-\infty, 1]^{\mathbb{N}}$ and $F = (f_1, \dots, f_n) \in (-\infty, 1]^{\mathbb{N}}$, let

$$E \oplus F = (\min\{e_1 + f_1, 1\}, \dots, \min\{e_n + f_n, 1\}).$$

Now we start the construction of y'.

First, we take $n_1 = 1$, $D_{n_1} = E_{n_1} = (0)$, and $n_2 = 10$ with $D_{n_2} = (0, \frac12, 1, \frac12, 0, 0, \frac12, 1, \frac12, 0)$ and $E_{n_2} = (0, -\frac12, -1, -\frac23, -2, -2, -\frac23, -1, -\frac12, 0)$.

We will construct n_k, D_{n_k}, and E_{n_k} inductively, which satisfy the following six conditions.

$(1)_k$ $2n_{k-1} | n_k$, $d_k \in \mathbb{N} \cup \{0\}$, $e_k \in \mathbb{N}$, and $\max_{1\le i\le n_k}\{(E_{n_k})_i\} = 0$, where

$$d_k = \Bigl|\min_{1\le i\le n_k}\{(D_{n_k})_i\}\Bigr| \qquad \text{and} \qquad e_k = \Bigl|\min_{1\le i\le n_k}\{(E_{n_k})_i\}\Bigr|.$$

$(2)_k$ For $1 \le j \le k-1$,

$$D_{n_j} = (D_{n_k})_1^{n_j} = (D_{n_k})_{n_k-n_j+1}^{n_k} \qquad \text{and} \qquad E_{n_j} = (E_{n_k})_1^{n_j} = (E_{n_k})_{n_k-n_j+1}^{n_k}.$$

$(3)_k$ For $1 \le j \le k-1$,

$$\max_{1\le i\le n_k}\{|(D_{n_k})_{n_j+i \pmod{n_k}} - (D_{n_k})_i|\} \le \frac{1}{2^j} \qquad \text{and}$$

$$\max_{1\le i\le n_k}\{|(E_{n_k})_{n_j+i \pmod{n_k}} - (E_{n_k})_i|\} \le \frac{1}{2^j}.$$

$(4)_k$ For any $1 \le j \le k-1$ and $l \in \mathbb{N}\cup\{0\}$ with $(l+1)n_{j+1} \le n_k$, there are $d_{k,j,l} \in \mathbb{R}$ and $e_{k,j,l} \in (-\infty, 0]$ such that

$$(D_{n_k})_{ln_{j+1}+\frac{n_{j+1}}{2}+1-n_j}^{ln_{j+1}+\frac{n_{j+1}}{2}} = E_{n_j} \oplus A_{n_j}(d_{k,j,l}) \qquad \text{and}$$

$$(E_{n_k})_{ln_{j+1}+\frac{n_{j+1}}{2}+1-n_j}^{ln_{j+1}+\frac{n_{j+1}}{2}} = E_{n_j} \oplus A_{n_j}(e_{k,j,l}).$$

$(5)_k$ For any $a \in \mathbb{R}$, we have

$$\min_{1\le i\le n_{k-1}}\{(D_{n_k} \oplus A_{n_k}(a))_i\} = \min\{1, a - e_{k-1}\} = 1 \qquad \text{or}$$

$$\max_{1\le i\le n_{k-1}}\Big\{(D_{n_k} \oplus A_{n_k}(a))_{\frac{n_k}{2}+1-i}\Big\} = \min\{1, a-(d_{k-1}+e_{k-1}+2)\} \le 0.$$

$(6)_k$ For $f_k = (2^{k-1}(d_{k-1}+1)+1)n_{k-1}$, we have

$$(D_{n_k})_{\frac{n_k}{2}+1-n_{k-1}}^{\frac{n_k}{2}} = E_{n_{k-1}} \qquad \text{and}$$

$$(D_{n_k})_{f_k+1-n_{k-1}}^{f_k} = D_{n_{k-1}} \oplus A_{n_{k-1}}(d_{k-1}+1) = A_{n_{k-1}}(1).$$

As $d_1 = e_1 = 0$, $d_2 = 0$, and $e_2 = 2$, it is easy to see that $(1)_2$–$(6)_2$ hold for $k = 2$.

Now assume that, for $2 \le k \le l$, we have constructed n_k, D_{n_k}, E_{n_k} satisfying $(1)_k$–$(6)_k$. To construct n_{l+1}, $D_{n_{l+1}}$, $E_{n_{l+1}}$, let

$$\begin{aligned}D_{n_{l+1}} = D_{n_l}&\Big(D_{n_l} \oplus A_{n_l}\Big(\frac{1}{2^l}\Big)\Big)\dots\Big(D_{n_l} \oplus A_{n_l}\Big(\frac{2^l(d_l+1)}{2^l}\Big)\Big)\\ &\Big(E_{n_l} \oplus A_{n_l}\Big(\frac{2^l(e_l+1)-1}{2^l}\Big)\Big)\dots\Big(E_{n_l} \oplus A_{n_l}\Big(\frac{1}{2^l}\Big)\Big)E_{n_l}\\ &E_{n_l}\Big(E_{n_l} \oplus A_{n_l}\Big(\frac{1}{2^l}\Big)\Big)\dots\Big(E_{n_l} \oplus A_{n_l}\Big(\frac{2^l(e_l+1)-1}{2^l}\Big)\Big)\\ &\Big(D_{n_l} \oplus A_{n_l}\Big(\frac{2^l(d_l+1)}{2^l}\Big)\Big)\dots\Big(D_{n_l} \oplus A_{n_l}\Big(\frac{1}{2^l}\Big)\Big)D_{n_l}\end{aligned}$$

and

$$\begin{aligned}E_{n_{l+1}} = E_{n_l}&\Big(E_{n_l} - A_{n_l}\Big(\frac{1}{2^l}\Big)\Big)\dots\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+1)}{2^l}\Big)\Big)\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+1)+1}{2^l}\Big)\Big)\\ &\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+1)+2}{2^l}\Big)\Big)\dots\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+e_l+2)}{2^l}\Big)\Big)\\ &\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+e_l+2)}{2^l}\Big)\Big)\dots\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+1)+1}{2^l}\Big)\Big)\\ &\Big(E_{n_l} - A_{n_l}\Big(\frac{2^l(d_l+1)}{2^l}\Big)\Big)\dots\Big(E_{n_l} - A_{n_l}\Big(\frac{1}{2^l}\Big)\Big)E_{n_l}.\end{aligned}$$

Now we verify that n_{l+1}, $D_{n_{l+1}}$, $E_{n_{l+1}}$ satisfy $(1)_{l+1}$–$(6)_{l+1}$.

By the construction, we have

$$n_{l+1}=2n_l(2^l(d_l+e_l+2)+1),\qquad d_{l+1}=e_l,\qquad e_{l+1}=2e_l+d_l+2,\qquad \text{and}$$
$$\max_{1\le i\le n_{l+1}}\{(E_{n_{l+1}})_i\}=\max_{1\le i\le n_l}\{(E_{n_l})_i\}=0.$$

Thus, $(1)_{l+1}$ is satisfied.

$(2)_{l+1}$ is satisfied by $(2)_l$ and the construction.

$(3)_{l+1}$ is satisfied by $(2)_l$, $(3)_l$ and the construction.

$(4)_{l+1}$ is satisfied by $(4)_l$ and the construction.

Since

$$\min_{1\le i\le n_l}\{(E_{n_l})_i\}=-e_l\qquad\text{and}\qquad \max_{1\le i\le n_l}\{(E_{n_l})_i\}=0,$$

we have

$$\min_{1\le i\le n_l}\{(E_{n_{l+1}}\oplus A_{n_{l+1}}(a))_i\}=\min_{1\le i\le n_l}\{(E_{n_l}\oplus A_{n_l}(a))_i\}=\min\{1,a-e_l\},$$
$$\max_{1\le i\le n_l}\Big\{(E_{n_{l+1}}\oplus A_{n_{l+1}}(a))_{\frac{n_{l+1}}{2}+1-i}\Big\}=\max_{1\le i\le n_l}\{((E_{n_l}-A_{n_l}(d_l+e_l+2))\oplus A_{n_l}(a))_i\}$$
$$=\min\{1,a-(d_l+e_l+2)\}.$$

As $a-e_l\ge(a-(d_l+e_l+2))+1$, we have $a-e_l\ge 1$ or $a-(d_l+e_l+2)\le 0$. Moreover, $\min\{1,a-e_l\}=1$ or $\min\{1,a-(d_l+e_j+2)\}\le 0$. Thus, $(5)_{l+1}$ is satisfied.

$(6)_{l+1}$ is satisfied by the construction.

Now, let $y'=\lim_{k\to\infty}E_{n_k}$, $y=(y_1,y_2,\ldots)$ with $y_i=\max\{y'_i,0\}$, $i\in\mathbb{N}$, and let Y be the orbit closure of y under the shift S.

Theorem 3.1. *(Y,S) is transitive, uniformly rigid, and has exactly two fixed points θ_0 and θ_1, which are the only minimal sets of S.*

Proof. By $(2)_k$ and $(3)_k$, we have $|y'_{n_j+i}-y'_i|\le\frac{1}{2^j}$ for $i,j\in\mathbb{N}$. Hence, $|y_{n_j+i}-y_i|\le\frac{1}{2^j}$ for $i,j\in\mathbb{N}$. This implies that (Y,S) is uniformly rigid.

By $(1)_k$, $(6)_k$, and $y_i=\max\{y'_i,0\}$, $i\in\mathbb{N}$, we know that, for any $j\in\mathbb{N}$, there are j_1 and j_2 such that $(y_{j_1+1},\ldots,y_{j_1+n_j})=A_{n_j}(0)$ and $(y_{j_2+1},\ldots,y_{j_2+n_j})=A_{n_j}(1)$. This means that θ_0 and θ_1 belong to Y.

By $(4)_k$, $(5)_k$, and $y_i=\max\{y'_i,0\}$, $i\in\mathbb{N}$, we know that, for any $j\in\mathbb{N}$, $l\in\mathbb{N}\cup\{0\}$, there are $m_1\in[ln_{j+2},(l+1)n_{j+2}-n_{j+1}]$ such that $(y_{m_1+1},\ldots,y_{m_1+n_j})=A_{n_j}(0)$ or $m_2\in[ln_{j+2},(l+1)n_{j+2}-n_{j+1}]$ such that $(y_{m_2+1},\ldots,y_{m_2+n_j})=A_{n_j}(1)$. Thus, using Lemma 2.2, one can easily show that θ_0 and θ_1 are the only minimal sets of S. $\square$

For a given $n\ge 3$, one easily modifies the above construction to obtain a transitive UR system containing only n minimal sets $\{\theta_{\frac{1}{i}}:1\le i\le n\}$. To get a transitive UR system containing countably many minimal sets $\{\theta_{\frac{1}{n}}:n\in\mathbb{N}\}\cup\{\theta_0\}$, we modify the construction of $D_{n_{l+1}}$ and $E_{n_{l+1}}$ such that $A_{n_l}(\frac{1}{i})$, $1\le i\le n$, appear in them.

4. PROOFS OF THE MAIN RESULTS

Theorem 4.1. *There is an AE system containing exactly n (for a given number n) or countably many or uncountably many minimal sets.*

Proof. Let (X,T) be a transitive UR system containing exactly n or countably many or uncountably many minimal sets that are fixed points. By Proposition 1.5 of [8], there is an AE system (Y, H) that is an extension of (X, T) with $\phi\colon Y \to X$ being the factor map. Let A be the set of fixed points of T. For each $a \in A$, $\phi^{-1}(a)$ is a closed invariant subset of Y. Collapsing each $\phi^{-1}(a)$ to a point, we get a factor (X_1, T_1) of (Y, H). As this factor map is minimal, by Proposition 3.7 of [3], (X_1, T_1) is AE. Clearly, the number of minimal sets in (X, T) is the same as in (X_1, T_1). □

Now we show that each transitive PR system has a factor that is a subsystem of the Bebutov system.

Proposition 4.2. *Each transitive PR system has a factor that is a subsystem of the Bebutov system.*

Proof. Assume that (X, T) is a transitive PR system that is embedded in some metric space with metric d. If (X, T) is minimal, let M be a point that is not in X, and if (X, T) is not minimal, let M be any minimal set of (X, T). Define $\phi\colon X \to \mathbb{R}^{\mathbb{N}}$ such that, for each $x \in X$, $\phi(x) = (d(x, M), d(T(x), M), d(T^2(x), M), \dots)$. It is clear that $\phi\colon X \to \phi(X)$ is continuous and $\phi T = S\phi$, where S is the shift. Hence, $(\phi(X), S)$ is a factor of (X, T) that is a subsystem of the Bebutov system. □

Let X be a compact metric space with metric d. Recall that if X is zero-dimensional, then each transitive PR system on X is minimal [10].

Theorem 4.3. *Let (X, T) be transitive and PR. If (X, T) contains a minimal set that is connected, then X is connected. Moreover, if each minimal set of (X, T) is connected, then, for each $x \in X$, $\omega(x, T)$ is connected.*

Proof. Assume the contrary. That is, X is not connected. Then $X = A_0 \cup A_1$, where A_i is a nonempty proper closed subset of X for $i = 0, 1$. Let $\Sigma = \Pi_{i \in \mathbb{N} \cup \{0\}} \{0, 1\}$ and S be the shift, where $\{0, 1\}$ has a discrete topology and Σ is equipped with the product topology. Define a map $\phi\colon X \to \Sigma$ as follows. For each $x \in X$, $\phi(x)_i = j$ if and only if $T^i(x) \in A_j$. It is easy to verify that $\phi\colon X \to \phi(X)$ is continuous and $\phi T = S\phi$. As (X, T) is transitive and PR, so is $(\phi(X), S)$. Since $\phi(X)$ is zero-dimensional, $(\phi(X), S)$ is minimal. Moreover, $(\phi(X), S)$ contains more than one point.

Let M be a connected minimal set in X. As $T(M) = M$, we have $S(\phi(M)) = \phi(M)$, and $\phi(M)$ is connected. This implies that $\phi(M)$ consists of one point. Since $(\phi(X), S)$ is minimal, we conclude that $\phi(X)$ consists of one point, a contradiction. This proves that X is connected.

Now assume that each minimal set of (X, T) is connected. Then, for each $x \in X$, $(\omega(x, T), T)$ is transitive and PR and contains a connected minimal set. Thus, $\omega(x, T)$ is connected. □

Corollary 4.4. *Let (X, T) be a transitive PR system. If (X, T) contains a minimal set that has n connected components for some $n \in \mathbb{N}$, then X has at most n connected components. Moreover, if, in addition, (X, T) is totally transitive, then X is connected.*

Proof. Let $x \in X$ be a transitive point and $X_i = \omega(T^i(x), T^n)$, $0 \le i \le n-1$. Then, $X = \bigcup_0^{n-1} X_i$. It is clear that (X_i, T^n) is transitive, PR, and contains a connected minimal set for each $0 \le i \le n-1$. By Theorem 4.3, X_i is connected, and, hence, X has at most n connected components.

If (X, T) is totally transitive, then $X_i = X$ for each $0 \le i \le n-1$. Thus, X is connected. □

5. A FINAL REMARK

There are some limitations on what kinds of minimal sets could appear in AE systems with additional conditions. If (X, T) is transitive and WAP, then (X, T) has a unique minimal set, which is equicontinuous. We say that a system is *scattering* if its product with any minimal set is transitive. By [3], there are scattering AE systems. We restate here a remark made in [4] and include a proof supplied by the authors.

Remark 4.5 of [4]. *Any minimal set in a scattering AE system is a fixed point.*

Proof. Assume that (X, T) is AE with an equicontinuity point x_0. By Lemma 1.2 of [8], given $\epsilon > 0$, there is $\eta > 0$ such that, if one has $d(T^n x_0, x_0) < \eta$ for some n, then $d(T^n y, y) < \epsilon$ for any $y \in X$.

Assume that there is a nontrivial minimal subsystem (Y, T) of (X, T) that is not a singleton, and let ϵ be such that there exist $y, y' \in Y$ with $d(y, y') > \epsilon$.

Choose η, $0 < \eta < \epsilon$, such that $d(T^n x_0, x_0) < \eta$ implies $d(T^n y, y) < \epsilon$ for any $y \in X$ and put $U = B(x_0, \eta_1)$, where η_1 is so small that $\operatorname{diam}(T^i(U)) < \eta_2$ for all $i \in \mathbb{N}$ and $\eta_1 + \eta_2 < \eta$. For any $y \in Y$, one also has $d(y, T^n y) \leq \epsilon$; now choose two small open sets $A, B \subset Y$ at a distance greater than ϵ: for any m such that $U \cap T^{-m}U \neq \varnothing$, this means $d(T^m(x_0), x_0) < \eta_1 + \eta_2 < \eta$, so that, for any $y \in Y$, one has $d(y, T^m y) < \epsilon$; therefore, $A \cap T^{-m}B = \varnothing$. This means that the Cartesian product of (X, T) and the minimal system (Y, T) is not transitive; this contradicts scattering. □

REFERENCES

1. *Akin E., Auslander J., Berg K.* When is a transitive map chaotic? // Convergence in ergodic theory and probability. Berlin: W. de Gruyter, 1996. P. 25–40. (Ohio State Univ. Math. Res. Inst. Publ.; V. 5).
2. *Akin E., Auslander J., Berg K.* Almost equicontinuity and the enveloping semigroup // Contemp. Math. 1998. V. 215. P. 75–81.
3. *Akin E., Glasner E.* Residual properties and almost equicontinuity // J. Anal. Math. 2001. V. 84. P. 243–286.
4. *Blanchard F., Host B., Maass A.* Topological complexity // Ergod. Th. and Dyn. Syst. 2000. V. 20. P. 641–662.
5. *Downarowicz T.* Weakly almost periodic flows and hidden eigenvalues // Contemp. Math. 1998. V. 215. P. 101–120.
6. *Ellis R., Nerurkar M.* Weakly almost periodic flows // Trans. Amer. Math. Soc. 1989. V. 313. P. 103–119.
7. *Glasner E., Maon D.* Rigidity in topological dynamics // Ergod. Th. and Dyn. Syst. 1989. V. 9. P. 309–320.
8. *Glasner E., Weiss B.* Sensitive dependence on initial conditions // Nonlinearity. 1993. V. 6. P. 1067–1075.
9. *Glasner E., Weiss B.* Locally equicontinuous dynamical systems // Colloq. Math. 2000. V. 84. P. 345–361.
10. *Katznelson Y., Weiss B.* When all points are recurrent/generic // Ergodic theory and dynamical systems. I. Boston: Birkhäuser, 1981. P. 195–210. (Progr. Math.; V. 10).