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RIEMANN–HILBERT APPROACH TO THE INVERSE PROBLEM FOR THE SCHRÖDINGER OPERATOR ON THE HALF-LINE

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A simple yet complete construction of the inverse problem for the Schrödinger operator on the half-line is presented in terms of the Riemann–Hilbert approach.

§1. Introduction

Our goal is to study the Riemann–Hilbert (R-H) problem for the half-line Schrödinger operator

$$L = -\frac{d^2}{dx^2} + v(x), \quad x \in [0, +\infty), \tag{1.1}$$

with Dirichlet boundary condition. Here, $v(x)$ is a real-valued smooth potential

$$v(x) \in C^\infty([0, \infty)), \tag{1.2}$$

($v(x)$ must have smooth continuation to the interval $[-\varepsilon, \infty)$, $\varepsilon > 0$) and quickly decaying as $x \to +\infty$:

$$\frac{d^l}{dx^l} v(x) = O(x^{-n}) \tag{1.3}$$

for any natural l and n. The inverse problem for this operator was studied in detail in several papers (see, e.g., [1] and [2, 3]). The role of the main tool in these investigations was played by the Gel′fand–Levitan equation.

In [4], the authors began the study of the inverse problem for a forth order operator. The growing technicalities make another (more algebraic) Riemann–Hilbert (R-H) approach more convenient. In the case of the Schrödinger operator on the entire line, the R-H approach was well described (see, e.g., [5, 6]) for various classes of the potentials v. Surprisingly, we did not find any references to the R-H approach on the half-line. Moreover, even though there are many results on the inverse problem on the half-line for various classes of the

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potentials v, most of the references we could find describe the inverse data in terms of the spectral function, the spectral shift function, resonances (see, e.g., [7] and the references therein), but not in terms of the reflection coefficient (see Lemma 2.6 below). At the same time, on the half-line, some additional interesting algebraic relations arise between the reflection coefficient and other inverse data. We also note that in the publications (see, e.g., [1, 2, 3]) where the reflection coefficient or the scattering matrix was employed to set the inverse data, another (Gel′fand–Levitan) approach was used, and other classes of potentials were considered.

All the above prompted us to write a short paper with simple yet complete construction of the inverse problem for the Schrödinger operator on the half-line for the potentials v in the Schwartz class in terms of the R-H approach. The formal main result (Corollary 5.3 on the one-to-one correspondence between the direct and inverse data) may or may not be new. As was mentioned above, after discussion with the specialists we could not find any direct reference, but the result could exist as a "folk" knowledge (for, e.g., the case of the Gel′fand–Levitan approach). In any case, the provided description of the R-H approach on the half-line is interesting by itself, and we believe that the present paper will be quite useful for further references. For definiteness, we use the Dirichlet boundary condition. Similar construction can be developed for other types of selfadjoint boundary conditions.

§2 is devoted to the description of the standard constructions of direct scattering for the Schrödinger equation on the half-line. Since the results about direct problem are not new and can be found for example in [1, 2, 3], we shall describe only the main properties of the inverse problem data, omitting many technical details in the corresponding proofs. In §3 we formulate several equivalent forms of the Riemann–Hilbert problem. §4 is devoted to the investigation of the R-H problem. Our main goal is to provide and investigate algebraic links between various objects associated with the R-H approach. The main results are formulated in Theorems 4.1–4.2 and Lemmas 4.3–4.6. Finally, in §5 we prove that the corresponding solution of the R-H problem yields a smooth quickly decaying potential $v(x)$ that solves the inverse problem.

§2. Direct scattering for the Schrödinger equation on the half-line

Here we recall the construction of the direct scattering problem for a Dirichlet Schrödinger operator on the half-line (see [1] and [2]).

We consider the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + v(x)$$

with a real-valued potential $v(x)$ satisfying (1.2) and (1.3).

We introduce two standard solutions $f(x,k)$ and $g(x,k)$ of the spectral equation

$$-\frac{d^2\psi}{dx^2} + v(x)\psi = k^2\psi, \tag{2.1}$$

such that

$$f(x,k) = e^{ikx}(1+o(1)), \quad x \to +\infty, \;\; k \in \overline{\mathbb{C}^+}, \tag{2.2}$$

and

$$g(0,k) = 0, \quad g'(0,k) = 1, \;\; k \in \mathbb{C}. \tag{2.3}$$

(This choice of the function g is specifically related to the Dirichlet boundary condition.) Standard computations show that $f(x,k)$ and $g(x,k)$ satisfy the following integral equations:

$$f(x,k) = \exp(ikx) - \int_x^{+\infty} \frac{\sin(k(x-y))}{k} v(y) f(y,k)\,dy \tag{2.4}$$

and

$$g(x,k) = \frac{\sin(kx)}{k} + \int_0^x \frac{\sin(k(x-y))}{k} v(y) g(y,k)\,dy. \tag{2.5}$$

We recall some properties of these solutions, starting with analytic properties of the functions $f(x,k)$ and $g(x,k)$.

Lemma 2.1. *For each $x \in [0,\infty)$, the solutions $f(x,k)$ and $g(x,k)$ satisfy the following conditions as functions of k:*

1) *$f(x,\cdot)$ is analytic in $\mathbb{C}^+$ and continuous up to the real line;*
2) *$f(x,\cdot) \in C^\infty(\mathbb{R})$;*
3) *$g(x,\cdot)$ is analytic in $\mathbb{C}$.*

Lemma 2.2. *The solutions $f(x,k)$ and $g(x,k)$ satisfy the following conditions as functions of x:*

1) *$f(\cdot,k) \in C^\infty([0,\infty))$ for $k \in \overline{\mathbb{C}^+}$;*
2) *$g(\cdot,k) \in C^\infty([0,\infty))$ for $k \in \mathbb{C}$.*

The proof of these elementary lemmas can be found, e.g., in [1].

We pass to the asymptotic properties of these functions as $|k| \to \infty$.

Lemma 2.3. *For all $x \in [0,\infty)$, the solutions $f(x,k)$ and $g(x,k)$ satisfy the following relations:*

1) $f(x,k) = \exp(ikx)(1+o(1)), \quad |k| \to \infty, \;\; k \in \overline{\mathbb{C}^+}$;
2) $g(x,k) = \frac{\sin(kx)}{k} + o(1/k)\exp(-ikx), \quad |k| \to \infty, \;\; k \in \overline{\mathbb{C}^+}$.

It is not hard to check that the function $g(x,k)$ for real k can be expressed in terms of $f(x,k)$ as follows:

$$g(x,k) = (a(k)\overline{f(x,k)} - \overline{a(k)}f(x,k))(-2ik)^{-1}.$$

Here $a(k)$ is also known as the Jost function. It can be found with the help of the Wronskian relation:

$$a(k) = f(0, k).$$

The next statement is a consequence of Lemmas 2.1–2.3.

Lemma 2.4. *The function* $a(k)$ *satisfies the following conditions*:

1) $a(k)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;
2) $a(k) \in C^\infty(\mathbb{R})$;
3) $a(k) = 1 + o(1), \quad |k| \to \infty, \ \ k \in \overline{\mathbb{C}^+}$;
4) $a(k) \neq 0$ *for each real* $k \neq 0$;
5) $a(k)$ *has a finite number of simple zeros in* $\mathbb{C}^+$:

$$a(k_l) = 0, \quad l = 1, 2, \dots, N, \ \ k_l = i\kappa_l, \ \ \kappa_l > 0.$$

6) $a(k)$ *may have zero at* $k = 0$; *if so, this zero must be simple*: $a'(0) \neq 0$.

It is not difficult to show that the zeros in $\mathbb{C}^+$ are associated with the eigenvalues of the operator L: $\lambda_l = -\kappa_l^2$.

We introduce a vector-valued function $\Psi := (\Psi_1, \Psi_2)^t$ by

$$\Psi(x, k) = \begin{pmatrix} f(x, k) \\ g(x, k)(-2ik)/a(k) \end{pmatrix}, \quad \operatorname{Im} k > 0, \tag{2.6}$$

$$\Psi(x, k) = \begin{pmatrix} \overline{g(x, \bar{k})}\overline{(2i\bar{k})}/\overline{(a(\bar{k}))} \\ \overline{f(x, \bar{k})} \end{pmatrix}, \quad \operatorname{Im} k < 0. \tag{2.7}$$

The components of this vector-valued function have the following properties.

Lemma 2.5. *For all* $x \in [0, \infty)$,

1) $\Psi_1(x, \cdot)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;
2) $\Psi_2(x, \cdot)$ *is analytic in* $\mathbb{C}^-$ *and continuous up to the real line*;
3) $\Psi_2(x, \cdot)$ *is meromorphic in* $\mathbb{C}^+$ (*and continuous up to the real line*) *with the simple poles* k_l, *and*

$$\operatorname{res}_{k=k_l} \Psi_2(x, k) = c_l \Psi_1(x, k_l), \quad c_l = i\tilde{c}_l, \ \ l = 1, 2, \dots, N.$$

Here the $\tilde{c}_l > 0$ *are some constants.*

4) $\Psi_1(x, \cdot)$ *is meromorphic in* $\mathbb{C}^-$ (*and continuous up to the real line*) *with the simple poles* $-k_l$, *and*

$$\operatorname{res}_{k=-k_l} \Psi_1(x, k) = \overline{c_l} \Psi_2(x, -k_l), \quad l = 1, 2, \dots, N.$$

Proof. All items 1–4 are obvious consequences of the construction of the $\Psi(x, k)$. The properties of the constants c_l can be found, e.g., in [1, formulas 1.28]. $\square$

We denote by Ψ^+ (Ψ^-) the limits of Ψ as k approaches the real axis so that $\operatorname{Im} k \to \pm 0$. The components of Ψ^+ and Ψ^- represent two bases of the solutions of the Schrödinger equation and, thus, are related by a linear transformation. Let $G(k)$ be the matrix that relates the vector-valued functions Ψ^+ and Ψ^-:

$$\Psi^+(x,k) = G(k)\Psi^-(x,k).$$

If $\operatorname{Im} k = +0$, then

$$\Psi^+(x,k) = \begin{pmatrix} e^{ikx} \\ \overline{e^{-ikx} + r(k)e^{ikx}} \end{pmatrix} (1 + o(1)), \quad x \to +\infty,$$

and respectively, for $\operatorname{Im} k = -0$ we have

$$\Psi^-(x,k) = \begin{pmatrix} e^{ikx} + r(k)e^{-ikx} \\ e^{-ikx} \end{pmatrix} (1 + o(1)), \quad x \to +\infty.$$

Here

$$r(k) = -a(k)(\overline{a(k)})^{-1}$$

is the reflection coefficient. Therefore,

$$G(k) = \begin{pmatrix} 1 & -r \\ \overline{r} & 1 - |r|^2 \end{pmatrix}. \tag{2.8}$$

Remark. When the inverse problem in terms of the Gel′fand–Levitan approach was considered, the scattering matrix $s(k) := -\overline{r}(k)$ was used (see, e.g., [2]). Here we use a slightly different object, simply to make it similar to the constructions of the R-H approach for the inverse problem on the entire line (see [5, 6]).

Lemma 2.6. *The function $r(k)$ satisfies the following conditions*:

1) $r(k) \in C^\infty(\mathbb{R})$;

2) $r(k)$ *admits the* (*differentiable*) *asymptotic expansion*

$$r(k) \sim -1 + \sum_{l=1}^{\infty} r_l k^{-l}, \quad k \to \pm\infty;$$

3) $r(k) = \overline{r(-k)}, \quad r_l = (-1)^l \overline{r_l}, \ k \in \mathbb{R}$;

4) $|r(k)| = 1, \quad k \in \mathbb{R}$;

5) *if* $a(0) \neq 0$, *then* $ind_{\mathbb{R}} r(k) = 2N$, *where* $ind_{\mathbb{R}} r(k) := ind(r)$ *is the index of the function* $r(k)$ *on the real axis*;

6) *if* $a(0) = 0$, *then* $ind_{\mathbb{R}} r(k) = 2N + 1$.

Statements 1–4 are obvious consequences of the construction of $r(k)$. Properties 5 and 6 can be found, e.g., in [2, formulas 3.3.22].

Now we consider a new vector-valued function $\widetilde{\Psi}(x,k)=\widetilde{\Psi}(x,k;M)$:

$$\widetilde{\Psi}(x,k)=\begin{pmatrix}1 & 0\\ 1+d_1(k)+s_1(k) & 1\end{pmatrix}\Psi(x,k)\ \text{if}\ \operatorname{Im}k\geqslant 0, \tag{2.9}$$

$$\widetilde{\Psi}(x,k)=\begin{pmatrix}1 & 1+d_2(k)+s_2(k)\\ 0 & 1\end{pmatrix}\Psi(x,k)\ \text{if}\ \operatorname{Im}k\leqslant 0, \tag{2.10}$$

where

$$d_1(k):=-\sum_{l=1}^{N}\frac{c_l}{k-k_l},\quad d_2(k)=\overline{d_1(\overline{k})},$$

and

$$s_1(k)=\sum_{m=1}^{M}\frac{\alpha_m}{(k+i)^m},\quad s_2(k)=\overline{s_1(\overline{k})},$$

with an arbitrarily given integer M, the constants k_l, c_l as in Lemmas 2.4 and 2.5, and coefficients α_m to be defined shortly.

The function $\widetilde{\Psi}$ satisfies the following jump condition at the real axis:

$$\widetilde{\Psi}^+(x,k)=\widetilde{G}(k)\widetilde{\Psi}^-(x,k).$$

It is not hard to show that the matrix $\widetilde{G}(k)$ is related to matrix $G(k)$ by the following formula:

$$\widetilde{G}(k)=\begin{pmatrix}1 & 0\\ 1 & 1+d_1+s_1\end{pmatrix}G(k)\begin{pmatrix}1 & -1-d_2-s_2\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -\widetilde{r}\\ \overline{\widetilde{r}} & 1-|\widetilde{r}|^2\end{pmatrix}, \tag{2.11}$$

with $\widetilde{r}=r+1+d_2+s_2$. The coefficients α_m can be chosen to guarantee that

$$\widetilde{r}(k)=O(k^{-M-1}),\ \text{i.e.,}\ \widetilde{G}(k)=I+O(k^{-M-1}),\quad k\to\pm\infty. \tag{2.12}$$

In what follows we fix such a choice of the coefficients α_m.

Finally, we introduce the function

$$\Phi(x,k)=\Phi(x,k;M):=\exp\{-i\sigma_3kx\}\widetilde{\Psi}(x,k;M). \tag{2.13}$$

We have

$$\Phi^+=\widetilde{\widetilde{G}}\Phi^-,\quad \widetilde{\widetilde{G}}(k)=\begin{pmatrix}1 & -\widetilde{r}e^{-2ikx}\\ \overline{\widetilde{r}e^{-2ikx}} & 1-|\widetilde{r}|^2\end{pmatrix},\quad k\in\mathbb{R}. \tag{2.14}$$

§3. Formulation of the Riemann–Hilbert problem

Our data of the inverse problem consist of the function r *and pairs* $\{k_l,c_l\}$, $l=1,\dots,N$, *such that*

1) *the function* r *possesses properties* 1–4 *listed in Lemma* 2.6;

2) *the pairs* $\{k_l,c_l\}$ *satisfy* $k_l\neq k_j$ *for* $l\neq j$, $k_l=i\kappa_l$, $\kappa_l>0$, *and* $c_l=i\tilde{c}_l$, $\tilde{c}_l>0$, $l=1,\dots,N$;

3) *either* $ind(r) = 2N$, *or* $ind(r) = 2N + 1$.

Let G be defined as in (2.8). We consider the following *Riemann–Hilbert problem* (*G-R-H problem*).

Given $\{r; k_l, c_l, l = 1, \dots, N\}$ *as described above, find a vector-valued function* $\Psi := (\Psi_1, \Psi_2)^t$ *such that*

1) $\Psi_1(x, \cdot)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;

2) $\Psi_2(x, \cdot)$ *is analytic in* $\mathbb{C}^-$ *and continuous up to the real line*;

3) $\Psi_2(x, \cdot)$ *is meromorphic in* $\mathbb{C}^+$ *and continuous up to the real line, with simple poles* k_l;

4) $\Psi_1(x, \cdot)$ *is meromorphic in* $\mathbb{C}^-$ *and continuous up to the real line, with simple poles* $-k_l$;

5) *the jump on the real axis is given by*

$$\Psi^+ = G\Psi^-, \quad k \in \mathbb{R}.$$

Here, as usual, the $\Psi^\pm$ *mean the limits of the function* Ψ *as* k *approaches the real axis from above and from below, respectively*;

6) *the asymptotic behavior at infinity is given by the formulas* (*here* $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$)

$$\Psi(x, k) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \exp(i\sigma_3 kx)(\mathbf{b} + o(1)), \quad \operatorname{Im} k \geqslant +0, \ |k| \to \infty;$$
$$\Psi(x, k) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \exp(i\sigma_3 kx)(\mathbf{b} + o(1)), \quad \operatorname{Im} k \leqslant -0, \ |k| \to \infty,$$

where either $\mathbf{b} = (0, 0)^t$ (*then the problem is said to be homogeneous*) *or* $\mathbf{b} = (1, 1)^t$ (*then the problem is nonhomogeneous*);

7) *the residues are given by*

$$res_{k=k_l} \Psi_2 = c_l \Psi_1(k_l), \quad res_{k=-k_l} \Psi_1 = \overline{c_l} \Psi_2(-k_l),$$
$$c_l = i\widetilde{c}_l, \ \widetilde{c}_l > 0, \ l = 1, \dots, N.$$

Thus, the results of §2 can be formulated as follows.

Theorem 3.1. *The vector-valued function* $\Psi(x, k)$ *defined in* §2 (*formulas* (2.6) *and* (2.7)) *satisfies the nonhomogeneous* G*-R-H problem.*

We shall need a regularized version of the R-H problem (with condition 7 removed). Let $\widetilde{G}$ be defined as in (2.11). We consider the following $\widetilde{G}$-R-H problem.

Find a vector-valued function $\widetilde{\Psi} := (\widetilde{\Psi}_1, \widetilde{\Psi}_2)^t$ *such that*

1) $\widetilde{\Psi}_1(x, \cdot)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;

2) $\widetilde{\Psi}_2(x, \cdot)$ *is analytic in* $\mathbb{C}^-$ *and continuous up to the real line*;

3) $\widetilde{\Psi}_2(x, \cdot)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;

4) $\widetilde{\Psi}_1(x,\cdot)$ *is analytic in* $\mathbb{C}^-$ *and continuous up to the real line*;

5) *the jump on the real axis is given by*

$$\widetilde{\Psi}^+ = \widetilde{G}\widetilde{\Psi}^-, \quad k \in \mathbb{R};$$

6) *the asymptotic behavior at infinity is given by*

$$\widetilde{\Psi}(x,k) = \exp(i\sigma_3 kx)(\mathbf{b} + o(1)), \quad |k| \to \infty,$$

where $\mathbf{b} = (0,0)^t$ *for the homogeneous problem and* $\mathbf{b} = (1,1)^t$ *for the nonhomogeneous one.*

Finally, let $\widetilde{\widetilde{G}}$ be defined as in (2.14). We formulate yet another equivalent form of our R-H problem ($\widetilde{\widetilde{G}}$*-R-H problem*).

Find a vector-valued function $\Phi := (\Phi_1, \Phi_2)^t$ *such that*

1) $\Phi_1(x,\cdot)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;

2) $\Phi_2(x,\cdot)$ *is analytic in* $\mathbb{C}^-$ *and continuous up to the real line*;

3) $\Phi_2(x,\cdot)$ *is analytic in* $\mathbb{C}^+$ *and continuous up to the real line*;

4) $\Phi_1(x,\cdot)$ *is analytic in* $\mathbb{C}^-$ *and continuous up to the real line*;

5) *the jump on the real axis is given by*

$$\Phi^+ = \widetilde{\widetilde{G}}\Phi^-, \quad k \in \mathbb{R};$$

6) *the asymptotic behavior at infinity is given by*

$$\Phi(x,k) = \mathbf{b} + o(1), \quad |k| \to \infty,$$

where $\mathbf{b} = (0,0)^t$ *for the homogeneous problem and* $\mathbf{b} = (1,1)^t$ *for the nonhomogeneous one.*

The explicit transformations (2.9), (2.10), and (2.13) establish one-to-one correspondence between solutions of the R-H problems formulated above. Thus, they are equivalent. Below, we shall use them all, because different questions may simplify for different versions of the R-H problem.

We notice that the passage from Ψ to $\widetilde{\Psi}$ allowed us to regularize the Riemann-Hilbert problem. More precisely, the term d_1 canceled the singularities and the term $1 + s_1$ improved the behavior at infinity.

§4. Uniqueness and solvability of the Riemann–Hilbert problem

Theorem 4.1. *Suppose* $ind(r) = 2N$. *If* $\widetilde{\Psi}(x,\cdot)$ *solves the homogeneous* $\widetilde{G}$*-R-H problem for some* $x \geqslant 0$, *then* $\widetilde{\Psi}(x,\cdot) = 0$.

Theorem 4.2. *Suppose* $ind(r) = 2N + 1$. *If* $\widetilde{\Psi}(x,\cdot)$ *solves the homogeneous* $\widetilde{G}$*-R-H problem for some* $x > 0$, *then* $\widetilde{\Psi}(x,\cdot) = 0$. *If* $\widetilde{\Psi}(0,\cdot)$ *solves the homogeneous* $\widetilde{G}$*-R-H problem for* $x = 0$ *and also satisfies the symmetry condition* $\widetilde{\Psi}(0,-k) = \Pi\widetilde{\Psi}(0,k)$, *where* $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, *then* $\widetilde{\Psi}(0,\cdot) = 0$.

Proof. STEP I. We can proceed with the proof of both theorems simultaneously. Let $\widetilde{\Psi}$ solve the homogeneous $\widetilde{G}$-R-H problem. From the symmetry of the $\widetilde{G}$-R-H problem it follows that the symmetric part

$$\widetilde{\Psi}_s(k) := \frac{1}{2}\big(\widetilde{\Psi}(k) + \Pi\widetilde{\Psi}(-k)\big)$$

of the solution and the antisymmetric part

$$\widetilde{\Psi}_a(k) := \frac{1}{2}\big(\widetilde{\Psi}(k) - \Pi\widetilde{\Psi}(-k)\big)$$

are also solutions. Now, assume that $\widetilde{\Psi}$ is symmetric, and let Ψ be corresponding (symmetric) solution of the G-R-H problem. For convenience we will use the (pretty standard) agreement that $\Psi^{\pm}$ denote not only the limits of Ψ on the real axis but rather the function Ψ in $\overline{\mathbb{C}^{\pm}}$, respectively. By the residue theorem and the symmetry of the solution, we have

$$\int_{-\infty}^{\infty} \Psi_1^+(k)\overline{\Psi_1^-(k)}dk = -2\pi\sum_{l=1}^{N}\widetilde{c}_l|\Psi_1^+(k_l)|^2.$$

At the same time,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi_1^+(k)\overline{\Psi_1^-(k)}dk &= \int_0^{\infty} \Psi_1^+(k)\overline{\Psi_1^-(k)}dk + \int_0^{\infty} \Psi_1^+(-k)\overline{\Psi_1^-(-k)}dk \\
&= \int_0^{\infty} \Psi_1^+(k)\overline{\Psi_1^-(k)}dk + \int_0^{\infty} \Psi_2^-(k)\overline{\Psi_2^+(k)}dk \\
&= \int_0^{\infty} (\Psi_1^-(k) - r(k)\Psi_2^-(k))\overline{\Psi_1^-(k)}dk + \int_0^{\infty} \Psi_2^-(k)r(k)\overline{\Psi_1^-(k)}dk \\
&= \int_0^{\infty} |\Psi_1^-(k)|^2 dk.
\end{aligned}$$

Thus,

$$\int_0^{\infty} |\Psi_1^-(k)|^2 dk + 2\pi\sum_{l=1}^{N}\widetilde{c}_l|\Psi_1^+(k_l)|^2 = 0,$$

which means that $\Psi_1^+(k_l) = 0$, $l = 1, \ldots, N$, and $\Psi_1^-(k) = 0$, $k \geqslant 0$. Using analyticity, the identity $\Psi_2^+ = \overline{r}\Psi_1^-$, and symmetry, we obtain

$$\Psi_1^+(k_l) = \Psi_2^-(-k_l) = 0, \;\; l = 1, \ldots, N; \quad \Psi_1^-(k) = \Psi_2^+(k) = 0.$$

The proof of similar relations for the antisymmetric solution is much the same.

STEP II. The two other components Ψ_1^+, Ψ_2^- of the solution satisfy the scalar homogeneous R-H problem

$$\Psi_1^+ = -r\Psi_2^- \text{ on } \mathbb{R}; \quad \Psi_1^+ \sim e^{ikx}o(1) \text{ in } \overline{\mathbb{C}^+}; \quad \Psi_2^- \sim e^{-ikx}o(1) \text{ in } \overline{\mathbb{C}^-}.$$

Moreover, $\Psi_1^+(k_l) = \Psi_2^-(-k_l) = 0$, $l = 1, \dots, N$. Thus, we can represent the solution in the form

$$\Psi_1^+(k) = \Big(\prod_{l=1}^{N} \frac{k-k_l}{k+k_l}\Big)\varphi^+(k), \quad \Psi_2^-(k) = \Big(\prod_{l=1}^{N} \frac{k+k_l}{k-k_l}\Big)\varphi^-(k),$$

where φ solves the R-H problem

$$\begin{aligned} &\varphi^+(k) = r_*(k)\varphi^-(k) \text{ on } \mathbb{R}; \quad \varphi^\pm \sim e^{\pm ikx}o(1) \text{ in } \overline{\mathbb{C}^\pm}; \\ &r_*(k) := -r(k)\prod_{l=1}^{N}\frac{(k+k_l)^2}{(k-k_l)^2}. \end{aligned} \tag{4.1}$$

Note that $ind(r_*) = 0$ if $ind(r) = 2N$ and $ind(r_*) = 1$ if $ind(r) = 2N+1$.

Consider the auxiliary problem: *find* ω (*denoted also by* $\omega^\pm$ *in* $\overline{\mathbb{C}^\pm}$) *analytic in* $\mathbb{C}^\pm$, *continuous in* $\overline{\mathbb{C}^\pm}$, *and such that*

$$\omega^+ = r_*\omega^- \ \textit{on}\ \mathbb{R}; \quad \omega \sim o(1) \ \textit{in}\ \mathbb{C}.$$

The case where $ind(r_*) = 0$. Then from [8, Chapter II] it follows that $\omega = 0$. This proves Theorem 4.1.

The case where $ind(r_*) = 1$. Then ω has the form (see [8, Chapter II])

$$\omega^\pm(k) = \frac{\alpha}{k \pm i}\widetilde{\omega}^\pm(k),$$

where α is a constant (it may depend on x) and $\widetilde{\omega}$ is a *unique* solution of the problem

$$\widetilde{\omega}^+(k) = r_*(k)\frac{k+i}{k-i}\widetilde{\omega}^-(k) \text{ on } \mathbb{R}; \quad \widetilde{\omega} \sim 1 \text{ in } \mathbb{C}.$$

If $x > 0$, then ω decays exponentially outside the real axis, so that $\alpha = 0$. If $x = 0$, then the uniqueness of the solution $\widetilde{\omega}$ implies its symmetry: $\widetilde{\omega}(k) = \widetilde{\omega}(-k)$. Thus, ω (and simultaneously φ and Ψ) will satisfy the symmetry condition only if $\alpha = 0$. Theorem 4.2 is proved. □

We are going to prove that the nonhomogeneous R-H problem is solvable and describe some properties of its solution as a function of x (smoothness, behavior at infinity, etc.). For this, we rewrite the R-H problem in terms of an equivalent system of singular integral equations.

Consider the following integral system:

$$\Phi^+(k) = \mathbf{b} + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{(I - \widetilde{\widetilde{G}}(\xi)^{-1})\Phi^+(\xi)}{\xi - k - i0} d\xi, \quad k \in \mathbb{R}. \tag{4.2}$$

Lemma 4.3. *The $\widetilde{\widetilde{G}}$-R-H problem is equivalent to the integral system* (4.2) *in the following sense. If Φ solves the $\widetilde{\widetilde{G}}$-R-H problem, then its boundary value Φ^+* (*the limit on the real axis from the upper half-plane*) *solves system* (4.2)*. Vice versa, if Φ^+ solves* (4.2)*, then it is the boundary value of a solution of the $\widetilde{\widetilde{G}}$-R-H problem.*

Proof. Let Φ be a solution of the $\widetilde{\widetilde{G}}$-R-H problem. Using the Cauchy formula and the residue theorem, we get

$$\begin{aligned} \Phi(z) - \mathbf{b} = &\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\Phi^+(\xi) - \mathbf{b}}{\xi - z} d\xi \\ &- \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\Phi^-(\xi) - \mathbf{b}}{\xi - z} d\xi, \quad z \in \mathbb{C} \setminus \mathbb{R}. \end{aligned} \tag{4.3}$$

Indeed, it suffices to close the first integral on the right-hand side of (4.3) in $\mathbb{C}^+$ and the second integral in $\mathbb{C}^-$. Next, since $\Phi^-(\xi) = (\widetilde{\widetilde{G}}(\xi))^{-1}\Phi^+(\xi)$, it is easy to obtain the corresponding integral system.

Assume now that $\Phi^+(k)$ is a solution of the integral system, and let $\Phi(z)$ be its analytic extension to $\mathbb{C}^+$. Put $\Phi^-(k) := (\widetilde{\widetilde{G}}(k))^{-1}\Phi^+(k)$, $k \in \mathbb{R}$. Then analytic continuation of the system to $\mathbb{C}^+$ gives (4.3) for $z \in \mathbb{C}^+$. In particular,

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\Phi^-(\xi) - \mathbf{b}}{\xi - z} d\xi = 0, \quad z \in \mathbb{C}^+.$$

Thus, Φ^- admits an analytic extension to $\mathbb{C}^-$, which we also denote by Φ. □

It is more convenient to transform the integral system (4.2) in the following way:

$$\widetilde{\Phi}^+(k) := \Phi^+(k) - \mathbf{b}.$$

Then the system reads as follows:

$$\begin{aligned}\widetilde{\Phi}^{+}(k) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(I-\widetilde{\widetilde{G}}(\xi)^{-1})\mathbf{b}}{\xi-k-i0}d\xi \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(I-\widetilde{\widetilde{G}}(\xi)^{-1})\widetilde{\Phi}^{+}(\xi)}{\xi-k-i0}d\xi, \quad k\in\mathbb{R}.\end{aligned} \tag{4.4}$$

The last system can be viewed as an equation $(I-K(x))\mathbf{y}=\mathbf{y}_0$ in the space $(L_2(\mathbb{R}))^2$ depending on the parameter $x \geqslant 0$. The operator K is bounded but not compact. Nevertheless, from [9] it follows that the corresponding system satisfies a Fredholm type theorem. We also note that, by the original R-H problem, the total index of the system is equal to zero.

The case where $ind(r) = 2N$. Theorem 4.1 shows that the homogeneous system has only trivial solution for any $x \geqslant 0$. Thus, the nonhomogeneous system has a unique solution for any $x \geqslant 0$, i.e., the operator $(I-K(x))$ is invertible. By (2.12), the operator $K(x)$ has M bounded derivatives. Now, a standard perturbation argument shows that the solution $\Phi(x,k;M)$ is also M times differentiable with respect to x. Thus, the same is true for $\Psi(x,k)$, which is independent of M. Since M was chosen arbitrarily, the function Ψ is infinitely smooth for $x \geqslant 0$. We note that, then, $\Phi(x,k;M)$ and $\widetilde{\Psi}(x,k;M)$ are also infinitely smooth in x for any M.

Now, we discuss the behavior of the solution at $x=0$ to verify that our inverse data correspond to the Dirichlet boundary condition.

Lemma 4.4. *Let* $\mathbf{b}=(1,1)^t$*, and let* Ψ *be a solution of the corresponding G-R-H problem. Then for* $x=0$ *we have* $\Psi_2^{+}(0,k)=\Psi_1^{-}(0,k)=0$.

Proof. Observe that, for $\mathbf{b}=(1,1)^t$ and $x=0$, the asymptotic behavior of Ψ is given by $\Psi^{+}(0,k)\sim\binom{1}{0}$ as $k\to\infty$ in $\overline{\mathbb{C}^{+}}$ and $\Psi^{-}(0,k)\sim\binom{0}{1}$ as $k\to\infty$ in $\overline{\mathbb{C}^{-}}$. Since a solution is unique, it suffices to find a solution for $x=0$ satisfying $\Psi_2^{+}(0,k)=\Psi_1^{-}(0,k)=0$. The existence of such a solution is equivalent to the solvability of the scalar R-H problem

$$\begin{gathered}\Psi_1^{+}=-r\Psi_2^{-} \text{ on } \mathbb{R};\quad \Psi_1^{+},\,\Psi_2^{-}\sim 1 \text{ in } \overline{\mathbb{C}^{\pm}};\\ \Psi_1^{+}(k_l)=\Psi_2^{-}(-k_l)=0,\quad l=1,\dots,N.\end{gathered} \tag{4.5}$$

Repeating the arguments from step II of the proof of Theorem 4.1 (the only difference is that now we deal with the nonhomogeneous problem) we see that the solution exists and is unique:

$$\Psi_1^+(k) = \left(\prod_{l=1}^{N} \frac{k-k_l}{k+k_l}\right)\varphi_*^+(k), \quad \Psi_2^-(k) = \left(\prod_{l=1}^{N} \frac{k+k_l}{k-k_l}\right)\varphi_*^-(k),$$

where φ_* is a unique solution of the R-H problem

$$\begin{aligned} &\varphi_*^+(k) = r_*(k)\varphi_*^-(k) \text{ on } \mathbb{R}; \quad \varphi_* \sim 1 \text{ in } \mathbb{C}; \\ &r_*(k) := -r(k)\prod_{l=1}^{N}\frac{(k+k_l)^2}{(k-k_l)^2}. \end{aligned} \tag{4.6}$$

□

The case where $ind(r) = 2N+1$. We start with an analog of Lemma 4.4.

Lemma 4.5. *Let $\mathbf{b} = (1,1)^t$; then for $x = 0$ the corresponding G-R-H problem admits a solution Ψ satisfying $\Psi_2^+(0,k) = \Psi_1^-(0,k) = 0$. Moreover, we can choose a solution satisfying also the symmetry condition $\widetilde{\Psi}(0,-k) = \Pi\widetilde{\Psi}(0,k)$ (here $\widetilde{\Psi}$ is the corresponding solution of the $\widetilde{G}$-R-H problem); then such a solution is unique.*

Proof. We will seek a solution satisfying $\Psi_2^+(0,k) = \Psi_1^-(0,k) = 0$ by using the construction of the proof of Lemma 4.4. The only difference is that at the last step a solution of the scalar R-H problem (4.6) exists but is not unique. This shows the existence of a solution Ψ satisfying $\Psi_2^+(0,k) = \Psi_1^-(0,k) = 0$. The symmetry of the data of the R-H problem implies that the symmetric part of this solution $\big(\widetilde{\Psi}_s(k) := \frac{1}{2}(\widetilde{\Psi}(k) + \Pi\widetilde{\Psi}(-k))\big)$ is a symmetric solution of the R-H problem. It is easy to check that the corresponding solution Ψ_s still satisfies the conditions $\Psi_{s,2}^+(0,k) = \Psi_{s,1}^-(0,k) = 0$. The uniqueness of such a solution follows from Theorem 4.2. □

As in the case where $ind(r) = 2N$, we can use Lemma 4.3 and Theorem 4.2 for $x > 0$ to establish the existence and uniqueness of a solution of the R-H problem. Also, the same arguments as above show that Ψ is smooth for $x > 0$. Obviously, this solution is symmetric ($\widetilde{\Psi}(x,-k) = \Pi\widetilde{\Psi}(x,k),\ x > 0$). Also, from Theorem 4.2 and Lemma 4.5 we know that there exists a unique symmetric solution of the R-H problem for $x = 0$. Unfortunately, since for $x = 0$ there exists a nontrivial kernel (consisting of antisymmetric solutions) of the integral system from Lemma 4.3, we cannot apply pertubation arguments to establish the smoothness of Ψ as $x \to 0$. For this reason, we shall modify our integral system slightly.

Applying the symmetry condition in the first component of the vector equation in system (4.4), we see that every (actually a unique) symmetric solution of the R-H problem solves the following system:

$$
\begin{aligned}
\widetilde{\Phi}_1^+(k) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{((I-\widetilde{\widetilde{G}}(\xi)^{-1})\mathbf{b})_1}{\xi-k-i0}d\xi \\
&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(\Pi(\widetilde{\widetilde{G}}(-\xi)^{-1}-I)\widetilde{\Phi}^+(-\xi))_1}{\xi-k-i0}d\xi, \\
\widetilde{\Phi}_2^+(k) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{((I-\widetilde{\widetilde{G}}(\xi)^{-1})\mathbf{b})_2}{\xi-k-i0}d\xi \\
&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{((I-\widetilde{\widetilde{G}}(\xi)^{-1})\widetilde{\Phi}^+(\xi))_2}{\xi-k-i0}d\xi.
\end{aligned}
\tag{4.7}
$$

Lemma 4.6. *The solution of system* (4.7) *exists and is unique for any* $x \geqslant 0$. *It coincides with the symmetric solution of the* $\widetilde{\widetilde{G}}$*-R-H problem.*

Proof. Existence and coincidence with the solution of the R-H problem were justified above. It suffices to prove uniqueness. Let $\widetilde{\Phi} = (\widetilde{\Phi}_1, \widetilde{\Phi}_2)$ be a solution of the homogeneous system (4.7) ($\mathbf{b} = 0$). Obviously, the analytic extension of $\widetilde{\Phi}_j$ to $\mathbb{C}^+$ belong to the Hardy space H_2^+. Thus, we have

$$
\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{\Phi}_1^+(\xi)-(\Pi(\widetilde{\widetilde{G}}(-\xi)^{-1}-I)\widetilde{\Phi}^+(-\xi))_1}{\xi-k-i0}d\xi = 0,
$$

$$
\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{\Phi}_2^+(\xi)-((I-\widetilde{\widetilde{G}}(\xi)^{-1})\widetilde{\Phi}^+(\xi))_2}{\xi-k-i0}d\xi = 0.
$$

This means that

$$
\begin{aligned}
\widetilde{\Phi}_1^+(\xi)-(\Pi(\widetilde{\widetilde{G}}(-\xi)^{-1}-I)\widetilde{\Phi}^+(-\xi))_1 \in H_2^-, \\
\widetilde{\Phi}_2^+(\xi)-((I-\widetilde{\widetilde{G}}(\xi)^{-1})\widetilde{\Phi}^+(\xi))_2 \in H_2^-.
\end{aligned}
\tag{4.8}
$$

Consider the first relation. After direct calculations, it reads as follows:

$$
\widetilde{\Phi}_1^+(\xi)+\overline{\widetilde{r}(-\xi)}e^{-2i\xi x}\widetilde{\Phi}_1^+(-\xi) \in H_2^-.
$$

Using the properties of r and the obvious fact that $e^{-2i\xi x}\widetilde{\Phi}_1^+(-\xi) \in H_2^-$, we get

$$
\widetilde{\Phi}_1^+(\xi)+r(\xi)e^{-2i\xi x}\widetilde{\Phi}_1^+(-\xi) \in H_2^-. \tag{4.9}
$$

Next, since $\widetilde{\Phi}_1^+(\xi) \in H_2^+$ and $\|\widetilde{\Phi}_1^+(\xi)\|_{L_2(\mathbb{R})} = \|r(\xi)e^{-2i\xi x}\widetilde{\Phi}_1^+(\xi)\|_{L_2(\mathbb{R})}$, relation (4.9) implies

$$\widetilde{\Phi}_1^+(\xi) = -r(\xi)e^{-2i\xi x}\widetilde{\Phi}_1^+(-\xi), \quad \widetilde{\Phi}_1^+(\xi) \in H_2^+. \tag{4.10}$$

Putting $\varphi^\pm(\xi) := e^{\pm i\xi x}\widetilde{\Phi}_1^+(\pm\xi)$, we see that $\varphi^\pm$ is a symmetric (i.e., $\varphi^+(\xi) = \varphi^-(-\xi)$) solution of the R-H problem of the form (4.1):

$$\varphi^+(\xi) = -r(\xi)\varphi^-(\xi) \text{ on } \mathbb{R}; \quad \varphi^\pm \sim e^{\pm i\xi x}o(1) \text{ in } \overline{\mathbb{C}^\pm}. \tag{4.11}$$

As was explained at the end of the proof of Theorem 4.2, such a problem has only the trivial symmetric solution. Thus, $\widetilde{\Phi}_1^+(\xi) = 0$. Then the second relation in (4.8) can be rewritten as $\widetilde{\Phi}_2^+(\xi) \in H_2^-$, so that $\widetilde{\Phi}_2^+(\xi) = 0$. □

Now, the arguments similar to those presented above show that our (symmetric) solution of the G-R-H problem is smooth as $x \to 0$.

§5. Inverse problem

Now we prove that the solution of the Riemann–Hilbert problem gives a solution of the inverse problem. Fix any large M.

Theorem 5.1. *There exists a real-valued smooth function $v(x)$, $v(x) \in C^\infty([0,\infty))$, satisfying*

$$-\frac{d^2\widetilde{\Psi}}{dx^2} + v(x)\widetilde{\Psi} - k^2\widetilde{\Psi} = 0. \tag{5.1}$$

Here $\widetilde{\Psi}$ is the solution of nonhomogeneous $\widetilde{G}$-R-H problem (with $\mathbf{b} = (1,1)^t$).

Proof. We recall that $\widetilde{\Psi}(\cdot,k) \in C^\infty([0,\infty))$. From the integral equations, it is not hard to see that $\widetilde{\Psi}$ has the following asymptotic expansion as $|k| \to \infty$:

$$\widetilde{\Psi}(x,k) = \exp(i\sigma_3 kx)\Big(\mathbf{b} + \sum_{j=1}^{M} \widetilde{\psi}_j(x)k^{-j} + o(k^{-M})\Big). \tag{5.2}$$

This expansion can be (partially) differentiated termwise with respect to x:

$$\frac{d^n}{dx^n}\big(\exp(-i\sigma_3 kx)\widetilde{\Psi}(x,k)\big) = \sum_{j=1}^{M-n} \frac{d^n}{dx^n}\widetilde{\psi}_j(x)k^{-j} + o(k^{-M+n}), \quad 1 \leqslant n \leqslant M. \tag{5.3}$$

We put $v(x) := 2i((\widetilde{\psi}_1)_1)'_x$, which is smooth and real-valued (by the symmetry of the R-H problem). More precisely (cf. the considerations above), v can be differentiated $M-1$ times, but since its definition does not depend on $M \geqslant 2$, v is infinitely smooth. We denote by q the vector-valued function

$$q(x,k) = -\frac{d^2\widetilde{\Psi}}{dx^2} + v(x)\widetilde{\Psi} - k^2\widetilde{\Psi}.$$

It is easy to check that q is holomorphic in $\mathbb{C}^\pm$ and satisfies the same junction condition at the real axis as the function $\widetilde{\Psi}$. Consider the asymptotic behavior of q as $|k| \to \infty$:

$$q(x,k) = \exp(i\sigma_3 kx)(-2i\sigma_3(\widetilde{\psi}_1)'_x + v(x)\mathbf{b} + o(1)).$$

Here, by the symmetry of the Riemann–Hilbert problem, we have $(\widetilde{\psi}_1)_1 = -(\widetilde{\psi}_1)_2$, so that

$$q(x,k) = \exp(i\sigma_3 kx)o(1), \quad |k| \to \infty.$$

This means that q solves the homogeneous $\widetilde{G}$-R-H problem, whence $q \equiv 0$ by uniqueness. ☐

We notice that the function $\Psi(x,k)$ also satisfies equation (5.1).

Theorem 5.2. *The function $v(x)$ defined in Theorem 5.1 satisfies (1.3).*

Proof. To prove this, we introduce a matrix-valued function H_{as} that "almost" satisfies the Riemann–Hilbert problem as $x \to +\infty$. Consider the following function $R(k,x)$ for $k \in \mathbb{C}^+$:

$$R(x,k) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\widetilde{r}(\lambda)}e^{2i\lambda x}}{\lambda - k} d\lambda.$$

It is easily seen that

$$R^+(x,k) = \overline{\widetilde{r}(k)}e^{2ikx} + O\Big(\frac{x^{-M}}{1+|k|}\Big).$$

We define a matrix-valued function H_{as} as follows:

$$H_{as} = \begin{pmatrix} 1 & 0 \\ R(x,k) & 1 \end{pmatrix}, \quad \operatorname{Im} k > 0,$$

$$H_{as} = \begin{pmatrix} 1 & \overline{R(x,\overline{k})} \\ 0 & 1 \end{pmatrix}, \quad \operatorname{Im} k < 0.$$

It is straightforward to check that H_{as} satisfies the following Riemann–Hilbert problem:

1) $H_{as}(x,\cdot)$ is analytic in $\mathbb{C}^\pm$;
2) $H_{as}(x,k) = I + o(1)$, $|k| \to \infty$;

3) at the real axis we have $H_{as}^+(x,k) = G_{as}(x,k)H_{as}^-(x,k)$ and the matrix G_{as} is related to the junction matrix $\widetilde{\widetilde{G}}$ by

$$G_{as}(x,k) = \widetilde{\widetilde{G}}(x,k) + O(x^{-M}/(1+|k|)).$$

We introduce

$$h_{as} = H_{as}\mathbf{b}.$$

Then

$$\Phi^+ - h_{as}^+ = G_{as}(\Phi^- - h_{as}^-) + (\widetilde{\widetilde{G}} - G_{as})\Phi^-.$$

Therefore,

$$\begin{aligned}\Phi(x,k) - h_{as}(x,k) = {} & \frac{H_{as}(x,k)}{2\pi i} \\ & \times \int_{-\infty}^{+\infty} \frac{(H_{as}^+(x,\xi))^{-1}}{\xi - k}(\widetilde{\widetilde{G}}(x,\xi) - G_{as}(x,\xi))\Phi^-(x,\xi)d\xi.\end{aligned} \tag{5.4}$$

As $x \to +\infty$, this relation may be regarded as a singular integral equation with a small kernel. Analyzing this equation, we easily show that

$$\Phi(x,k) - h_{as}(x,k) = O(x^{-M}/(1+|k|)). \tag{5.5}$$

On the other hand, as $|k| \to \infty$ we have

$$\Phi = \exp\{-i\sigma_3 kx\}\widetilde{\Psi} = \mathbf{b} + \sum_{j=1}^{M} \widetilde{\psi}_j(x)k^{-j} + o(k^{-M}).$$

This and (5.5) imply that $\widetilde{\psi}_1(x)$ decays. To show the required decay of $v(x) = 2i((\widetilde{\psi}_1)_1)'_x$ and its derivatives, it suffices to differentiate (5.4) and apply the same arguments as above. □

Corollary 5.3. *There is a one-to-one correspondence between the Dirichlet–Schrödinger operators with real-valued potentials v in the Schwartz class and the data of the inverse problem $\{r; k_l, c_l, l = 1, \dots, N\}$, where*

$$k_l \neq k_j \text{ for } l \neq j; \quad k_l = i\kappa_l,\ \kappa_l > 0; \quad c_l = i\widetilde{c}_l,\ \widetilde{c}_l > 0, \quad l = 1, \dots, N,$$

the coefficient r enjoys properties 1–4 listed in Lemma 2.6, and either $\mathrm{ind}_{\mathbb{R}} r(k) = 2N$ or $\mathrm{ind}_{\mathbb{R}} r(k) = 2N+1$.

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