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A SPECTRUM ASSOCIATED WITH
MINKOWSKI DIAGONAL continued fraction

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Let α be real irrational number. The function $\mu_\alpha(t)$ is defined as follows. The Legendre theorem states that if

$$\left| \alpha - \frac{A}{Q} \right| < \frac{1}{2Q^2}, \quad (A, Q) = 1 \quad (1)$$

then the fraction $\frac{A}{Q}$ is a convergent fraction for the continued fraction expansion of α . The converse statement is not true. It may happen that $\frac{A}{Q}$ is a convergent to α but (1) is not valid. One should consider the sequence of the denominators of the convergents to α for which (1) is true. Let this sequence be

$$Q_0 < Q_1 < \dots < Q_n < Q_{n+1} < \dots .$$

Then for $\alpha \notin \mathbb{Q}$ the function $\mu_\alpha(t)$ is defined by

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot \|Q_n \alpha\| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot \|Q_{n+1} \alpha\|, \quad Q_n \leq t \leq Q_{n+1}.$$

From the other hand, for every ν one of the consecutive convergent fractions $\frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}$ to α satisfies (1). So either

$$(Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1})$$

for some ν , or

$$(Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1})$$

for some ν .

Actually the function $\mu_\alpha(t)$ was considered by Minkowski [4]. There exists an alternative geometric definition of $\mu_\alpha(t)$. Some related facts were discussed in [3, 6].

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The quantity

$$\mathbf{m}(\alpha) = \limsup_{t \rightarrow +\infty} t \cdot \mu_\alpha(t).$$

was considered in [6]. An explicit formula for the value of $\mathbf{m}(\alpha)$ in terms of continued fraction expansion for α was proved in [6]. It is as follows. Put

$$\mathbf{m}_n(\alpha) = \begin{cases} G(\alpha_\nu^*, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1}) \text{ with some } \nu, \\ F(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1}) \text{ with some } \nu, \end{cases} \quad (2)$$

where

$$G(x, y) = \frac{x + y + 1}{4}, \quad F(x, y) = \frac{(1 - xy)^2}{4(1 + xy)(1 - x)(1 - y)}$$

and α_ν, α_ν^* come from continued fraction expansion to

$$\alpha = [a_0; a_1, a_2, \dots, a_t, \dots]$$

in such a way:

$$\alpha_\nu = [a_\nu; a_{\nu+1}, \dots], \quad \alpha_\nu^* = [0; a_\nu, a_{\nu-1}, \dots, a_1].$$

Then

$$\mathbf{m}(\alpha) = \limsup_{n \rightarrow +\infty} \mathbf{m}_n(\alpha),$$

The spectrum

$$\mathbb{M} = \{m \in \mathbb{R} : \exists \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } m = \mathbf{m}(\alpha)\}.$$

was studied in [6]. It was proven there that $\mathbb{M} \subset [\frac{1}{4}, \frac{1}{2}]$ and that $\frac{1}{4}, \frac{1}{2} \in \mathbb{M}$. However no further structure of the spectrum \mathbb{M} is known.

In this paper we consider the spectrum

$$\mathbb{I} = \{m \in \mathbb{R} : \exists \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } \mathbf{i}(\alpha) = m\},$$

where

$$\mathbf{i}(\alpha) = \liminf_{n \rightarrow \infty} \mathbf{m}_n(\alpha)$$

(however, compared to $\mathbf{m}(\alpha)$, this quantity has no clear Diophantine sense).

It is clear that

$$\min \mathbb{I} = \frac{1}{4}, \quad \max \mathbb{I} = \frac{1}{2}.$$

Theorem. *There exists positive ω_0 such that*

$$\left[\frac{1}{4}, \omega_0 \right] \subset \mathbb{I}.$$

The proof is based on M. Hall's ideas (see [2]). It uses technique from [5].

Remark. An explicit formula for ω_0 may be obtained from the proof below. It is interesting to get optimal estimates for the value of ω_0 .

We need some well known results.

Recall the definition of a τ -set $\mathcal{F} \subset \mathbb{R}$. The set \mathcal{F} must be of the form

$$\mathcal{F} = \mathcal{S} \setminus \left(\bigcup_{\nu+1}^{\infty} \Delta_{\nu} \right),$$

where $\mathcal{S} \subset \mathbb{R}$ is a segment, and $\Delta_{\nu} \subset \mathcal{S}$, $\nu = 1, 2, 3, \dots$ is an ordered sequence of disjoint intervals. Moreover for every t if

$$\mathcal{S} \setminus \left(\bigcup_{\nu+1}^{t-1} \Delta_{\nu} \right) = \bigcup_{j=1}^r \mathcal{M}_j$$

is a union of segments \mathcal{M}_j and $\Delta_t \subset \mathcal{M}_{j^*}$ then

$$\mathcal{M}_{j^*} = \mathcal{N}^1 \sqcup \Delta_t \sqcup \mathcal{N}^2,$$

and

$$\min(|\mathcal{N}^1|, |\mathcal{N}^2|) \geq \tau |\Delta_t|.$$

Consider the set

$$\mathcal{F}_5 = \{ \alpha = [0; b_1, b_2, b_3, \dots] : b_{\nu} \leq 5 \quad \forall \nu \}$$

consisting of all irrational real numbers from the unit interval $(0, 1)$ with partial quotients bounded by 5. One can easily see that

$$\begin{cases} A = \min \mathcal{F}_5 = [0; \overline{5, 1}] = \frac{\sqrt{45}-5}{10} = 0.1708^+, \\ B = \max \mathcal{F}_5 = [0; \overline{1, 5}] = \frac{\sqrt{45}-5}{2} = 0.85410^+. \end{cases} \quad (3)$$

Put

$$\mathcal{S}_5 = [A, B] \subset [0, 1].$$

The following lemma comes from the results of the papers [1] or [7].

Lemma 1. The set \mathcal{F}_5 is a τ -set with $\tau = \tau_5 = 1.788^+$.

Let $H(x, y) : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ be a function in two variables of the class $G \in C^1(\mathcal{S} \times \mathcal{S})$. Consider the set

$$\mathcal{J} = \{ z \in \mathbb{R} : \exists x, y \in \mathcal{S} \quad z = H(x, y) \}.$$

By continuity argument \mathcal{J} is a segment.

Lemma 2. Suppose that the derivatives $\partial H/\partial x, \partial H/\partial y$ do not take zero values on the box $\mathcal{S} \times \mathcal{S}$. Suppose that \mathcal{F} is τ -set and $\mathcal{S} = [\min \mathcal{F}, \max \mathcal{F}]$. Suppose that

$$\tau \geq \max_{x, y \in \mathcal{S}} \max \left(\left| \frac{\partial H/\partial x}{\partial H/\partial y} \right|, \left| \frac{\partial H/\partial y}{\partial H/\partial x} \right| \right). \quad (4)$$

Then

$$\{z : \exists x, y \in \mathcal{F} \text{ such that } z = H(x, y)\} = \mathcal{J}.$$

Lemma 2 is a straightforward generalization of a result from [5]. We do not give its proof here as the proof follows the argument from [5] word-by-word.

Now we are able to conclude the proof of Theorem.

We consider pairs of integers (R_1, R_2) of the form

$$(R_1, R_2) = (R, R) \text{ or } (R, R + 1) \quad (5)$$

with $R \geq 6$. Consider a function

$$H_{R_1, R_2}(x, y) = F\left(\frac{1}{R_1 + x}, \frac{1}{R_2 + y}\right).$$

For R_1, R_2 under consideration the function $H_{R_1, R_2}(x, y)$ decreases both in x and in y .

For $0 < x, y < 1$ put

$$\varphi(x, y) = (1 - 3x + 3xy - x^2y)(1 - x).$$

For any $y \in (0, 1)$ the function $\varphi(x, y)$ decreases in x . For any $x \in (0, 1)$ the function $\varphi(x, y)$ increases in y . Now

$$\frac{\partial F / \partial y}{\partial F / \partial x} = \frac{\varphi(x, y)}{\varphi(y, x)},$$

and

$$\left| \frac{\partial H_{R_1, R_2} / \partial y}{\partial H_{R_1, R_2} / \partial x} \right| = \frac{\varphi\left(\frac{1}{R_1 + x}, \frac{1}{R_2 + y}\right)}{\varphi\left(\frac{1}{R_2 + y}, \frac{1}{R_1 + x}\right)} \left(\frac{R_1 + x}{R_2 + y} \right)^2.$$

Easy calculation shows that for $R_1, R_2 \geq 6$ one has

$$\begin{aligned} & \max_{x, y \in \mathcal{S}_5} \max \left(\left| \frac{\partial H_{R_1, R_2} / \partial x}{\partial H_{R_1, R_2} / \partial y} \right|, \left| \frac{\partial H_{R_1, R_2} / \partial y}{\partial H_{R_1, R_2} / \partial x} \right| \right) = \\ & = \frac{\varphi\left(\frac{1}{R_1 + B}, \frac{1}{R_2 + A}\right)}{\varphi\left(\frac{1}{R_2 + A}, \frac{1}{R_1 + B}\right)} \left(\frac{R_1 + B}{R_2 + A} \right)^2 \leq \frac{\varphi\left(\frac{1}{R_1 + B}, \frac{1}{R_1 + A}\right)}{\varphi\left(\frac{1}{R_1 + A}, \frac{1}{R_1 + B}\right)} \left(\frac{R_1 + B}{R_1 + A} \right)^2 \leq \\ & \leq \frac{\varphi\left(\frac{1}{6 + B}, \frac{1}{6 + A}\right)}{\varphi\left(\frac{1}{6 + A}, \frac{1}{6 + B}\right)} \left(\frac{6 + B}{6 + A} \right)^2 = 1.363^+ < \tau_5. \end{aligned}$$

Here A and B are defined in (3) and in the last inequalities we use the bounds $6 \leq R_1 \leq R_2$ which follows from (5).

We see that for any R_1, R_2 under consideration and for τ_5 -set \mathcal{F}_5 the condition (4) is satisfied. We apply Lemma 2 to see that the image of the set $\mathcal{F}_5 \times \mathcal{F}_5$ under the mapping $H_{R_1, R_2}(x, y)$ is just the segment

$$\mathcal{J}_{R_1, R_2} = [H_{R_1, R_2}(B, B), H_{R_1, R_2}(A, A)].$$

But

$$H_{R, R}(B, B) < H_{R, R+1}(A, A)$$

and

$$H_{R, R+1}(B, B) < H_{R+1, R+1}(A, A).$$

That is why if we put

$$\omega_0 = H_{R_0, R_0}(A, A).$$

with $R_0 \geq 6$ we get

$$\bigcup_{R \geq R_0} \mathcal{J}_{R, R} \cup \bigcup_{R \geq R_0} \mathcal{J}_{R, R+1} = (1/4, \omega_0].$$

Take $m \in (0, \omega_0]$. Then there exists R_1, R_2 such that

$$m \in \mathcal{J}_{R_1, R_2}$$

and there exist

$$\beta = [0; b_1, b_2, \dots, b_\nu, \dots], \quad \gamma = [0; c_1, c_2, \dots, c_\nu, \dots], \quad \beta, \gamma \in \mathcal{F}_5,$$

such that

$$F\left(\frac{1}{R_1 + \alpha}, \frac{1}{R_2 + \beta}\right) = m.$$

Now we take

$$\alpha = [0; \underbrace{a_1, R_1, R_2, b_1}_1, \underbrace{a_2, a_1, R_1, R_2, b_1, b_2}_2, \dots, \underbrace{a_\nu, a_{\nu-1}, \dots, a_2, a_1, R_1, R_2, b_1, b_2, \dots, b_{\nu-1}, b_\nu}_\nu, \dots].$$

Standard argument shows that for n_ν defined from

$$\frac{p_{n_\nu}}{q_{n_\nu}} = [0; a_1, R_1, R_2, b_1, a_2, a_1, R_1, R_2, b_1, b_2, \dots, a_\nu, a_{\nu-1}, \dots, a_2, a_1, R_1]$$

one has

$$\lim_{\nu \rightarrow +\infty} F(\alpha_{n_\nu}^*, \alpha_{n_\nu+1}^{-1}) = m.$$

At the same time for $F(\cdot, \cdot)$ and $G(\cdot, \cdot)$ we have

$$\inf_{n \neq n_\nu \forall \nu} F(\alpha_n^*, \alpha_{n+1}) > \omega_0$$

and

$$\inf_{n \in \mathbb{Z}_+} G(\alpha_n^*, \alpha_{n+2}^{-1}) > \omega_0,$$

for large R_0 . So $\mathfrak{i}(\alpha) = m$ and everything is proved. \square

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