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p-Adic and Adelic Quantum Mechanics[1]

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p-Adic mathematical physics emerged as a result of efforts to find a non-Archimedean approach to the space–time and string dynamics at the Planck scale. One of its main achievements is a successful formulation and development of p-adic and adelic quantum mechanics, which have complex-valued wave functions of p-adic and adelic arguments, respectively. Various aspects of these quantum mechanics are reviewed here. In particular, the corresponding Feynman's path integrals, some minisuperspace cosmological models, and relevant approaches to string theory are presented. As a result of an adelic approach, p-adic effects exhibit a space–time and some other discreteness, which depend on the adelic quantum state of the physical system under consideration. In addition to the review, this article also contains some new results.

1. INTRODUCTION

At the turn of the 20th century, two great parts of fundamental science were born: *Quantum Physics* and *p-Adic Mathematics*. Developing quite independently till the last two decades, they started to interact rather successfully, so that not only some quantum but also classical p-adic models have been constructed and investigated. As a result, in 1987, *p-Adic Mathematical Physics* emerged, which is a basis to explore various p-adic aspects of modern theoretical physics. Among the main achievements of this new branch of contemporary mathematical physics are *p-Adic and Adelic Quantum Mechanics*.

There are many physical and mathematical motivations for employing p-adic numbers and adeles in the investigation of mathematical and theoretical aspects of modern quantum physics. Some primary of them are the following:

(i) the field of rational numbers $\mathbb{Q}$, which contains all observational and experimental numerical data, is a dense subfield not only in the field of real numbers $\mathbb{R}$ but also in the fields of p-adic numbers $\mathbb{Q}_p$;

(ii) there is a sufficiently well developed analysis [1] within and over $\mathbb{Q}_p$, analogous to the one related to $\mathbb{R}$;

(iii) the local–global (Hasse–Minkowski) principle, which states that, usually, when something is valid on all local fields ($\mathbb{R}$, $\mathbb{Q}_p$), it is also valid on the global field ($\mathbb{Q}$);

(iv) fundamental physical laws and relevant general mathematical methods should be invariant [9] under the interchange of the number fields $\mathbb{R}$ and $\mathbb{Q}_p$;

(v) the question "Which aspects of the Universe cannot be described without using p-adic numbers?";

(vi) there is a generic quantum-gravity uncertainty Δx (see (1)) for possible measurements of distances approaching the Planck length ℓ_0, which restricts the priority of the Archimedean geometry based on real numbers and gives rise to the employment of the non-Archimedean one related to p-adic numbers;

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(vii) it seems to be quite reasonable to extend the standard Feynman's path integral over real space to p-adic spaces; and

(viii) the adelic quantum mechanics [10, 11], which is a quantum mechanics on an adelic space and contains standard, as well as all p-adic, quantum mechanics, is consistent with all the above assertions.

It is worth explaining in some detail the above motivation (vi). Namely, according to various considerations that take together standard quantum and gravitational principles, there is a strong generic restriction on the experimental investigation of the space–time structure at very short distances due to the relation

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\,\mathrm{cm}, \tag{1}$$

where Δx is the uncertainty in measuring a distance, ℓ_0 is the Planck length, $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant, G is Newton's gravitational constant, and c is the speed of light in vacuum. The uncertainty (1) means that one cannot measure distances smaller than ℓ_0 and ℓ_0 can be regarded as a minimal (fundamental) length. However, this result is derived under the assumption that the whole space–time can be described only by real numbers and Archimedean geometry. But we cannot *a priori* exclude p-adic numbers and their non-Archimedean geometric properties. To get a more complete insight into the structure of space–time at the Planck scale, it is quite natural to use the adelic approach, which treats simultaneously and on an equal footing real (Archimedean) and p-adic (non-Archimedean) aspects.

According to (1), an approach based only on Archimedean geometry and real numbers predicts its own breakdown at the Planck scale and gives rise to a p-adic non-Archimedean sector of possible geometries. Namely, recall that, having two segments of different lengths a and b on a straight line, where $a < b$, one can overpass the longer b by applying the smaller a some n times along b. In other words, if a and b are two positive real numbers and $a < b$, then there exists a sufficiently large natural number n such that $na > b$. This is an evident property of Euclidean spaces (and the field of real numbers), which is known as the Archimedean postulate and can be extended to standard Riemannian spaces. One of the axioms of metric spaces is the triangle inequality, which reads

$$d(x,y) \leq d(x,z) + d(z,y), \tag{2}$$

where $d(x,y)$ is a distance between points x and y. However, there is a subclass of metric spaces for which the triangle inequality is stronger in such a way that

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} \leq d(x,z) + d(z,y). \tag{3}$$

The metric spaces with the strong triangle inequality (3) are called non-Archimedean or ultrametric spaces. Since a measurement means a quantitative comparison of a given observable with respect to a fixed value taken as its unit, it follows that a realization of the Archimedean postulate is practically equivalent to the measurement of distances. According to the uncertainty (1), it is not possible to handle distances shorter than 10^{-33} cm, and consequently, one cannot apply only Archimedean geometry beyond the Planck length. Hence, for the mathematical modeling of space–time, as well as matter (strings and branes), it is necessary to employ adeles when approaching the Planck scale.

Before proceeding to the systematic investigation of a possible adelic quantum theory at the Planck scale, it is useful to explore various aspects of p-adic and adelic quantum mechanics. These quantum mechanics are well formulated and so far elaborated at the level that promises their successful extension toward *Adelic Superstring/M-theory.* This article contains a brief systematic presentation of quantum mechanics on real, p-adic, and adelic spaces.

For the necessary information on the usual properties of p-adic numbers and related analysis, one can see [1–6].

2. QUANTUM MECHANICS ON A REAL SPACE

This is the ordinary (or standard) quantum mechanics (OQM). It has four main sectors: Hilbert space, Quantization, Evolution, and Interpretation. Some of them can be formulated in a few different ways, which are equivalent. The Hilbert space of the OQM consists of square integrable complex-valued functions of real arguments, which are mainly the coordinates of D-dimensional space and time, and is usually denoted by $L_2(\mathbb{R}^D)$.

To physical observables, there correspond linear self-adjoint operators in $L_2(\mathbb{R}^D)$. Classical dynamical expressions, which depend on the canonical variables x_i and k_j of the phase space, become operators by the quantization procedure usually initiated by the Heisenberg algebra with the commutation relations

$$[\widehat{x}_i, \widehat{k}_j] = i\hbar\delta_{ij}, \qquad [\widehat{x}_i, \widehat{x}_j] = 0, \qquad [\widehat{k}_i, \widehat{k}_j] = 0, \tag{4}$$

where $i, j = 1, 2, \ldots, D$. Note that, instead of (4), one can use an equivalent quantization based on the group relations ($h = 1$)

$$\chi_\infty(-\alpha_i\widehat{x}_i)\,\chi_\infty(-\beta_j\widehat{k}_j) = \chi_\infty(\alpha_i\beta_j\delta_{ij})\,\chi_\infty(-\beta_j\widehat{k}_j)\,\chi_\infty(-\alpha_i\widehat{x}_i), \tag{5}$$

$$\chi_\infty(-\alpha_i\widehat{x}_i)\,\chi_\infty(-\alpha_j\widehat{x}_j) = \chi_\infty(-\alpha_j\widehat{x}_j)\,\chi_\infty(-\alpha_i\widehat{x}_i), \tag{6}$$

$$\chi_\infty(-\beta_i\widehat{k}_i)\,\chi_\infty(-\beta_j\widehat{k}_j) = \chi_\infty(-\beta_j\widehat{k}_j)\,\chi_\infty(-\beta_i\widehat{k}_i), \tag{7}$$

where $\chi_\infty(u) = \exp(-2\pi iu)$ is a real additive character and (α_i, β_j) is a point of the classical phase space. The quantization of expressions that contain products of x_i and k_j is not unique. According to the Weyl quantization [12], any function $f(k, x)$, of classical canonical variables k and x, which has the Fourier transform $\widetilde{f}(\alpha, \beta)$, becomes a self-adjoint operator in $L_2(\mathbb{R}^D)$ in the following way:

$$\widehat{f}(\widehat{k}, \widehat{x}) = \int \chi_\infty(-\alpha\widehat{x} - \beta\widehat{k})\widetilde{f}(\alpha, \beta)\, d^D\alpha\, d^D\beta. \tag{8}$$

The evolution of the elements $\Psi(x, t)$ of $L_2(\mathbb{R}^D)$ is usually given by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,\Psi(x, t) = H(\widehat{k}, x)\,\Psi(x, t), \tag{9}$$

where H is a Hamiltonian and $\widehat{k}_j = -i\hbar\frac{\partial}{\partial x_j}$. Besides this differential equation, there is also the following integral form:

$$\psi(x'', t'') = \int \mathcal{K}_\infty(x'', t''; x', t')\,\psi(x', t')\,d^D x', \tag{10}$$

where $\mathcal{K}_\infty(x'', t''; x', t')$ is the kernel of the unitary representation of the evolution operator $U_\infty(t'', t')$, which is postulated by Feynman to be a path integral [13]

$$\mathcal{K}_\infty(x'', t''; x', t') = \int_{(x',t')}^{(x'',t'')} \chi_\infty(-S[q])\,\mathcal{D}q, \tag{11}$$

where the functional $S[q] = \int_{t'}^{t''} L(\dot{q}, q, t)\,dt$ is the action for a path $q(t)$ in the classical Lagrangian $L(\dot{q}, q, t)$ and $x'' = q(t'')$ and $x' = q(t')$ are the end points with the notation $x = (x_1, x_2, \ldots, x_D)$ and $q = (q_1, q_2, \ldots, q_D)$. The kernel $\mathcal{K}_\infty(x'', t''; x', t')$ is also known as the probability amplitude for a quantum particle to pass from position x' at time t' to another point x'' at t'' and is closely related to the quantum-mechanical propagator and Green's function. The integral in (11) has an intuitive meaning that a quantum-mechanical particle may pass from x' to x'' along infinitely many paths that connect these two points and that one has to sum the probability amplitudes over all of them. Thus, the Feynman path integral means a continual (functional) summation of single transition amplitudes $\exp\bigl(\frac{i}{\hbar}S[q]\bigr)$ over all possible continual paths $q(t)$ connecting $x' = q(t')$ and $x'' = q(t'')$. In Feynman's formulation, the path integral (11) is the limit of an ordinary multiple integral over N variables $q_i = q(t_i)$ when $N \to \infty$. Namely, the time interval $t'' - t'$ is divided into $N+1$ equal subintervals and integration is performed for every $q_i \in (-\infty, +\infty)$ at a fixed time t_i. We will consider Feynman's path integral also in the next sections and, especially, in Section 4.

The interpretation of the OQM is related to the scalar products of complex-valued functions in $L_2(\mathbb{R}^D)$ and will not be discussed here; however, it can be found in standard books on quantum mechanics, including [12].

3. QUANTUM MECHANICS ON p-ADIC AND ADELIC SPACES

It is remarkable that quantum mechanics on a real space can be generalized to p-adic spaces for any prime number p. However, there is no unique way to perform this generalization. As a result, there are two main approaches, those with complex-valued and p-adic-valued elements of the Hilbert space on $\mathbb{Q}_p^D$. The approach with p-adic-valued wave functions has been introduced by Vladimirov and Volovich [14, 1] and developed by Khrennikov [15, 16]. The p-adic quantum mechanics with complex-valued wave functions is more suitable for combining with the ordinary quantum mechanics, and in the sequel, we will refer only to this kind of quantum mechanics on p-adic spaces.

Since wave functions are complex-valued, one cannot construct a direct analogue of the Schrödinger equation (9) with a p-adic version of the Heisenberg algebra (4). According to the Weyl approach to quantization, the canonical noncommutativity in the p-adic case should be introduced by the operators ($h = 1$)

$$\widehat{Q}_p(\alpha)\psi_p(x) = \chi_p(-\alpha x)\psi_p(x), \qquad \widehat{K}_p(\beta)\psi_p(x) = \psi_p(x+\beta), \tag{12}$$

which satisfy

$$\widehat{Q}_p(\alpha)\widehat{K}_p(\beta) = \chi_p(\alpha\beta)\widehat{K}_p(\beta)\widehat{Q}_p(\alpha), \tag{13}$$

where $\chi_p(u) = \exp(2\pi i\{u\}_p)$ is an additive character on the field $\mathbb{Q}_p$ and $\{u\}_p$ is the fractional part of $u \in \mathbb{Q}_p$.

Let $\widehat{x}$ and $\widehat{k}$ be some operators of position x and momentum k, respectively. Let us define operators $\chi_v(\alpha\widehat{x})$ and $\chi_v(\beta\widehat{k})$ by the formulas

$$\chi_v(\alpha\widehat{x})\,\chi_v(ax) = \chi_v(\alpha x)\,\chi_v(ax), \qquad \chi_v(\beta\widehat{k})\,\chi_v(bk) = \chi_v(\beta k)\,\chi_v(bk), \tag{14}$$

where the index v denotes real ($v = \infty$) and any p-adic case, i.e., $v = \infty, 2, \dots, p, \dots$, taking into account all nontrivial and inequivalent valuations on $\mathbb{Q}$. It is evident that these operators also act on a function $\psi_v(x)$, which has the Fourier transform $\widetilde{\psi}(k)$, in the following way:

$$\chi_v(-\alpha\widehat{x})\,\psi_v(x) = \chi_v(-\alpha\widehat{x}) \int \chi_v(-kx)\,\widetilde{\psi}(k)\,d^Dk = \chi_v(-\alpha x)\,\psi_v(x), \tag{15}$$

$$\chi_v(-\beta\widehat{k})\,\psi_v(x) = \int \chi_v(-\beta k)\,\chi_v(-kx)\,\widetilde{\psi}(k)\,d^Dk = \psi_v(x+\beta), \tag{16}$$

where the integration in the p-adic case is with respect to the Haar measure dk with the properties $d(k+a) = dk$, $d(ak) = |a|_p\,dk$, and $\int_{|k|_p\le 1} dk = 1$. Comparing (12) with (15) and (16), we conclude that $\widehat{Q}_p(\alpha) = \chi_p(-\alpha\widehat{x})$ and $\widehat{K}_p(\beta) = \chi_p(-\beta\widehat{k})$. Now, the group relations (5)–(7) can be straightforwardly generalized, including p-adic cases, by formally replacing the index ∞ by v. Thus, we have

$$\chi_v(-\alpha_i\widehat{x}_i)\,\chi_v(-\beta_j\widehat{k}_j) = \chi_v(\alpha_i\beta_j\delta_{ij})\,\chi_v(-\beta_j\widehat{k}_j)\,\chi_v(-\alpha_i\widehat{x}_i), \tag{17}$$

$$\chi_v(-\alpha_i\widehat{x}_i)\,\chi_v(-\alpha_j\widehat{x}_j) = \chi_v(-\alpha_j\widehat{x}_j)\,\chi_v(-\alpha_i\widehat{x}_i), \tag{18}$$

$$\chi_v(-\beta_i\widehat{k}_i)\,\chi_v(-\beta_j\widehat{k}_j) = \chi_v(-\beta_j\widehat{k}_j)\,\chi_v(-\beta_i\widehat{k}_i). \tag{19}$$

One can introduce the unitary operator

$$W_v(\alpha\widehat{x},\beta\widehat{k}) = \chi_v\left(\frac{1}{2}\alpha\beta\right)\chi_v(-\beta\widehat{k})\,\chi_v(-\alpha\widehat{x}), \tag{20}$$

which satisfies the Weyl relation

$$W_v(\alpha\widehat{x},\beta\widehat{k})\,W_v(\alpha'\widehat{x},\beta'\widehat{k}) = \chi_v\left(\frac{1}{2}(\alpha\beta'-\alpha'\beta)\right)W_v\big((\alpha+\alpha')\widehat{x},(\beta+\beta')\widehat{k}\big) \tag{21}$$

and is a unitary representation of the Heisenberg–Weyl group. Recall that this group consists of elements (z,η) with the group product

$$(z,\eta)\cdot(z',\eta') = \left(z+z',\eta+\eta'+\frac{1}{2}B(z,z')\right), \tag{22}$$

where $z=(\alpha,\beta)\in\mathbb{Q}_v\times\mathbb{Q}_v$ and $B(z,z') = \alpha\beta'-\beta\alpha'$ is a skew-symmetric bilinear form on the phase space. Using the operator $W_v(\alpha\widehat{x},\beta\widehat{k})$, one can generalize the Weyl formula for quantization (8) so that it reads

$$\widehat{f}_v(\widehat{k},\widehat{x}) = \int W_v(\alpha\widehat{x},\beta\widehat{k})\,\widetilde{f}_v(\alpha,\beta)\,d^D\alpha\,d^D\beta. \tag{23}$$

It is worth noting that equation (16) suggests introducing

$$\{\beta\widehat{k}\}_p^n\psi_p(x) = \int\{\beta k\}_p^n\chi_p(-kx)\widetilde{\psi}_p(k)\,d^Dk, \tag{24}$$

which may be regarded as a new kind of the p-adic pseudodifferential operator (for a successful Vladimirov pseudodifferential operator, see [1]). Also, equation (17) suggests a p-adic analogue of the Heisenberg algebra in the form $(h=1)$

$$\{\alpha_i\widehat{x}_i\}_p\,\{\beta_j\widehat{k}_j\}_p - \{\beta_j\widehat{k}_j\}_p\,\{\alpha_i\widehat{x}_i\}_p = -\frac{i}{2\pi}\,\delta_{ij}\{\alpha\beta\}_p. \tag{25}$$

As a basic instrument to treat the dynamics of a p-adic quantum model, it is natural to take the kernel $\mathcal{K}_p(x'',t'';x',t')$ of the evolution operator $U_p(t'',t')$. This kernel is obtained by the generalization of its real analogue starting from (10) and (11), i.e.,

$$\psi_v(x'',t'') = \int\mathcal{K}_v(x'',t'';x',t')\,\psi_v(x',t')\,d^Dx', \tag{26}$$

and

$$\mathcal{K}_v(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')}\chi_v\left(-\int_{t'}^{t''}L(\dot{q},q,t)\,dt\right)\mathcal{D}q. \tag{27}$$

According to Vladimirov and Volovich [14, 17, 1], p-adic quantum mechanics is given by a triple

$$\big(L_2(\mathbb{Q}_p), W_p(z), U_p(t)\big), \tag{28}$$

where $W_p(z)$ corresponds to our $W_p(\alpha\widehat{x}, \beta\widehat{k})$. A similar formulation is done in [18], where the evolution operator for one-dimensional systems is presented by a unitary representation of an Abelian subgroup of $SL(2,\mathbb{Q}_p)$ instead of the path integral for the kernel $\mathcal{K}_p(x'',t'';x',t')$ (see also [19]).

The adelic quantum mechanics [10, 11] is a natural generalization of the above formulation of the ordinary and p-adic quantum mechanics. Recall that an adele x [6–8] is an infinite sequence

$$x = (x_\infty, x_2, \dots, x_p, \dots), \tag{29}$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that, for all but a finite set $\mathbf{S}$ of primes p, one has $x_p \in \mathbb{Z}_p = \{x_p \in \mathbb{Q}_p\colon |x_p|_p \le 1\}$. Componentwise addition and multiplication are standard arithmetical operations on the ring of adeles $\mathcal{A}$, which can be defined as

$$\mathcal{A} = \bigcup_{\mathbf{S}} \mathcal{A}(\mathbf{S}), \qquad \mathcal{A}(\mathbf{S}) = \mathbb{R} \times \prod_{p\in\mathbf{S}} \mathbb{Q}_p \times \prod_{p\notin\mathbf{S}} \mathbb{Z}_p. \tag{30}$$

Rational numbers are naturally embedded in the space of adeles. $\mathcal{A}$ is a locally compact topological space.

There are two kinds of analysis over the topological ring of adeles $\mathcal{A}$, which are generalizations of the corresponding analyses over $\mathbb{R}$ and $\mathbb{Q}_p$. The first one is related to the mapping $\mathcal{A} \to \mathcal{A}$, and the other one, to $\mathcal{A} \to \mathbb{C}$. In complex-valued adelic analysis, it is worth mentioning an additive character

$$\chi(x) = \chi_\infty(x_\infty) \prod_p \chi_p(x_p); \tag{31}$$

a multiplicative character

$$|x|^s = |x_\infty|_\infty^s \prod_p |x_p|_p^s, \qquad s \in \mathbb{C}; \tag{32}$$

and elementary functions of the form

$$\varphi_{\mathbf{S}}(x) = \varphi_\infty(x_\infty) \prod_{p\in\mathbf{S}} \varphi_p(x_p) \prod_{p\notin\mathbf{S}} \Omega(|x_p|_p), \tag{33}$$

where $\varphi_\infty(x_\infty)$ is an infinitely differentiable function on $\mathbb{R}$ such that $|x_\infty|_\infty^n \varphi_\infty(x_\infty) \to 0$ as $|x_\infty|_\infty \to \infty$ for any $n \in \{0,1,2,\dots\}$, $\varphi_p(x_p)$ are some locally constant functions with compact support, and

$$\Omega(|x_p|_p) = \begin{cases} 1, & |x_p|_p \le 1, \\ 0, & |x_p|_p > 1. \end{cases} \tag{34}$$

All finite linear combinations of elementary functions (33) make up a set $S(\mathcal{A})$ of the Schwartz–Bruhat adelic functions. The Fourier transform of $\varphi(x) \in S(\mathcal{A})$, which maps $S(\mathcal{A})$ onto $S(\mathcal{A})$, is

$$\widetilde{\varphi}(y) = \int_{\mathcal{A}} \varphi(x)\chi(xy)\,dx, \tag{35}$$

where $\chi(xy)$ is defined by (31) and $dx = dx_\infty\,dx_2\,dx_3\dots$ is the Haar measure on $\mathcal{A}$.

The Hilbert space $L_2(\mathcal{A})$ contains complex-valued functions of adelic argument with the following scalar product and norm:

$$(\psi_1,\psi_2)=\int_{\mathcal{A}}\overline{\psi}_1(x)\,\psi_2(x)\,dx,\qquad \|\psi\|=(\psi,\psi)^{1/2}<\infty.$$

A basis of $L_2(\mathcal{A})$ may be given by the set of orthonormal eigenfunctions in a spectral problem of the evolution operator $U(t)$, where $t\in\mathcal{A}$. Such eigenfunctions have the form

$$\psi_{\mathbf{S},\alpha}(x,t)=\psi_n^{(\infty)}(x_\infty,t_\infty)\prod_{p\in\mathbf{S}}\psi_{\alpha_p}^{(p)}(x_p,t_p)\prod_{p\notin\mathbf{S}}\Omega(|x_p|_p),\tag{36}$$

where $\psi_n^{(\infty)}\in L_2(\mathbb{R})$ and $\psi_{\alpha_p}^{(p)}\in L_2(\mathbb{Q}_p)$ are eigenfunctions in the ordinary and p-adic cases, respectively. The indices $n,\alpha_2,\dots,\alpha_p,\dots$ are related to the corresponding real and p-adic eigenvalues of the same observable in a physical system. $\Omega(|x_p|_p)$ is an element of $L_2(\mathbb{Q}_p)$, defined by (34), which is invariant under a transformation of an evolution operator $U_p(t_p)$ and provides the convergence of the infinite product (36). For a fixed $\mathbf{S}$, the states $\psi_{\mathbf{S},\alpha}(x,t)$ in (36) are eigenfunctions of $L_2(\mathcal{A}(\mathbf{S}))$, where $\mathcal{A}(\mathbf{S})$ is a subset of adeles $\mathcal{A}$ defined by (30). The elements of $L_2(\mathcal{A})$ may be regarded as the superpositions $\psi(x)=\sum_{\mathbf{S},\alpha}C(\mathbf{S},\alpha)\,\psi_{\mathbf{S},\alpha}(x)$, where $\psi_{\mathbf{S},\alpha}(x)\in L_2(\mathcal{A}(\mathbf{S}))$ (36) and $\sum_{\mathbf{S},\alpha}|C(\mathbf{S},\alpha)|_\infty^2=1$.

The theory of p-adic generalized functions is presented in [1]. There is not yet a theory of generalized functions on adelic spaces, but there is some progress within adelic quantum mechanics [20] (see also [21]).

The adelic evolution operator $U(t)$ is defined by

$$U(t'')\,\psi(x'')=\int_{\mathcal{A}}\mathcal{K}(x'',t'';x',t')\,\psi(x',t')\,dx'=\prod_v\int_{\mathbb{Q}_v}\mathcal{K}_v(x_v'',t_v'';x_v',t_v')\,\psi_v(x_v',t_v')\,dx_v',\tag{37}$$

where $v=\infty,2,3,\dots,p,\dots$. The eigenvalue problem for $U(t)$ reads

$$U(t)\,\psi_{\mathbf{S},\alpha}(x)=\chi(E_\alpha t)\,\psi_{\mathbf{S},\alpha}(x),\tag{38}$$

where $\psi_{\mathbf{S},\alpha}(x)$ are adelic eigenfunctions (36) and $E_\alpha=(E_\infty,E_2,\dots,E_p,\dots)$ is the corresponding adelic energy.

The adelic quantum mechanics takes into account ordinary as well as p-adic quantum effects and may be regarded as a starting point for constructing a more complete quantum cosmology, quantum field theory, and string/M-theory. In the limit of large distances, the adelic quantum mechanics effectively becomes the ordinary one [22].

4. p-ADIC AND ADELIC PATH INTEGRALS AND SOME SIMPLE QUANTUM MODELS

A p-adic path integral was introduced in [17] and was computed for the harmonic oscillator [23] and for a particle in a constant field [24] by subdividing the time interval. An analytic evaluation of path integrals for quantum-mechanical systems with general-form quadratic Lagrangians for real and p-adic cases is performed in the same way in [25, 26].

Starting from (26), one can easily derive the following three general properties:

$$\int\mathcal{K}_v(x'',t'';x,t)\mathcal{K}_v(x,t;x',t')\,dx=\mathcal{K}_v(x'',t'';x',t'),\tag{39}$$

$$\int\overline{\mathcal{K}}_v(x'',t'';x',t')\mathcal{K}_v(y,t'';x',t')\,dx'=\delta_v(x''-y),\tag{40}$$

$$\mathcal{K}_v(x'',t'';x',t'')=\lim_{t'\to t''}\mathcal{K}_v(x'',t'';x',t')=\delta_v(x''-x').\tag{41}$$

Quantum fluctuations lead to deformations of a classical trajectory, and any quantum history may be presented as $q(t) = x(t) + y(t)$, where $y' = y(t') = 0$ and $y'' = y(t'') = 0$. For Lagrangians $L(\dot q, q, t)$ that are quadratic polynomials in $\dot q$ and q, the corresponding Taylor expansion of $S[q]$ around a classical path $x(t)$ is

$$S[q] = S[x] + \frac{1}{2!}\delta^2 S[x] = S[x] + \frac{1}{2}\int_{t'}^{t''} \left(\dot y \frac{\partial}{\partial \dot q} + y\frac{\partial}{\partial q}\right)^2 L(\dot q, q, t)\, dt, \tag{42}$$

where we used $\delta S[x] = 0$. Hence, we get

$$\mathcal{K}_v(x'', t''; x', t') = \chi_v\left(-\frac{1}{h}\overline{S}(x'', t''; x', t')\right) \int_{(y'\to 0, t')}^{(y''\to 0, t'')} \chi_v\left(-\frac{1}{2h}\int_{t'}^{t''} \left(\dot y \frac{\partial}{\partial \dot q} + y\frac{\partial}{\partial q}\right)^2 L(\dot q, q, t)\, dt\right) \mathcal{D}y, \tag{43}$$

where $\overline{S}(x'', t''; x', t') = S[x]$.

From (43) it follows that $\mathcal{K}_v(x'', t''; x', t')$ has the form

$$\mathcal{K}_v(x'', t''; x', t') = N_v(t'', t')\chi_v\left(-\frac{1}{h}\overline{S}(x'', t''; x', t')\right), \tag{44}$$

where $N_v(t'', t')$ does not depend on the end points x'' and x'.

To calculate $N_v(t'', t')$, one can use (39) and (40) (see, e.g., [26]). Then, one obtains that the v-adic kernel $\mathcal{K}_v(x'', t''; x', t')$ of the unitary evolution operator for one-dimensional systems with quadratic Lagrangians has the form

$$\begin{aligned}\mathcal{K}_v(x'', t''; x', t') = \lambda_v\left(-\frac{1}{2h}\frac{\partial^2}{\partial x''\partial x'}\overline{S}(x'', t''; x', t')\right) \left|\frac{1}{h}\frac{\partial^2}{\partial x''\partial x'}\overline{S}(x'', t''; x', t')\right|_v^{1/2} \\ \times \chi_v\left(-\frac{1}{h}\overline{S}(x'', t''; x', t')\right),\end{aligned} \tag{45}$$

where the λ_v-functions are defined in [1].

For practical considerations, we define an adelic path integral in the form

$$\mathcal{K}_{\mathcal{A}}(x'', t''; x', t') = \prod_v \int_{(x_v', t_v')}^{(x_v'', t_v'')} \chi_v\left(-\frac{1}{h}\int_{t_v'}^{t_v''} L(\dot q_v, q_v, t_v)\, dt_v\right) \mathcal{D}q_v. \tag{46}$$

By an adelic Lagrangian, one means an infinite sequence

$$L_{\mathcal{A}}(\dot q, q, t) = \big(L(\dot q_\infty, q_\infty, t_\infty), L(\dot q_2, q_2, t_2), L(\dot q_3, q_3, t_3), \dots, L(\dot q_p, q_p, t_p), \dots\big), \tag{47}$$

where $|L(\dot q_p, q_p, t_p)|_p \le 1$ for all primes p but a finite set $\mathbf{S}$ of them.

When one has a system with more than one dimension and the coordinates are uncoupled, the total v-adic path integral is a product of the corresponding one-dimensional path integrals. The investigation of the coupled case is in progress.

As an illustration of p-adic and adelic quantum-mechanical models, the following one-dimensional systems with quadratic Lagrangians were considered:

(1) $L(\dot{q}, q) = \frac{m}{2}\,\dot{q}^2$, a free particle [1, 10, 11];

(2) $L(\dot{q}, q) = \frac{m}{2}\,\dot{q}^2 + aq$, a particle in a constant field [24];

(3) $L(\dot{q}, q) = \frac{m}{2}\,\dot{q}^2 - \frac{m\omega^2}{2}\,q^2$, a harmonic oscillator [1, 10, 11];

(4) $L(\dot{q}, q) = -mc^2\sqrt{\dot{q}_\mu \dot{q}^\mu}$, a free relativistic particle [22]; and

(5) $L(\dot{q}, q) = \frac{m}{2}\,\dot{q}^2 - \frac{m\omega^2(t)}{2}\,q^2$, a harmonic oscillator with time-dependent frequency [27].

Note that, when time is real and trajectories are p-adic, and vice versa, possible functional integrals were considered by Parisi [28]. There is another proposal for a path integral formulation of some evolution operators with p-adic time [29]. For an approach to adelic path space integrals with real time, see [30].

5. p-ADIC AND ADELIC QUANTUM COSMOLOGY

The main task of quantum cosmology is to describe the very early stage in the evolution of the Universe. At this stage, the Universe was in a quantum state, which should be described by a wave function. Usually, one assumes that this wave function is complex-valued and depends on some real parameters. Since quantum cosmology is related to the Planck-scale phenomena, it is natural to reconsider its foundations. Here, we adhere to the standard point of view that the wave function takes complex values, but we treat its arguments (space–time coordinates, gravitational and matter fields) to be not only real but also p-adic and adelic.

There is no p-adic generalization of the Wheeler–De Witt equation for cosmological models. Instead of a differential approach, Feynman's path integral method was exploited [33], and minisuperspace cosmological models are investigated as models of adelic quantum mechanics [34, 35].

p-Adic and adelic minisuperspace quantum cosmology is an application of p-adic and adelic quantum mechanics to cosmological models, respectively. In the path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method; i.e., the amplitude of transition from one state with an intrinsic metric h'_{ij} and a matter configuration ϕ' on an initial hypersurface Σ' to another state with a metric h''_{ij} and a matter configuration ϕ'' on a final hypersurface Σ'' is given by the path integral

$$\mathcal{K}_\infty(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi])\, \mathcal{D}(g_{\mu\nu})_\infty\, \mathcal{D}(\Phi)_\infty \qquad (48)$$

over all four-geometries $g_{\mu\nu}$ and matter configurations Φ that interpolate between the initial and final configurations. In (48), $S_\infty[g_{\mu\nu}, \Phi]$ is an Einstein–Hilbert action for the gravitational and matter fields. This action can be calculated using metric in the standard $3 + 1$ decomposition

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -(N^2 - N_i N^i)\, dt^2 + 2N_i\, dx^i\, dt + h_{ij}\, dx^i\, dx^j, \qquad (49)$$

where N and N_i are the lapse and shift functions, respectively. To perform p-adic and adelic generalization, we first construct a p-adic counterpart of the action using the form-invariance under the change from the real to p-adic number fields. Then, we generalize (48) and introduce a p-adic complex-valued cosmological amplitude

$$\mathcal{K}_p(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \chi_p(-S_p[g_{\mu\nu}, \Phi])\, \mathcal{D}(g_{\mu\nu})_p\, \mathcal{D}(\Phi)_p. \qquad (50)$$

Since the space of all three-metrics and matter field configurations on a three-surface, called superspace, has infinitely many dimensions, one uses approximation in computation. A useful approximation is to truncate the infinite degrees of freedom to a finite number $q_\alpha(t)$, $\alpha = 1, 2, \dots, n$. In this way, one obtains a minisuperspace model. Usually, one restricts the four-metric to be of the form (49), with $N^i = 0$ and h_{ij} approximated by $q_\alpha(t)$. For the homogeneous and isotropic cosmologies, the metric is a Robertson–Walker one, whose spatial sector reads

$$h_{ij}dx^i dx^j = a^2(t)\,d\Omega_3^2 = a^2(t)\big[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\,d\varphi^2)\big], \tag{51}$$

where $a(t)$ is a scale factor. If we also use a single scalar field ϕ as a matter content of the model, the minisuperspace coordinates are a and ϕ. The models may also be homogeneous and anisotropic.

For the boundary condition $q_\alpha(t'') = q''_\alpha$, $q_\alpha(t') = q'_\alpha$ in the gauge $N = 1$, we have a v-adic minisuperspace propagator

$$\mathcal{K}_v(q''_\alpha|q'_\alpha) = \int dt\,\mathcal{K}_v(q''_\alpha, t''; q'_\alpha, t'), \qquad t = t'' - t', \tag{52}$$

where

$$\mathcal{K}_v(q''_\alpha, t''; q'_\alpha, t') = \int \chi_v(-S_v[q_\alpha])\,\mathcal{D}q_\alpha \tag{53}$$

is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates (q'_α, q''_α) at fixed times. S_v is the v-adic-valued action of the minisuperspace model, which has the form

$$S_v[q_\alpha] = \int_{t'}^{t''} dt\left[\frac{1}{2}f_{\alpha\beta}(q)\dot q^\alpha \dot q^\beta - U(q)\right], \tag{54}$$

where $f_{\alpha\beta}$ is a minisuperspace metric ($ds_m^2 = f_{\alpha\beta}\,dq^\alpha\,dq^\beta$) with a signature $(-,+,+,\dots)$. This metric includes gravitational and matter degrees of freedom.

The standard minisuperspace ground-state wave function in the Hartle–Hawking (no-boundary) proposal [31] is defined by the functional integration in the Euclidean version of

$$\psi_\infty[h_{ij}] = \int \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi])\,\mathcal{D}(g_{\mu\nu})_\infty\,\mathcal{D}(\Phi)_\infty \tag{55}$$

over all compact four-geometries $g_{\mu\nu}$ that induce h_{ij} at the compact three-manifold. This three-manifold is the only boundary of all the four-manifolds. Extending the Hartle–Hawking proposal to the p-adic minisuperspace [32], we obtain that the adelic Hartle–Hawking wave function is the infinite product

$$\psi[h_{ij}] = \prod_v \int \chi_v(-S_v[g_{\mu\nu}, \Phi])\,\mathcal{D}(g_{\mu\nu})_v\,\mathcal{D}(\Phi)_v, \tag{56}$$

where the path integration must be performed over both Archimedean and non-Archimedean geometries. When the evaluation of the corresponding functional integrals for a minisuperspace model yields $\psi(q_\alpha)$ in the form (36), we say that such a cosmological model is a Hartle–Hawking adelic one. It is shown [33] that the de Sitter minisuperspace model in $D = 4$ space–time dimensions is the Hartle–Hawking adelic one.

It is shown in [34, 35] that p-adic and adelic generalization of the minisuperspace cosmological models can be successfully performed in the framework of p-adic and adelic quantum mechanics [10, 11] without using the Hartle–Hawking approach. The following cosmological models are investigated: the de Sitter model, a model with a homogeneous scalar field, an anisotropic Bianchi model with three scale factors, and some two-dimensional minisuperspace models. As a result of p-adic effects and the adelic approach, in these models, there is some discreteness of minisuperspace and cosmological constant. This kind of discreteness was obtained for the first time in the context of the Hartle–Hawking adelic de Sitter quantum model [33].

6. TOWARD ADELIC STRING THEORY

The notion of p-adic string was introduced in [36], where a hypothesis on the existence of non-Archimedean geometry at the Planck scale was put forward and a string theory with p-adic numbers was initiated. In particular, a generalization of the usual Veneziano and Virasoro–Shapiro amplitudes with complex-valued multiplicative characters over various number fields was proposed, and a p-adic-valued Veneziano amplitude was constructed by means of p-adic interpolation. Very successful p-adic analogues of the Veneziano and Virasoro–Shapiro amplitudes were proposed in [37] as the corresponding Gel'fand–Graev [6] beta functions. Using this approach, Freund and Witten obtained [38] an attractive adelic formula

$$A_\infty(a,b)\prod_p A_p(a,b) = 1, \tag{57}$$

which states that the product of the crossing symmetric Veneziano (or Virasoro–Shapiro) amplitude and its all p-adic counterparts equals unity (or a definite constant). This makes it possible to consider an ordinary four-point function, which is rather complicated, as an infinite product of its inverse p-adic analogues, which have simpler forms. These first papers aroused interest in various aspects of p-adic string theory (for a review, see [39, 1]). Recent interest in p-adic string theory has been mainly related to the tachyon condensation [40], nonlinear dynamics [41], and an extension of p-adic and adelic path integrals to string amplitudes [42].

Like in the ordinary string theory, the starting point in the p-adic string theory is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following four forms:

$$A_\infty(k_1,\dots,k_4) \equiv A_\infty(a,b) = g^2\int_{\mathbb{R}} |x|_\infty^{a-1}\,|1-x|_\infty^{b-1}\,dx \tag{58}$$

$$= g^2\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right] \tag{59}$$

$$= g^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)} \tag{60}$$

$$= g^2\int \mathcal{D}X \exp\Big(-\frac{i}{2\pi}\int d^2\sigma\,\partial^\alpha X_\mu\,\partial_\alpha X^\mu\Big)\prod_{j=1}^4\int d^2\sigma_j \exp\big(ik_\mu^{(j)}X^\mu\big), \tag{61}$$

where $\hbar = 1$, $T = 1/\pi$, $a = -\alpha(s) = -1-\frac{s}{2}$, $b = -\alpha(t)$, and $c = -\alpha(u)$, with the condition $s+t+u = -8$, i.e., $a+b+c = 1$.

To introduce a p-adic Veneziano amplitude, one can consider a p-adic analogue of any of the above four expressions. A p-adic generalization of the first expression was proposed in [37], and it reads

$$A_p(a,b) = g_p^2\int_{\mathbb{Q}_p} |x|_p^{a-1}\,|1-x|_p^{b-1}\,dx, \tag{62}$$

where $|\cdot|_p$ denotes the p-adic absolute value. In this case, only the string world-sheet parameter x is treated as a p-adic variable, while all the other quantities maintain their usual (real) valuation. An attractive adelic formula (57) was found in [38]. A similar product formula holds also for the Virasoro–Shapiro amplitude. These infinite products are divergent, but they can be successfully regularized. Unfortunately, there is a problem of extending this formula to the higher point functions.

p-Adic analogues of (58) and (59) were also proposed in [36] and [43], respectively. In these cases, world-sheet, string momenta, and amplitudes are manifestly p-adic. Since string amplitudes are p-adic-valued functions, their physical interpretation is not yet clear.

Expression (61) is based on Feynman's path integral method, which is generic for all quantum systems and has a successful p-adic generalization. A p-adic counterpart of (61) was proposed in [42] and was partially elaborated in [44] and [45]. Note that, in this approach, the p-adic string amplitude is complex-valued, while not only the world-sheet parameters but also the target space coordinates and the string momenta are p-adic variables. Such a p-adic generalization is a natural extension of the formalism of p-adic [17] and adelic [10, 11] quantum mechanics to string theory. This is a promising subject; it should be investigated in detail and applied to the branes and the M-theory, which is presently the best candidate for the fundamental physical theory at the Planck scale.

7. CONCLUDING REMARKS

Among very interesting and fruitful recent developments, there have been noncommutative geometry and noncommutative field theory, which may be regarded as a deformation of the ordinary one in which standard field multiplication is replaced by the Moyal (star) product

$$(f \star g)(x) = \exp\left[\frac{i\hbar}{2}\theta^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right] f(y)g(z)\bigg|_{y=z=x}, \tag{63}$$

where $x = (x^1, x^2, \dots, x^D)$ is a spatial point and $\theta^{ij} = -\theta^{ji}$ are noncommutativity parameters. Replacing the ordinary product between noncommutative coordinates by the Moyal product (63), one obtains

$$x^i \star x^j - x^j \star x^i = i\hbar\theta^{ij}. \tag{64}$$

It is worth noting that one can introduce [45] the Moyal product in p-adic quantum mechanics, and it reads

$$(f * g)(x) = \int\limits_{\mathbb{Q}_p^D}\int\limits_{\mathbb{Q}_p^D} d^D k\, d^D k'\, \chi_p\Big(-(x^i k_i + x^j k'_j) + \frac{1}{2}k_i k'_j\,\theta^{ij}\Big)\,\widetilde{f}(k)\widetilde{g}(k'), \tag{65}$$

where D denotes the spatial dimensionality and $\widetilde{f}(k)$ and $\widetilde{g}(k')$ denote the Fourier transforms of $f(x)$ and $g(x)$, respectively. Some real, p-adic, and adelic aspects of the noncommutative scalar solitons are investigated in [46].

An extension of the above formalism with Feynman's path integral to p-adic and adelic quantum field theory is considered in [47].

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