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ON ASYMPTOTIC EXPANSION FOR MATHEMATICAL EXPECTATION OF A RENEWAL–REWARD PROCESS WITH DEPENDENT COMPONENTS AND HEAVY-TAILED INTER-ARRIVAL TIMES[1)]

В статье исследуется процесс восстановления с вознаграждением с зависимыми компонентами. Получено асимптотическое разложение для математического ожидания процесса в случае, когда интервалы времени между приходами имеют распределения с тяжелыми хвостами.

Ключевые слова и фразы: процесс восстановления, функция восстановления, процесс восстановления с вознаграждением, распределение с тяжелыми хвостами, субэкспоненциальное распределение.

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1. Introduction. In this paper a renewal–reward process is studied. There are some important properties and asymptotic results for the renewal–reward process and for its moments in the literature, for example, in [2], [7], [12]–[14]. Consider a sequence of independent random vectors $\{(T_i, X_i)\}$, $i \geqslant 1$, with possibly dependent coordinates, where (T_i, X_i) are identically distributed. Assume that $\{T_i\}$ is a renewal sequence. Define $S_n = \sum_{i=1}^{n} T_i$ so that $\{S_n\}$, $n \geqslant 1$, are the renewal times and let $N(t) = \max\{n\colon S_n \leqslant t\}$. The renewal–reward process $C(t)$, $t \geqslant 0$, is defined as follows:

$$C(t) = \sum_{i=1}^{N(t)} X_i. \tag{1.1}$$

Renewal–reward processes occur in various stochastic optimization models, particularly, in Markov and semi-Markov decision processes (see, for example, [1], [3], [4]). Many results for renewal processes generalize to renewal–reward processes. Some of these are the strong law of large numbers and elementary renewal theorem, central limit theorem, and Blackwell and key renewal theorems.

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In [7] an asymptotic expansion as $t \to \infty$ for the mathematical expectation of the renewal–reward process has been derived under the assumptions that F is nonlattice and μ_2, λ_1, $n_{1,1}$ exist:

$$D(t) \equiv \mathbf{E}C(t) = at + b + o(1),$$

where $a = \lambda_1/\mu_1$, $b = (\mu_2\lambda_1 - 2\mu_1 n_{1,1})/(2\mu_1^2)$, $\mu_1 \equiv \mathbf{E}T_1$, $\mu_2 \equiv \mathbf{E}T_1^2$, $\lambda_1 = \mathbf{E}X_1$, $n_{1,1} = \mathbf{E}T_1X_1$.

In [2] the remainder term of the asymptotic result for the mathematical expectation in [7] was sharpened from $o(1)$ to $o(t^{-k})$ under some conditions.

In this study, our aim is to investigate the mathematical expectation of the process $C(t)$ with the interclaim times having strong subexponential integrated tail distribution and with nonnegative claims. Let us introduce some notation and known results from the literature.

We denote by F the distribution of T_1 and by F_{I} the integrated tail function:

$$F_{\mathrm{I}}(t) = \frac{1}{\mu_1}\int_0^t \overline{F}(s)\,ds, \qquad t > 0, \tag{1.2}$$

where $\overline{F}(t) = 1 - F(t)$ and $F(t) = \mathbf{P}\{T_1 \leqslant t\}$. The integrated tail function $F_{\mathrm{I}}(t)$ is also called as equilibrium function in the literature, which plays an important role in renewal theory, in queueing theory, and in the ruin theory.

Let functions G_1 and G_2 be defined on $[0, \infty]$. We denote the convolution of the functions G_1 and G_2 by $G_1 * G_2$:

$$G_1 * G_2 = \int_0^t G_1(t-s)\,dG_2(s).$$

If G_1 and G_2 satisfy $G_1(0) = G_2(0) = 0$, then $G_1 * G_2 = G_2 * G_1$.

Now let us introduce some important classes of distributions.

Definition 1.1. A distribution F is said to be *heavy-tailed* ($F \in \mathcal{H}$) if and only if

$$\int_{-\infty}^{+\infty} e^{\lambda t}\,dF(t) = \infty \quad \text{for all} \quad \lambda > 0.$$

Definition 1.2. A distribution F is said to be *long-tailed* ($F \in \mathcal{L}$) if F has right-unbounded support and, for any fixed $a > 0$, $\overline{F}(t+a) \sim \overline{F}(t)$ as $t \to \infty$.

Definition 1.3. A distribution G on the positive half-line is said to be subexponential ($F \in \mathcal{S}$) if $\overline{F * F}(t) \sim 2\overline{F}(t)$ as $t \to \infty$.

The theory of subexponential distributions is well established by now. Its relevance is obvious from applications in various areas of applied probability. For recent reviews of applications of subexponentiality the reader is referred to [5], [6].

Definition 1.4. A distribution F is said to be strong subexponential ($F \in \mathcal{S}^*$) if F has a finite expectation μ_1 and

$$\lim_{t\to\infty} \int_0^t \frac{\overline{F}(t-s)}{\overline{F}(t)} \overline{F}(s)\, ds = 2\mu_1. \tag{1.3}$$

The standard examples of these distributions are the lognormal, Weibull with $\overline{F}(t) = \exp(-t^{\alpha})$, $0 < \alpha < 1$, and Pareto with $\overline{F}(t) = (\beta/(t+\beta))^{\alpha}$, $t \geqslant 0$, $\beta > 0$, $\alpha > 1$, distributions.

It is well known that $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}$. The fact that $F_{\mathrm{I}} \in \mathcal{S}^*$ implies that F has a finite second moment μ_2 (see [10]). More information on relations between these classes of distributions can be found in [9]–[11].

For investigation of the renewal–reward process we need also a renewal function $H(t)$, which is a mathematical expectation of the renewal process. Since $\mathbf{P}\{N(t) \geqslant n\} = \mathbf{P}\{S_n \leqslant t\}$, it holds that

$$H(t) \equiv \mathbf{E}N(t) = \sum_{n=1}^{\infty} \mathbf{P}\{N(t) \geqslant n\} = \sum_{n=1}^{\infty} \mathbf{P}\{S_n \leqslant t\} = \sum_{n=1}^{\infty} F^{*(n)}(t),$$

where $F^{*(n)}$ is an n-fold convolution of F with $F^{*(1)} \equiv F$.

Let us introduce the following results from the literature.

Theorem 1.1 ([10]). *Suppose F is nonarithmetic and $F_{\mathrm{I}} \in \mathcal{S}^*$. Then the following asymptotic relation holds as $t \to \infty$:*

$$H(t) - \frac{t}{\mu_1} - \frac{\mu_2 - 2\mu_1^2}{2\mu_1^2} = -\frac{1}{\mu_1} \int_t^{\infty} \overline{F}_{\mathrm{I}}(s)\, ds + O(\overline{F}_{\mathrm{I}}(t)). \tag{1.4}$$

Theorem 1.2 ([8]). *Let $F \in \mathcal{S}$ and so $F \in \mathcal{L}$. Suppose that distributions $G_1, \ldots, G_n$ are such that, individually for each i, either (i) $G_i \in \mathcal{L}$ and $\overline{G}_i(t) = O(\overline{F}(t))$ as $t \to \infty$ or (ii) $\overline{G}_i(t) = o(\overline{F}(t))$ as $t \to \infty$. Then as $t \to \infty$*

$$\overline{G_1 * \cdots * G_k}(t) = \sum_{i=1}^{k} \overline{G}_i(t) + o(\overline{F}(t)). \tag{1.5}$$

2. Main results. Our main purpose is to obtain an asymptotic expansion for the mathematical expectation of the renewal–reward process $C(t)$ with the nonnegative claims $\{X_n\}$, $n \geqslant 1$, under some assumptions using (1.4).

Let us introduce the following functions:

$$G_{1,1}(t) = \frac{1}{\lambda_1} \int_0^t g(s)\, dF(s); \qquad \overline{G}_{1,1}(t) = 1 - G_{1,1}(t), \qquad t \geqslant 0,$$

$$G_{2,1}(t) = \frac{1}{n_{1,1}} \int_0^t s g(s)\, dF(s); \qquad \overline{G}_{2,1}(t) = 1 - G_{2,1}(t), \qquad t \geqslant 0,$$

$$G_{\mathrm{I}}(t) = \frac{\lambda_1}{n_{1,1}} \int_0^t \overline{G}_{1,1}(s)\, dF(s); \qquad \overline{G}_{\mathrm{I}}(t) = 1 - G_{\mathrm{I}}(t), \qquad t \geqslant 0,$$

where $g(s) = \mathbf{E}(X_1 \mid T_1 = s)$ is a conditional mathematical expectation of X_1 for given $T_1 = s$,

$$n_{1,1} \equiv \mathbf{E}T_1X_1 = \int_0^\infty sg(s)\,dF(s), \qquad \lambda_1 \equiv \mathbf{E}X_1 = \int_0^\infty g(s)\,dF(s).$$

Since we assume that claims are nonnegative, we can say that $G_{1,1}$, $G_{2,1}$, and G_{I} have the properties of distribution function and

$$\overline{G}_{\mathrm{I}}(t) = \overline{G}_{2,1}(t) - \frac{\lambda_1}{n_{1,1}}\, t\overline{G}_{1,1}(t).$$

We formulate the main result of this paper as the following theorem.

Theorem 2.1. *Let the following assumptions hold*:

1) F *is nonarithmetic*;
2) $F_{\mathrm{I}} \in \mathcal{S}^*$;
3) λ_1 *and* $n_{1,1}$ *exist*;
4) $g(s) \leqslant L$ *as* $s \to \infty$ *for some* $L > 0$.

Then the following asymptotic relation holds as $t \to \infty$ *for the mathematical expectation of the renewal–reward process with nonnegative claims*:

$$D(t) = \frac{\lambda_1}{\mu_1}t + \frac{\mu_2\lambda_1 - 2\mu_1 n_{1,1}}{2\mu_1^2} - \frac{\lambda_1}{\mu_1}\int_t^\infty \overline{F}_{\mathrm{I}}(s)\,ds + O(\overline{F}_{\mathrm{I}}(t)). \tag{2.1}$$

Before proceeding with proof of this theorem, we should prove the following lemmas.

Lemma 2.1. *Suppose the assumptions of Theorem 2.1 are satisfied, then* $\overline{G}_{1,1}(t) = o(\overline{F}_{\mathrm{I}}(t))$ *and* $\overline{G}_{\mathrm{I}}(t) = O(\overline{F}_{\mathrm{I}}(t))$, *as* $t \to \infty$.

Proof. Since F has a finite mean μ_1 and $F_{\mathrm{I}} \in \mathcal{S}^* \subset \mathcal{L}$, it follows that $\overline{F}(t) = o(\overline{F}_{\mathrm{I}}(t))$ as $t \to \infty$ (see [9, Lemma 2.25]). On the other hand, since $g(s) \leqslant L$ as $s \to \infty$ for some $L > 0$, we can write as $t \to \infty$

$$\overline{G}_{1,1}(t) \leqslant \frac{1}{\lambda_1}\int_t^\infty g(s)\,dF(s) \leqslant \frac{L}{\lambda_1}\overline{F}(t).$$

So $\overline{G}_{1,1}(t) = O(\overline{F}(t)) = o(\overline{F}_{\mathrm{I}}(t))$ as $t \to \infty$.

For $\overline{G}_{\mathrm{I}}(t)$ as $t \to \infty$ we can write

$$\overline{G}_{\mathrm{I}}(t) \leqslant \frac{\lambda_1}{n_{1,1}}\int_t^\infty \overline{G}_{1,1}(s)\,ds \leqslant \frac{L}{n_{1,1}}\int_t^\infty \overline{F}(s)\,ds = \frac{\mu_1 L}{n_{1,1}}\overline{F}_{\mathrm{I}}(t),$$

which means that $\overline{G}_{\mathrm{I}}(t) = O(\overline{F}_{\mathrm{I}}(t))$ as $t \to \infty$.

This completes the proof of Lemma 2.1.

Lemma 2.2. *Suppose the assumptions of Theorem 2.1 are satisfied, then*

$$\begin{aligned} A(t) &= \int_0^t \int_{t-s}^\infty \overline{F}_{\mathrm{I}}(\tau)\,d\tau\,dG_{1,1}(s) \\ &= -\frac{\mu_2}{2\mu_1}\overline{G}_{1,1}(t) + \int_t^\infty \overline{F}_{\mathrm{I}}(s)\,ds - \frac{n_{1,1}}{\lambda_1}\overline{G}_{\mathrm{I}}(t) + \frac{n_{1,1}}{\lambda_1}\overline{F_{\mathrm{I}} * G_{\mathrm{I}}}(t). \end{aligned} \tag{2.2}$$

Proof. Integrating by parts gives us

$$\begin{aligned} A(t) &= G_{1,1}(t)\int_0^\infty \overline{F}_{\mathrm{I}}(\tau)\,d\tau - \int_0^t G_{1,1}(s)\overline{F}_{\mathrm{I}}(t-s)\,ds \\ &= \frac{\mu_2}{2\mu_1}G_{1,1}(t) - \int_0^t \overline{F}_{\mathrm{I}}(s)\,ds + \frac{n_{1,1}}{\lambda_1}\int_0^t \overline{F}_{\mathrm{I}}(t-s)\,dG_{\mathrm{I}}(s) \\ &= -\frac{\mu_2}{2\mu_1}\overline{G}_{1,1}(t) + \int_t^\infty \overline{F}_{\mathrm{I}}(s)\,ds + \frac{n_{1,1}}{\lambda_1}G_{\mathrm{I}}(t) - \frac{n_{1,1}}{\lambda_1}F_{\mathrm{I}} * G_{\mathrm{I}}(t) \\ &= -\frac{\mu_2}{2\mu_1}\overline{G}_{1,1}(t) + \int_t^\infty \overline{F}_{\mathrm{I}}(s)\,ds - \frac{n_{1,1}}{\lambda_1}\overline{G}_{\mathrm{I}}(t) + \frac{n_{1,1}}{\lambda_1}\overline{F_{\mathrm{I}} * G_{\mathrm{I}}}(t). \end{aligned}$$

This completes the proof of Lemma 2.2.

Proof of Theorem 2.1. By the definition

$$D(t) \equiv \mathbf{E}C(t) = \mathbf{E}\sum_{i=1}^{N(t)} X_i. \tag{2.3}$$

By Wald's identity (see, for example, [14, p. 134]),

$$D(t) = \mathbf{E}\sum_{i=1}^{N(t)+1} X_i - \mathbf{E}X_{N(t)+1} = \lambda_1(H(t)+1) - \mathbf{E}X_{N(t)+1}. \tag{2.4}$$

It is not difficult to see that (see, for example, [14, p. 134])

$$\begin{aligned} \mathbf{E}X_{N(t)+1} &= \int_t^\infty g(s)\,dF(s) + \int_0^t \int_{t-\tau}^\infty g(s)\,dF(s)\,dH(\tau) \\ &= \lambda_1\overline{G}_{1,1}(t) + A_1(t), \end{aligned} \tag{2.5}$$

where $H(t)$ is a renewal function and

$$A_1(t) = \int_0^t \int_{t-\tau}^\infty g(s)\,dF(s)\,dH(\tau).$$

The function $A_1(t)$ can be written as follows:

$$\begin{aligned} A_1(t) &= \int_0^\infty g(s)\,dF(s)\int_0^t dH(\tau) - \int_0^t g(s)\,dF(s)\int_0^{t-s} dH(\tau) \\ &= \lambda_1 H(t) - A_2(t), \end{aligned} \tag{2.6}$$

where

$$A_2(t) = \int_0^t H(t-s)g(s)\,dF(s).$$

Using (1.4) we can write $A_2(t)$ as follows:

$$\begin{aligned}A_2(t) &= \frac{t}{\mu_1}\int_0^t g(s)\,dF(s) - \frac{1}{\mu_1}\int_0^t sg(s)\,dF(s) + \frac{\mu_2-2\mu_1^2}{2\mu_1^2}\int_0^t g(s)\,dF(s)\\ &\quad - \frac{1}{\mu_1}\int_0^t\int_{t-s}^\infty \overline{F}_{\mathrm{I}}(\tau)\,d\tau g(s)\,dF(s) + \int_0^t M(t-s)g(s)\,dF(s)\\ &= \frac{\lambda_1}{\mu_1}t - \frac{\lambda_1}{\mu_1}tG_{1,1}(t) - \frac{n_{1,1}}{\mu_1} + \frac{n_{1,1}}{\mu_1}\overline{G}_{2,1}(t) + \frac{\mu_2-2\mu_1^2}{2\mu_1^2}\lambda_1 G_{1,1}(t)\\ &\quad - \frac{1}{\mu_1}A_3(t) + A_4(t)\\ &= \frac{\lambda_1}{\mu_1}t + \frac{\mu_2-2\mu_1^2}{2\mu_1^2}\lambda_1 G_{1,1}(t) - \frac{n_{1,1}}{\mu_1} + \frac{n_{1,1}}{\mu_1}\overline{G}_{\mathrm{I}}(t) - \frac{1}{\mu_1}A_3(t) + A_4(t),\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}A_3(t) &= \lambda_1\int_0^t\int_{t-s}^\infty \overline{F}_{\mathrm{I}}(\tau)\,d\tau\,dG_{1,1}(s),\\ A_4(t) &= \lambda_1\int_0^t M(t-s)\,dG_{1,1}(s),\end{aligned}$$

and $M(t) = O(\overline{F}_{\mathrm{I}}(t))$ as $t\to\infty$.

Let us investigate $A_3(t)$ and $A_4(t)$ separately. For $A_3(t)$, using Lemma 2.2, we can write the following relation as $t\to\infty$:

$$A_3(t) = -\frac{\mu_2\lambda_1}{2\mu_1}\overline{G}_{1,1}(t) + \lambda_1\int_t^\infty \overline{F}_{\mathrm{I}}(s)\,ds - n_{1,1}\overline{G}_{\mathrm{I}}(t) + n_{1,1}\overline{F_{\mathrm{I}} * G_{\mathrm{I}}}(t).$$

Since, as $t\to\infty$ we have $\overline{G}_{\mathrm{I}}(t) = O(\overline{F}_{\mathrm{I}}(t))$ by Lemma 2.1 and $F_{\mathrm{I}}\in\mathcal{S}$ by the assumption of the theorem, it follows that $G_{\mathrm{I}}\in\mathcal{L}$ and the assumptions of Theorem 1.2 are satisfied since $G_1\equiv G_{\mathrm{I}}$, $G_2\equiv F_{\mathrm{I}}$, and $F\equiv F_{\mathrm{I}}$, so as $t\to\infty$

$$\overline{F_{\mathrm{I}} * G_{\mathrm{I}}}(t) = \overline{F}_{\mathrm{I}}(t) + \overline{G}_{\mathrm{I}}(t) + o(\overline{F}_{\mathrm{I}}(t)) = O(\overline{F}_{\mathrm{I}}(t)). \tag{2.8}$$

Using (2.8) and Lemma 2.1, for $A_3(t)$, as $t\to\infty$ we can write

$$A_3(t) = \lambda_1\int_t^\infty \overline{F}_{\mathrm{I}}(s)\,ds + O(\overline{F}_{\mathrm{I}}(t)). \tag{2.9}$$

Since $M(t) = O(\overline{F}_{\mathrm{I}}(t))$ as $t\to\infty$, for $A_4(t)$, the following relation as $t\to\infty$ can be written:

$$\begin{aligned}|A_4(t)| &\leqslant \lambda_1\int_0^t |M(t-s)|\,dG_{1,1}(s)\\ &\leqslant C\lambda_1\int_0^t \overline{F}_{\mathrm{I}}(t-s)\,dG_{1,1}(s) = C\lambda_1\overline{F}_{\mathrm{I}} * G_{1,1}(t),\end{aligned} \tag{2.10}$$

where $C > 0$ is some constant. The assumptions of Theorem 1.2 are satisfied since $G_1 \equiv G_{1,1}$, $G_2 \equiv F_{\mathrm{I}}$ and $F \equiv F_{\mathrm{I}}$, so as $t \to \infty$

$$\overline{F_{\mathrm{I}} * G_{1,1}}(t) = \overline{F}_{\mathrm{I}}(t) + \overline{G}_{1,1}(t) + o(\overline{F}_{\mathrm{I}}(t)) = O(\overline{F}_{\mathrm{I}}(t)). \qquad (2.11)$$

Taking into account

$$\overline{F}_{\mathrm{I}} * G_{1,1}(t) = G_{1,1}(t) - F_{\mathrm{I}} * G_{1,1}(t) = \overline{F_{\mathrm{I}} * G_{1,1}}(t) - \overline{G}_{1,1}(t)$$

and (2.11) in (2.10), for $A_4(t)$, as $t \to \infty$ we obtain

$$A_4(t) = O(\overline{F}_{\mathrm{I}}(t)). \qquad (2.12)$$

Taking into account Lemma 2.1, (2.9), and (2.12) in (2.7), the following relation as $t \to \infty$ for $A_2(t)$ can be written:

$$A_2(t) = \frac{\lambda_1}{\mu_1} t + \frac{\mu_2 \lambda_1 - 2\mu_1 n_{1,1}}{2\mu_1^2} - \lambda_1 - \frac{\lambda_1}{\mu_1} \int_t^\infty \overline{F}_{\mathrm{I}}(s)\, ds + O(\overline{F}_{\mathrm{I}}(t)). \qquad (2.13)$$

Taking into account (2.13) in (2.6) for $A_1(t)$ the following relation as $t \to \infty$ can be written:

$$A_1(t) = \lambda_1 (H(t) + 1) - \frac{\lambda_1}{\mu_1} t - \frac{\mu_2 \lambda_1 - 2\mu_1 n_{1,1}}{2\mu_1^2} + \frac{\lambda_1}{\mu_1} \int_t^\infty \overline{F}_{\mathrm{I}}(s)\, ds + O(\overline{F}_{\mathrm{I}}(t)). \qquad (2.14)$$

Finally, taking into account (2.14) in (2.4), we can obtain (2.1).

This completes the proof of Theorem 2.1.

Remark 2.1. If we take $X_1 \equiv 1$, we can easily obtain (1.4) from (2.1). For this, we just need to put $\lambda_1 = 1$ and $n_{1,1} = \mu_1$.

Example 2.1. Suppose that T_1 has Pareto distribution with scale parameter $\beta > 0$ and shape parameter $\alpha > 2$:

$$\mathbf{P}\{T_1 > t\} = \overline{F}(t) = \left(\frac{\beta}{t + \beta} \right)^\alpha, \qquad t \geqslant 0.$$

It is well known that in this case Pareto distribution belongs to the class $\mathcal{S}^*$ and

$$\mu_1 = \frac{\beta}{\alpha - 1}, \qquad \mu_2 = \frac{2\beta^2}{(\alpha - 2)(\alpha - 1)}.$$

For $\overline{F}_{\mathrm{I}}$ we can write

$$\overline{F}_{\mathrm{I}}(t) = \frac{1}{\mu_1} \int_t^\infty \overline{F}(s)\, ds = \left(\frac{\beta}{t + \beta} \right)^{\alpha - 1}, \qquad t \geqslant 0.$$

It is clear that F_{I} is Pareto distribution with scale parameter $\beta > 0$ and shape parameter $\alpha - 1$. So, $F_{\mathrm{I}} \in \mathcal{S}^*$.

Suppose that X_n is dependent on T_n since

$$X_n = \frac{LT_n}{T_n + \beta}, \qquad L > 0, \quad n \geqslant 1.$$

It is not difficult to see that

$$g(s) = \frac{Ls}{s+\beta}, \qquad \lambda_1 = \mathbf{E}X_1 = \int_0^\infty g(s)\,dF(s) = \frac{L}{\alpha+1},$$
$$n_{1,1} = \mathbf{E}T_1X_1 = \int_0^\infty sg(s)\,dF(s) = \frac{L\beta}{\alpha^2-1}.$$

The assumptions of Theorem 2.1 are satisfied, so, as $t \to \infty$

$$D(t) = \frac{L(\alpha-1)}{\beta(\alpha+1)}t + \frac{L}{(\alpha+1)(\alpha-2)} - \frac{L(\alpha-1)}{(\alpha+1)(\alpha-2)}\left(\frac{\beta}{t+\beta}\right)^{\alpha-2} + O\left(\left(\frac{\beta}{t+\beta}\right)^{\alpha-1}\right).$$

Since

$$\left(\frac{\beta}{t+\beta}\right)^{\alpha} \sim \left(\frac{\beta}{t}\right)^{\alpha}$$

as $t \to \infty$, therefore, the following relation as $t \to \infty$ can be written:

$$D(t) = \frac{L(\alpha-1)}{\beta(\alpha+1)}t + \frac{L}{(\alpha+1)(\alpha-2)} - \frac{L(\alpha-1)}{(\alpha+1)(\alpha-2)}\left(\frac{\beta}{t+\beta}\right)^{\alpha-2} + O(t^{1-\alpha}).$$

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