Записки научных
семинаров ПОМИ
Том 331, 2006 г.

D. Sternheimer

SOME REFLECTIONS ON MATHEMATICIANS' VIEWS OF QUANTIZATION

Abstract. We start with a short presentation of the difference in attitude between mathematicians and physicists even in their treatment of physical reality, and look at the paradigm of quantization as an illustration. In particular we stress the differences in motivation and realization between the Berezin and deformation quantization approaches, exposing briefly Berezin's view of quantization as a functor. We continue with a schematic overview of deformation quantization and of its developments in contrast with the latter and discuss related issues, in particular the spectrality question. We end by a very short survey of two main avatars of deformation quantization, quantum groups and quantum spaces (especially noncommutative geometry) presented in that perspective.

**This paper is dedicated
to the memory of "Alik" Berezin**

1. Introduction

Mathematics arose as an abstraction of our representation of the physical universe. The language it developed was, and still is, in turn seminal for a better formulation of that representation, but a Babel tower effect can be perceived almost from the start. Indeed, as Sir Michael Atiyah said in his closing lecture of the 2000 International Congress in Mathematical Physics (with examples taken from algebraic geometry), Mathematics and Physics are two communities separated by a common language.

The way of using mathematical formalism is generically very different in both. Not many can (like Faddeev and Flato) speak at ease that common language with both accents and styles. In the best cases, physicists speak the mathematical language with a distinctive accent that mathematicians may have a hard time to understand. On the other hand, Johann Wolfgang von Goethe wrote [50]: *Die Mathematiker sind eine Art*

Franzosen: Redet man mit ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anders. (Mathematicians are a kind of Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different[1].) An English translation of the quote can be seen on the walls of the UCLA Department of Mathematics, among many others, but that one struck me. Being originally a French mathematician, thus with inborn *esprit de contradiction* and feeling for logic, I shall do my best to bring a counterexample to that affirmation, even more so since my first name may give me a predisposition to decipher and interpret writings on the wall. In fact, both mathematicians and physicists are "guilty" of some form of misinterpretation of the other point of view, that can be seminal.

The present contribution is in great part an edited version, with some additions, of [70] (of which the author has kept Copyright to himself), in the volume of *Letters in Mathematical Physics* dedicated to Alik Berezin.

2. The Berezin paradigm

In the 70's, the last decade of his too short life, the main subject of interest for Alik Berezin was [60] to develop "supermathematics," a subject he had implicitely touched as early as 1961 [7], writing many mathematical works (some of them posthumous, see, e.g., [10, 14, 12]). He certainly was (mildly speaking) one of the founders of that domain, albeit (like some others) more as a mathematician, which may be one reason why he does not seem to have received in physics the recognition he deserves.

At the same time and apparently as part of a larger goal, he addressed the question of understanding quantization, in as precise a way as possible. The pursuit of precision was a constant feature in all his works. It is essential in modern mathematical physics, but should be exercised with care. He had tackled the issue of quantization before, especially in relation with second quantization [8].

Independently, Moshé Flato and a number of coworkers (among which I am happy to have been) had exactly the same objectives, with the opposite order of priority. Moshé and I had met Alik a couple of times (in 1966 and 1972 in Moscow) and appreciated him. Those were difficult times as far as the communication between East and West was concerned. As an

[1] Goethe studied science and even wrote a visionary botany essay in 1790, but in physics he had e.g. a running battle against Newton's theory of color and his bias against both Frenchmen and mathematicians is obvious here.

unfortunate consequence, neither of us was really aware of the efforts of the other group until about 1979–80. We "flirted with" supersymmetry in 1967–70 [39] before the terminology was coined, but that was from the point of view of (traditional) Lie group theory, introducing a "Poincaré–Weyl" group with both vector and spinor translations. In the course of developing some possible applications we had, in 1970, to multiply the spinorial translations by an anticommuting operator F, obtaining implicitly the Poincaré supersymmetry before the notion was introduced. But at the time we did not pursue that direction further: our main focus had shifted towards more elaborate topics in representation theory and especially towards deformation theory and what eventually was called deformation quantization.

There is however a deeper reason, of scientific origin, for the "orthogonality" of these efforts and different focus of interest: Our approaches, in spite of an apparent similitude in techniques, in particular the use of phase space methods, have very different starting points. (The same applies to "supermathematics.")

Indeed, a scientist should answer three questions: *why, what,* and *how.* It is generally recognized that work is 1% inspiration and 99% perspiration. The latter applies to *how.* Knowing what one is doing is very important and represents the greater part of the inspiration. But it is essential to know why one pursues such a research. That is where our two approaches differed in an essential way.

Alik was basically a mathematician, even if he was deeply interested in physics, motivated by it, knew a lot of physics and could even [60] speak with physicists using their language – albeit probably not with their accent. In particular, like almost all mathematicians (and physicists for that matter) he took as a God-given axiom the operator formulation of quantum theories that is common fare among physicists. His aim was therefore to understand mathematically the correspondence principle. That led him, e.g. in his beautiful papers [10] to which we really paid attention a number of years later, to consider quantization (roughly speaking, his formulation is as usual more precise) as a functor between a category of algebras of classical observables (on phase space) and a category of algebras of operators (in Hilbert space). That is very nice, but it is a kind of translation of the physical reality into mathematical language, while supermathematics was for him a much more challenging enterprise, especially as a mathematician.

Flato knew first-hand, having interacted in Israel with top-level physicists and mathematicians already as an undergraduate student, that physicists are neither God nor Jesus, but that at least the best of them (e.g. Dirac with his "delta function") sense very well where are the stones when they walk over mathematical waters. Mathematicians would try to build a bridge following the physicists' path (the theory of distributions in the above simple example), often ending with beautiful mathematics of interest in itself, with many other developments. But, as far as the original physical problem is concerned, that may take years if at all feasible. There are examples galore of the phenomenon. A true mathematical physicist may first "get his feet wet" in order to identify the aim of the physicists, then find his way around the lake to reach it.

We therefore looked more precisely into why do physicists need quantization, and more generally why does physics evolve from one stage to another, and what happens when the need appears. That is where deformation theory intervenes. In Berezin's case, though the modern understanding of quantization as a deformation can now be seen [60] in a number of expressions he developed, that was not his approach.

3. The deformation philosophy

One should never forget that physical theories have their domain of applicability defined by the relevant distances, velocities, energies, etc. involved. Some of the brightest theoretical physicists of the last century did pursue the "Holy Grail" of a unified theory, whether a unified equation or (in a more sophisticated setting) a "theory of everything," but the quest remained inconclusive. Furthermore, had it been crowned with total success, the need to focus on various ranges would remain, if only for practical reasons: It is hypothetically conceivable, and obviously practically impossible, to explain sociological phenomena by decomposing human beings into elementary particles and applying the laws of physics!

The passage from one domain (of distances, etc.) to another does not happen in an uncontrolled way. One may imagine hypothetical universes, but these become interesting for physics only when confronted with reality. At some point, experimental phenomena appear that cause a paradox and contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified. Then the attached structures (symmetries, observables, states, etc.) *deform* the initial structure. Namely, we have a new structure which in the limit, when the new parameter goes to zero, coincides with the previous formalism. The question is therefore, in which category do we seek for deformations? Usually physics is rather conservative and if we start e.g. with the category of associative or Lie algebras, we tend to deform in the same category. But there are important examples of generalizations of this principle: e.g. quantum groups are deformations of Hopf algebras.

The discovery of the nonflat nature of Earth may be the first example of this phenomenon. Closer to us, the paradox coming from the Michelson and Morley experiment (1887) was resolved in 1905 by Einstein with the special theory of relativity: In our context, one can express that by saying that the Galilean geometrical symmetry group of Newtonian mechanics is deformed to the Poincaré group, the new fundamental constant being c^{-1} where c is the velocity of light in vacuum.

It is interesting to note that a first mathematical example of deformations was introduced at around the same time with the Riemann surface theory, though deformations became systematically studied in the mathematical literature only at the end of the fifties with the profound works of Kodaira and Spencer [55] on deformations of complex analytic structures (that were triggered by [42]). Now, when one has an action on a geometrical structure, it is natural to try and "linearize" it by inducing from it an action on an algebra of functions on that structure. This is implicitly what Gerstenhaber did shortly afterwards [47] with his definition and thorough study of deformations of rings and algebras (see also the too little-known comparison [48] between both, using an extension of his algebraic deformation theory from rings to sheaves of rings).

It is in the Gerstenhaber sense that the Galileo group is deformed to the Poincaré group; that operation is the inverse of the notion of group contraction introduced ten years before, empirically, by Ínönü and Wigner [53], an earlier example of which can be found in [67]. This fact triggered strong interest for deformation theory in France among a number of theoretical physicists, including Flato who had just arrived from the Racah school and knew well the effectiveness of symmetry in physical problems. He was soon to realize that, however important symmetry is as a notion and a tool in a mathematical treatment of physical problems, it is not the only one and should be complemented with other (often related) concepts: The notion of deformation can be applied to a variety of categories that are used to express mathematically the physical reality. For example, the buck does not stop at the Poincaré group. Indeed, within the Lie groups category, it can be deformed to the de Sitter and anti de Sitter groups; the latter has important consequences that we shall not develop here (see, e.g., [36, 37, 69]). Within the Hopf algebras category one can deform further to quantum groups, and eventually deform (quantize) space-time itself.

4. The correspondence functor

4.1. Conventional approaches. Around 1900, as a last resort to explain the black body radiation, Planck proposed the quantum hypothesis: The energy of light is not emitted continuously but in quanta proportional to its frequency. He wrote h for the proportionality constant which bears his name. This paradoxical situation got a beginning of a theoretical basis when, in the *annus mirabilis* 1905, Einstein came with the theory of the photoelectric effect which eventually led in 1923 Prince Louis de Broglie to his discovery of the duality of waves and particles, which he described in his celebrated Thesis published in 1925, and to what he called *mécanique ondulatoire*. For him, waves and particles were two aspects of the same physical reality. Leading scientists writing in German, in particular Hermann Weyl, Werner Heisenberg, Erwin Schrödinger, and Niels Bohr (whose empirical image of the atom preceded all in 1913) transformed that intuitive view into the quantum mechanics that we know (in part developed independently) where the observables are operators in spaces of wave functions endowed with a scalar product structure defined around 1915 by David Hilbert. If I may write so, forthwith it became something apparently quite different!

Intuitively, classical mechanics is the limit of quantum mechanics when $\hbar = \frac{h}{2\pi}$ goes to zero, and "quantization is an algorithm by which a quantum system corresponds to a classical dynamical one" [10]. That algorithm was systematized very early, in particular by Hermann Weyl in 1927 [72].

4.1.1. Weyl quantizations in a flat phase space. If we start with a classical observable $u(p,q)$, some function on a (flat) phase space $\mathbb{R}^{2\ell}$ (with $p,q \in \mathbb{R}^\ell$), one can associate to it an operator (the corresponding quantum observable) $\Omega(u)$ in the Hilbert space $L^2(\mathbb{R}^\ell)$ by the following general recipe:

$$u \mapsto \Omega_w(u) = \int_{\mathbb{R}^{2\ell}} \tilde{u}(\xi,\eta)\exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)w(\xi,\eta)\ d^\ell\xi d^\ell\eta, \tag{4.1}$$

where $\tilde{u}$ is the inverse Fourier transform of u, $\hat{p}_\alpha$ and $\hat{q}_\alpha$ are operators satisfying the canonical commutation relations $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta}$ $(\alpha, \beta = 1, ..., \ell)$, w is a weight function, and the integral is taken in the weak operator topology. What is now called normal ordering corresponds to choosing the weight $w(\xi, \eta) = \exp(-\frac{1}{4}(\xi^2 \pm \eta^2))$, standard ordering (the case of the usual pseudodifferential operators in mathematics, for which one orders polynomials by writing first the position q, then the momentum p) to the weight $w(\xi, \eta) = \exp(-\frac{i}{2}\xi\eta)$, and the original Weyl (symmetric) ordering is obtained by the original formula of Weyl, where $w = 1$.

An inverse formula was found shortly afterwards by Eugene Wigner [73]. It maps an operator into what mathematicians call its symbol by a kind of trace formula. For example, Ω_1 (where $w = 1$) defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2\ell})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^\ell)$ with inverse given by

$$u = (2\pi\hbar)^{-\ell}\ \mathrm{Tr}[\Omega_1(u)\exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)], \tag{4.2}$$

and if $\Omega_1(u)$ is of trace class one has $\mathrm{Tr}(\Omega_1(u)) = (2\pi\hbar)^{-\ell}\int u\,\omega^\ell \equiv \mathrm{Tr}_\mathrm{M}(u)$, the "Moyal trace," where ω^ℓ is the (symplectic) volume dx on $\mathbb{R}^{2\ell}$. Numerous developments followed in the direction of phase-space methods, many of which can be found described in [1]. In particular, it was noticed very early that the Poisson bracket P of two classical observables (defined below, if needed) does not in general transform under (4.1) into the commutator of the quantized observables. Indeed, formula (4.3) below expresses the inverse image M of the commutator, but surprisingly it was obtained only in 1949 by Moyal [61] (as a tool with another motivation); a similar formula for the symbol of a product $\Omega_1(u_1)\Omega_1(u_2)$ had been found a little earlier [51] and can now be written more clearly, see (4.4), as a (Moyal) *star product* $*_M$:

$$\begin{aligned} M(u_1, u_2) &= \nu^{-1}\sinh(\nu P)(u_1, u_2) \\ &= P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{\nu^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2) \end{aligned} \tag{4.3}$$

$$u_1 *_M u_2 = \exp(\nu P)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{\nu^r}{r!} P^r(u_1, u_2), \tag{4.4}$$

where $2\nu = i\hbar$, $P^r(u_1, u_2) = \Lambda^{i_1 j_1} \ldots \Lambda^{i_r j_r}(\partial_{i_1 \ldots i_r} u_1)(\partial_{j_1 \ldots j_r} u_2)$ is the r^{th} power ($r \geq 1$) of the Poisson bracket bidifferential operator P, $i_k, j_k = 1, \ldots, 2\ell$, $k = 1, \ldots, r$, and $(\Lambda^{i_k j_k}) = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. To fix ideas, we may assume here $u_1, u_2 \in C^\infty(\mathbb{R}^{2\ell})$ and the sums are taken as formal series.

4.1.2. Nonflat phase spaces. In general, the phase space of a classical mechanical system needs not be a flat space $\mathbb{R}^{2\ell}$. Physicists are well aware of the fact, but too often work in a local chart and consider only canonical quantization with p's and q's. A notable exception is Dirac who developed [27] an ingenious method of quantization for systems where the geometry of the phase space is more complicated, due to the presence of constraints.

Recall that a *Poisson manifold* (M, Λ) is (see, e.g., [5]) a differentiable manifold M on which a skew-symmetric 2-tensor Λ is defined such that the Poisson bracket $P(u,v) \equiv \Lambda(du \wedge dv)$, $u, v \in C^\infty(M)$, satisfies the Jacobi identity (or, equivalently, that $[\Lambda, \Lambda] = 0$ in the sense of the Schouten–Nijenhuis bracket). A Poisson manifold is *symplectic* if in addition Λ is everywhere nondegenerate, or, equivalently, if the (then defined) inverse of Λ is a nondegenerate closed two-form ω. Dirac's "second class constraints" (pairs of conjugate constraints) reduce a flat phase space to a symplectic submanifold (where the conjugate constraints vanish), while "first class constraints" further reduce it to a Poisson submanifold. Dirac (and many after him) worked out a number of such cases "by hand."

4.1.3. Geometric quantization. A more systematic approach, which we mention for completeness but shall not develop here since it has been extensively developed elsewhere (see, e.g., [58, 74]), is known under the name *geometric quantization*. It is mainly a by-product of Lie group representations theory, where it gave very significant results. It turned out to be geometric all right, but its scope as far as quantization is concerned has been rather limited since few classical observables could be quantized, except in situations which amount essentially to the Weyl case. In a nutshell one considers a phase-space W as a coadjoint orbit of some Lie group (the Weyl case corresponds to the Heisenberg group with the canonical commutation relations and $\mathfrak{h}_\ell$ as Lie algebra); there one defines a "prequantization" on the Hilbert space $L^2(W)$ and tries to halve the number of degrees of freedom by using polarizations (often complex ones, which is not an innocent operation as far as physics is concerned) to get a Lagrangian submanifold $\mathcal{L}$ of dimension half of that of W and quantized observables as operators in $L^2(\mathcal{L})$. A main advantage of the method is that it makes more natural the appearance of a phenomenon of quantization (e.g. only a discrete family of coadjoint orbits can be quantized) due to the constraints imposed by the Hilbert space formalism.

4.2. The Berezin approach to quantization. In a series of very interesting papers [9, 10], also developed in the 70's, Berezin considers a number of examples, including Euclidean and Lobatchevsky planes, cylinder, torus and sphere, Kähler manifolds and duals of Lie algebras. That led him in [10] to propose a very general definition of quantization:

Definition 4.1. *Let (M, Λ) be a Poisson manifold (phase space). An associative algebra $\mathfrak{A}$ with involution is called a quantization of the classical system (M, Λ) if*

1) There is a family A_h of associative algebras, indexed by $h \in E \subset \mathbb{R}_+$ (with limit point 0 not included), such that $\mathfrak{A}$ consists of functions $f(h)$ valued in A_h (the involution and multiplication of both algebras being in natural correspondence).

2) There is an algebra homomorphism $\phi\colon \mathfrak{A} \to C^\infty(M)$ such that $\mathfrak{A}$ is "large enough" (i.e., $\phi(\mathfrak{A})$ separates points on M), and ϕ maps $(1/ih)$ times the commutator of two elements $f, g \in \mathfrak{A}$ into $P(\phi(f), \phi(g)) + O(h)$ and the conjugate of $f \in \mathfrak{A}$ into the complex conjugate of $\phi(f)$.

The quantization is called a special quantization if in addition

3) $A_h = C^\infty(M)$.

4) $\mathfrak{A} = \{f(h, x); f(h, x) \in A_h \quad \text{for fixed} \quad h\}$.

5) ϕ is given by $\phi(f) = \lim_{h\to 0} f(h, x)$.

Most (but not all, e.g. the 2-dimensional cylinder and torus, both of which have a nontrivial topology) examples of quantizations treated by Berezin are special. These have the additional properties that the algebras A_h have a unit (the function 1) and a trace defined by integration over M with respect to some measure on it.

Remark. In deformation quantization one can show the equivalence (as deformations) between the case of general star products and those for which the unit is preserved (that is a general feature for associative algebra deformations [49]) and/or that are closed [22, 64, 63]. Since the algebras appearing in Definition 4.1 (and in noncommutative geometry) are not explicitly required to be algebras of operators, this shows that deformation quantization can be considered as a special quantization in Berezin's sense if the Planck constant is regarded as a formal parameter, and be cast [18] into the formalism of noncommutative geometry.

This allows Berezin to set the stage for the definition of a *functor of quantization* as a correspondence $(M, \Lambda) \rightsquigarrow \mathfrak{A}$. In order to achieve that, he has to look at the correspondence between morphisms of classical mechanics (M, Λ) and morphisms of "quantized algebras" $\mathfrak{A}$. Given two such algebras $\mathfrak{A}_i$ with defining sets E_i and algebras $A_h^{(i)}$, $i = 1, 2$, an algebra homomorphism $\psi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is said admissible if it is generated by homomorphisms ψ_h of the algebras $A_h^{(i)}$ (then necessarily $E_1 \subset E_2$).

He defines admissible isomorphisms in a similar way and looks at a set $\mathcal{E}$ of classical mechanics with a given quantization $\mathfrak{A}$. The correspondence $(M, \Lambda) \rightsquigarrow \mathfrak{A}$ is then called a functor of quantization if the expected diagram is commutative (see [10]). Similarly he can define the equivalence of two quantizations of the same classical mechanics (M, Λ) and look at the action of groups of motions (transformation groups of M). There is a notion of "naturalness" of a functor for special quantizations, but he shows in [9] that there is no quantization functor which is "natural" with respect to the category of all the mappings of classical mechanics – a result which is not so surprising for experts.

We shall not develop more that direction, however interesting it may be, since that was done in some extent by Berezin in his papers which are a kind of *symphonie inachevée*. More could of course be done, but should perhaps be considered in the light of deformation quantization, including the examples he treated – many of which have been treated in deformation quantization without comparison to his works. That direction of research is definitely worth considering.

5. Deformations and quantization

5.1. The Gerstenhaber theory of deformations of algebras. A concise formulation of a Gerstenhaber deformation (with formal series parameter ν) of an algebra (associative, Lie, bialgebra, etc.) is [47, 15]:

Definition 5.1. *A deformation of an algebra A over a field $\mathbb{K}$ is an algebra $\tilde{A}$ (flat) over $\mathbb{K}[[\nu]]$ such that $\tilde{A}/\nu\tilde{A} \approx A$. Two deformations $\tilde{A}$ and $\tilde{A}'$ are said equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$, and $\tilde{A}$ is called trivial if it is isomorphic to $A[[\nu]] = A \otimes_{\mathbb{K}} \mathbb{K}[[\nu]]$, that is, if $\tilde{A}$ is obtained from A by base field extension from $\mathbb{K}$ to $\mathbb{K}[[\nu]]$.*

Note that more general deformations exist [62]. For associative (resp. Lie) algebras, the above definition tells us that there exists a new product $*$ (resp. bracket $[\cdot,\cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{\cdot,\cdot\}$), this means that, for $u_1, u_2 \in A$ (we can extend this to $A[[\nu]]$ by $\mathbb{K}[[\nu]]$-linearity) we have the formal series expansions:

$$u_1 * u_2 = u_1 u_2 + \sum_{r=1}^{\infty} \nu^r C_r(u_1, u_2), \tag{5.1}$$

$$[u_1, u_2] = \{u_1, u_2\} + \sum_{r=1}^{\infty} \nu^r B_r(u_1, u_2), \tag{5.2}$$

where the C_r are Hochschild 2-cochains and the B_r are Chevalley–Eilenberg (skew-symmetric) 2-cochains such that for $u_1, u_2, u_3 \in A$ we have, denoting by $\mathcal{S}$ the summation over cyclic permutations, $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$ and $\mathcal{S}[[u_1, u_2], u_3] = 0$.

For a (topological) *bialgebra* (an associative algebra A where we have in addition a coproduct $\Delta : A \longrightarrow A \otimes A$ and the obvious compatibility relations), denoting by $\otimes_\nu$ the tensor product of $\mathbb{K}[[\nu]]$-modules, we can identify $\tilde{A}\,\hat{\otimes}_\nu \tilde{A}$ with $(A\,\hat{\otimes} A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some topology (e.g. projective for Fréchet nuclear topology on A). Then we have also a deformed coproduct $\tilde{\Delta} = \Delta + \sum_{r=1}^{\infty} \nu^r D_r$, $D_r \in \mathcal{L}(A, A\hat{\otimes}A)$, satisfying $\tilde{\Delta}(u_1 * u_2) = \tilde{\Delta}(u_1) * \tilde{\Delta}(u_2)$. In this context appropriate cohomologies can be introduced and there are natural additional requirements for Hopf algebras.

5.2. Deformation quantization. One recognizes in (4.4) a special case of (5.1), and similarly for the Moyal bracket. So, via a Weyl quantization map, the algebra of quantized observables can be viewed as a deformation of that of classical observables. The trace of an operator is replaced by the integral over the phase space of the corresponding function. It can be defined even in the absence of a (possibly generalized) Weyl mapping Ω, for which a formula similar to the Moyal trace of Sec. 4.1.1 holds.

What we call "deformation quantization" relates to (and generalizes) what in the conventional (operator) formulation are the Heisenberg picture and Weyl's quantization procedure. We want to stress that deformation quantization is not merely, as many think, "a reformulation of quantizing a mechanical system" [33], e.g. in the framework of Weyl quantization: *The process of quantization itself is a deformation.* Note that the framework of deformations contains more than those deformations that are related to quantization, but in our view *Quantization* $\subset$ *Deformation*.

Of course, one cannot argue with success. In order to convince (open-minded) physicists that deformation quantization is the right way to look at quantization in general situations, we had to treat in an *autonomous* manner significant physical examples, without recourse to the traditional operator formulation of quantum mechanics. After that it is a matter of practical feasibility of calculations, when there are Weyl and Wigner maps to intertwine between both formalisms, to choose to work with operators in Hilbert spaces or with functional analysis methods (distributions etc.) When there are no such maps (recent results on a fractional analytic index [59] point also to that direction) one does not have much choice but to work with functional analysis. Unless (what too many do) one works in an open chart and ignores the problems that may arise at the boundary of charts – or one tries and follows the difficult direction of research outlined by Berezin.

5.2.1. Spectrality in deformation quantization. The autonomous treatment was achieved in [5] with the paradigm of the harmonic oscillator and more, including the angular momentum and the hydrogen atom. The role of the unitary time evolution operator of a quantized system is played here by the "star exponential" of its classical Hamiltonian H (expressed as a usual exponential series but with "star powers" of $tH/i\hbar$, t being the time, and computed as a distribution both in phase space variables and in time). What corresponds to the spectrum of the quantum operator associated with H is the support of the Fourier–Stieltjes transform (in t) of the star exponential, which Laurent Schwartz called the spectrum of that distribution. We thus get the discrete spectrum $(n+\frac{\ell}{2})\hbar$, $n=0,1,\ldots$, with the right multiplicities (e.g. $\frac{1}{2}(n+1)(n+2)$ for $\ell=3$) of the *harmonic oscillator* $H=\frac{1}{2}(p^2+q^2)$ in $\mathbb{R}^\ell$. One can also obtain in this way the continuous spectrum $\mathbb{R}$ for the dilation generator pq. The eigenprojectors are given [5] by known special functions on the phase space (generalized Laguerre and hypergeometric, multiplied by some exponential).

Other examples can be brought to this case by functional manipulations [5]. For instance, the Casimir element of $\mathfrak{so}(\ell)$ representing the *angular momentum* has spectrum $n(n+(\ell-2))\hbar^2$. For the *hydrogen atom*, with Hamiltonian $H=\frac{1}{2}p^2-|q|^{-1}$, the Moyal product on $\mathbb{R}^8$ induces a star product on $W=T^*S^3$; the energy levels, solutions of $(H-E)*\varphi=0$, are found to be (as they should) $E=\frac{1}{2}(n+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E\in\mathbb{R}^+$ for the continuous spectrum. We thus have recovered, in a completely *autonomous* manner entirely within deformation quantization, the results of "conventional" quantum mechanics in these typical examples.

An advantage of spectral theory in deformation quantization is that it can be meaningful even in cases in which the corresponding operator, if there exists one via some Weyl mapping, is "not spectrable," e.g. is symmetric with deficiency indices (0,1) as is the case of the momentum operator $p=id/dx$ on $L^2[0,+\infty)$ which corresponds to a phase space with an infinite potential barrier at $x=0$. Unfortunately, that aspect of deformation quantization, however important for applications, has not been studied with the same intensity as its more abstract developments (sketched in Sec. 5.2.3 below).

5.2.2. Pseudospectra. In the 70's and 80's, numerical analysts invented at least 5 times, independently (see the history in the Pseudospectra Gateway `http://web.comlab.ox.ac.uk/projects/pseudospectra/`), a new notion to provide an analytical and graphical alternative for investigating nonnormal matrices and operators. It is now extensively developed (see, e.g., [71] and the 851 references therein), but may have remained ignored by most mathematicians if E. B. Davies (see, e.g., [24]), a mathematical physicist, had not found a way of relating the concept to semi-classical analysis, and thereby to interest pure mathematicians in a subject that is in rapid development. See, e.g., [25] where the authors show in particular that the resolvent is large inside the pseudospectrum. Roughly speaking, the notion of pseudospectrum provides thus a way to get around what I will dare to call the Procrustean bed of conventional spectral theory of operators by considering regions where the resolvent is large enough. The relation with spectral theory in deformation quantization remains to be studied.

5.2.3. Developments. Further examples were (already in [44]) and are being developed, including (see, e.g., [28, 29, 30]) in the direction of field theory. That has also led, which is natural in view of the essential role played by the interconnection between group theory and quantum mechanics almost from the beginning (see [72]), to the development of *star representations of Lie groups* as deformations of algebras of functions on coadjoint orbits (see the reviews quoted below and references therein). Here also some quantization of orbits appears, in particular (see [44]) in the fact that (even when a star representation can be built on a continuum of isomorphic orbits) the analytic form of the representations is considerably simplified when the orbit is adapted to the representation.

That aspect of deformation theory has in the past 30 years or so been extended considerably. It now includes general symplectic [26, 35, 64] and Poisson (finite-dimensional) manifolds [56, 17]. In symplectic manifolds, symplectic connections play an essential role; that led to the notion of *Fedosov manifolds* [46]. Results include *existence and classification* up to equivalence (in the sense of Definition 5.1). Equivalence classes are in one-to-one correspondence with equivalence classes of formal Poisson tensors $\sum_{k=0}^{\infty} \nu^k \Lambda_k$. For symplectic manifolds, these are simply formal series in ν with coefficients in the second de Rham cohomology (of dimension b_2) of the manifold. There are partial results for infinite-dimensional manifolds, the case of interest for quantum field theory which has been a constant preoccupation for Berezin since [8], including string and M-theory [33], for "manifolds with singularities" [45], and for algebraic varieties [57]. Work is still in progress for these and other difficult cases.

Potentially important issues are those of the uniqueness of quantum mechanics and of deformations of deformations: In what sense does "the buck stop there" (that was shown to hold, in a natural framework, in [4]), or, in other words, can one deform further in some category? In particular, is there a further deformation of quantum mechanics? Maybe some form of determinism could be reinstated there. Then, what happens with Bell's inequalities [6]? Their violation was experimentally established in [2], thereby confirming the usual quantum mechanical formalism. But that does not exclude nonlocal hidden variables [40]. And one wonders what happens in topologically nontrivial phase spaces (with $b_2 > 0$) when many nonequivalent quantizations are theoretically possible, especially in view of the huge amount of inequivalent Bell-type inequalities (related to polytopes [3]) that have appeared in quantum information theory.

The domain has many far-reaching ramifications in both mathematics and physics. One which we have not yet touched here but which is worth mentioning briefly is the connection with path integrals, a subject of strong interest for Berezin (see, e.g., [13] and [11], one of his last publications, where incidentally he mentions for the first time our publications on deformations of Poisson brackets that preceded deformation quantization). It was known almost from the beginning [65] that, in Weyl ordering, the Feynman path integral (FPI) is the Fourier transform over the momentum of the star exponential. Furthermore, for normal ordering [28], the star exponential coincides with the FPI, and in [17] the Kontsevich star product for Poisson manifolds is written as a path integral.

We shall not here go further in the details of the developments of deformation quantization, for which we refer to the original papers and a number of recent reviews (cf., e.g., [16, 31, 68, 69]). In the next section we briefly overview a couple of famous "avatars" and suggest a "functional" definition of quantization.

6. Avatars of deformation quantization and epilogue

In the 80's two major developments occurred, that turned out to have some strong connection with deformation quantization but that unfortunately Berezin could not see – even though in retrospect one can trace some origin earlier.

6.1. Quantum groups. Quantum groups can (since Drinfeld [34]) be viewed as an avatar of deformation quantization when the category in question is that of Hopf algebras, in this case an algebra N of functions on a Poisson–Lie group G or a topologically dual algebra N', "completed" enveloping algebra of a Lie bialgebra. The topological theory is well established in the semi-simple case [15, 16], where one has preferred deformations (only the product, or the coproduct, are deformed), but the approach is general. [The approach of Jimbo [54] is slightly different but the general features remain the same.] The main algebras considered here are $C^\infty(G)$, the algebra of smooth functions on a (semi-simple) Poisson–Lie group G, its dual (which contains distributions with support at the identity, i.e., the enveloping algebra $\mathsf{U}\mathfrak{g}$ of G), or a subspace of $C^\infty(G)$ such as the algebra generated by the coefficients of a suitable set of unitary irreducible representations of G, and its dual (a space of generalized distributions). We shall not develop this aspect here, referring to the original papers, reviews such as [41, 16], and references quoted therein.

6.2. Noncommutative geometry and quantized manifolds.

6.2.1. Noncommutative Geometry versus Deformation Quantization. Noncommutative differential geometry (NCG) is a novel approach to geometry, in part originating from, and aimed at, applications in physics. It was founded in the early eighties by Alain Connes [18] on the basis of his fundamental works in operator algebras. Quoting Connes [19]: "*Its origin is twofold. On the one hand, there is a wealth of spaces whose coordinate algebra is no longer commutative but which have obvious relevance in physics or mathematics. On the other hand, the stretching of geometric thinking imposed by passing to noncommutative spaces forces one to rethink most of our familiar notions.*" It is now a very active branch of mathematics with current and potential applications to a number of domains in physics, from solid state to quantization of gravity. The strategy is to formulate usual differential geometry in a somewhat unusual manner, using in particular operator algebras and related concepts, so as to be able to "plug in" noncommutativity in a natural way. Algebraic tools such as K-theory and (cyclic) cohomology and homology play an important role. Noncommutativity is then introduced in particular by deforming algebras used in the equivalent description of usual geometry and modifying accordingly associated notions. A similar strategy can be found in the approach to noncommutative algebraic geometry by noncommutative motives, now being developed by Kontsevich.

NCG can thus be viewed (again, if I may write so) as a very elaborate avatar of deformation quantization. As for quantum groups, a richer structure is considered there, bringing important conceptual and technical differences. For instance, the Hochschild cohomology complex is complemented by another, leading to cyclic cohomology. But the associative algebra formulation of deformation quantization can be adapted to that context. As a matter of fact, algebras of functions endowed with closed star products [22] are an important example [18] of the NCG approach; (strongly) closed star products are those for which integration on the manifold defines a trace; they exist [64], and all star products on symplectic manifolds are equivalent to closed ones [63].

Conversely in a way, the Hopf algebraic approach to renormalization and Feynman graphs developed by Connes and Kreimer (see, e.g., [23]) may in part be understood heuristically as a "cohomological renormalization" where one would pass (for infinitely many degrees of freedom) from e.g. the normal star product to another, by subtraction of some cocycle, in order to get a finite answer (see [29] for a preliminary example).

6.2.2. Noncommutative manifolds. A very elaborate beginning of the theory of noncommutative manifolds, especially in dimension 3 and 4, can be found in [21] (and work in progress); they play an increasing role in modern theoretical physics, including string and M-theory, where star products play a role [32]. Prior to these papers only a few significant examples, mainly noncommutative tori and the (flat) Moyal deformation, of noncommutative manifolds (spectral triples) were known. First a noncommutative analog of the 4-sphere (with a modified Dirac operator) was introduced. Then a complete list of 3-dimensional spherical manifolds was given. After that the structure of these noncommutative spaces was analyzed and their moduli space described.

The notion of large N limit appears frequently in the theoretical physics literature in various contexts, in particular in string theory. Mathematically, one formulation is that, in some sense, the algebras of complex $N \times N$ matrices "converge" to (functions on) the sphere S^2 for $N \to \infty$, in a way which suggests that one is dealing with "quantum metric spaces." Recently Rieffel [66] has shown how to give a precise meaning to that notion by means of Berezin quantization [10] using coherent states, and of the quantum Gromov–Hausdorff distance to express the limit.

We have mentioned at the end of Sec. 3 that our Minkowskian spacetime can be deformed to anti de Sitter AdS_4. There massless particles become composite (of singletons, and for the photon that is compatible with QED [36]). It is becoming increasingly fashionable to "quantize" Minkowski space-time. So why not go one step further [69] and assume that in some regions, at very small distances (e.g. the Planck length) our universe is deformed, not only by being curved to AdS_4 but also by being quantized? At this stage, the idea is still largely Science Fiction as far as physics is concerned, but the mathematical problems are interesting in themselves and are being studied.

6.3. Epilogue. We are now in position to give a possibly surprising answer (*"both"*) to the question: Is quantization a deformation or a functor? That can be seen as a schizophrenic answer, but physical reality has multiple facets. That is even more true when dealing with quantum theories. Louis de Broglie, in establishing his *mécanique ondulatoire*, started from the idea that waves and particles are two manifestations of the same physical reality. The particle aspect is more related to the correspondence functor (and, more traditionally, to the Schrödinger representation), while the wave aspect, when taking into account quantum effects, is more related to the deformation quantization aspect (and, more traditionally, to the Heisenberg picture).

In our view both are complementary, not contradictory. In a way, that might settle the continuing quest for hidden variables. Einstein and de Broglie were not at ease with the probabilistic Copenhagen interpretation, in spite of its successes. For Einstein, "God does not play dice with the Universe." Recently 't Hooft [52] made challenging contributions to that ancient controversy. Our answer goes in the same direction.

Another too often overlooked question is the fact that quantization started (e.g. when looking at atomic spectra) with discretization of continua. That is a starting point in Connes' motivations [18, 19, 20] for noncommutative geometry. Recently Jürg Fröhlich [43], studying the mean-field limit of quantum many-body systems, showed how (here also a "large N limit" appears), starting from certain Hamiltonian continuum theories of matter, the passage to an atomistic description can be understood as a deformation quantization.

To conclude, we may suggest the following "functional definition":

Definition 6.1. *Quantization is expressed by the passage from commutative to noncommutative mathematical structures, having as a consequence a discretization of continua.*

References

1. G. S. Agarwal and E. Wolf, *Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics* I, II, III. — Phys. Rev. **D2** (1970), 2161–2186, 2187–2205, 2206–2225.
2. A. Aspect, R. Dalibard, and G. Roger, *Experimental test of Bell's inequalities using time-varying analyzers.* — Phys. Rev. Lett. **49** No. 25 (1982), 1804–1807.
3. D. Avis, H. Imai, T. Ito, and Y. Sasaki, *Two-party Bell inequalities derived from combinatorics via triangular elimination.* — J. Phys. A **38**, No. 50 (2005), 10971–10987 (`quant-ph/0505060`).
4. H. Basart, M. Flato, A. Lichnerowicz, and D. Sternheimer, *Deformation theory applied to quantization and statistical mechanics.* — Lett. Math. Phys. **8** (1984), 483–494.
5. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization I. Deformations of symplectic structures.* — Ann. Physics **111** (1978), 61–110; II. Physical applications, *ibid*, 111–151.
6. J. S. Bell, *On the problem of hidden variables in quantum mechanics.* — Rev. Modern Phys. **38** (1966), 447–452.
7. F. A. Berezin, *Canonical operator transformation in representation of secondary quantization.* — Dokl. Akad. Nauk SSSR **137** (1961), 311–314 (Russian); translated as Soviet Physics Dokl. **6** (1961), 212–215.
8. F. A. Berezin, *The method of second quantization.* Nauka, Moscow (1965). [English transl., Academic Press 1966]
9. F. A. Berezin, *Quantization.* — Izv. Akad. Nauk SSSR, Ser. Mat. **38** (1974), 1116–1175.
   F. A. Berezin, *Quantization in complex symmetric spaces.* — Izv. Akad. Nauk SSSR, Ser. Mat. **39** (1975), 363–402, 472. [English translations: Math. USSR-Izv. **38** (1975), 1109–1165 and **39** (1976), 341–379.]
10. F. A. Berezin, *General concept of quantization.* — Comm. Math. Phys. **40** (1975), 153–174.
11. F. A. Berezin, *Feynman path integrals in a phase space.* — Usp. Fiz. Nauk **132** (1980), 493–548.
12. F. A. Berezin, *Introduction to superanalysis.* — edited and with a foreword by A. A. Kirillov, with an appendix by V. I. Ogievetsky, translated from the Russian by J. Niederle and R. Kotecký, translation edited by D. Leĭtes. Mathematical Physics and Applied Mathematics, **9**, xii+424 pp., D. Reidel Publishing Co., Dordrecht (1987).
13. F. A. Berezin and M. A. Shubin, *The Schrödinger equation.* — xi+555 pages, Kluwer Acad. Publ., Dordrecht (1991).
14. F. A. Berezin and D. A. Leites, *Supermanifolds.* — Dokl. Akad. Nauk SSSR **224**, No. 3 (1975), 505–508.
15. P. Bonneau, M. Flato, M. Gerstenhaber, and G. Pinczon, *The hidden group structure of quantum groups: strong duality, rigidity and preferred deformations.* — Comm. Math. Phys. **161** (1994), 125–156.
16. P. Bonneau, M. Gerstenhaber, A. Giaquinto, and D. Sternheimer, *Quantum groups and deformation quantization: explicit approaches and implicit aspects.* — J. Math. Phys. **45** (2004), 3703–3741.

17. A. S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula.* — Comm. Math. Phys. **212** (2000), 591–611 (math.QA/9902090). *On the Globalization of Kontsevich's Star Product and the Perturbative Poisson Sigma Model.* — Prog. Theor. Phys. Suppl. **144** (2001), 38–53 (hep-th/0111028).
18. A. Connes, *Noncommutative Geometry.* Academic Press, San Diego (1994). *Noncommutative differential geometry.* — Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.
19. A. Connes, *A short survey of noncommutative geometry.* — J. Math. Phys. **41** (2000), 3832–3866 (hep-th/0003006).
20. A. Connes, *Cyclic cohomology, noncommutative geometry and quantum group symmetries.* — Noncommutative geometry 1–71, Lecture Notes in Math. **1831**, Springer, Berlin (2004).
21. A. Connes and M. Dubois-Violette, *Moduli space and structure of noncommutative 3-spheres.* — Lett. Math. Phys. **66** (2003), 99–121 (math.QA/0308275).
22. A. Connes, M. Flato, and D. Sternheimer, *Closed star-products and cyclic cohomology.* — Lett. Math. Phys. **24** (1992), 1–12.
23. A. Connes and D. Kreimer, *From local perturbation theory to Hopf and Lie algebras of Feynman graphs.* — EuroConférence Moshé Flato 2000, Part I (Dijon), Lett. Math. Phys. **56** (2001), 3–15.
24. E. B. Davies, *Semi-classical states for non-self-adjoint Schrödinger operators.* — Comm. Math. Phys. **200** (1999), 35–41. *Semi-classical analysis and pseudo-spectra.* — J. Differential Equations **216** (2005), 153–187.
25. N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudospectra of semiclassical (pseudo-) differential operators.* — Comm. Pure Appl. Math. **57** (2004), 384–415.
26. M. De Wilde and P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds.* — Lett. Math. Phys. **7** (1983), 487–496.
27. P. A. M. Dirac, *Lectures on Quantum Mechanics.* Belfer Graduate School of Sciences Monograph Series No. 2, Yeshiva University, New York (1964). *Generalized Hamiltonian dynamics.* — Canad. J. Math. **2** (1950), 129–148.
28. J. Dito, *Star-product approach to quantum field theory: the free scalar field.* — Lett. Math. Phys. **20** (1990), 125–134. *Star-products and nonstandard quantization for K-G equation.* — J. Math. Phys. **33** (1992), 791–801.
29. J. Dito, *An example of cancellation of infinities in star-quantization of fields.* — Lett. Math. Phys. **27** (1993), 73–80. *Star-produits en dimension infinie : le cas de la théorie quantique des champs.* Thesis, Université de Bourgogne, Dijon (1993).
30. G. Dito (=J. Dito), *Deformation quantization of covariant fields.* — Deformation quantization (Strasbourg 2001), IRMA Lectures in Math. Theoret. Phys. **1** (2002), 55–66, Walter de Gruyter, Berlin (math.QA/0202271).
31. G. Dito and D. Sternheimer, *Deformation quantization: genesis, developments and metamorphoses*, Deformation quantization (Strasbourg 2001), IRMA Lect. Math. Theor. Phys., **1** (2002), 9–54, Walter de Gruyter, Berlin (math.QA/0201168).
32. M. R. Douglas, H. Liu, G. Moore, and B. Zwiebach, *Open string star as a continuous Moyal product.* — J. High Energy Phys. 0204 (2002), No. 22, 27 pp. (hep-th/0202087).

33. M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory.* — Rev. Mod. Phys. **73** (2001), 977–1029 (`hep-th/0106048`).
34. V. Drinfeld, *Quantum Groups.* Proc. ICM86 Berkeley **1** (1987), 101–110, Amer. Math. Soc., Providence.
35. B. V. Fedosov, *A simple geometrical construction of deformation quantization.* — J. Diff. Geom. **40** (1994), 213–238. *Deformation quantization and index theory.* Mathematical Topics **9**, Akademie Verlag, Berlin (1996). *The Atiyah-Bott-Patodi method in deformation quantization.* — Comm. Math. Phys. **209** (2000), 691–728.
36. M. Flato and C. Fronsdal, *Composite Electrodynamics.* — J. Geom. Phys. **5** (1988), 37–61.
37. M. Flato, C. Frønsdal, and D. Sternheimer, *Singletons, physics in AdS universe and oscillations of composite neutrinos.* — Lett. Math. Phys. **48** (1999), 109–119.
38. M. Flato, L. K. Hadjiivanov, and I. T. Todorov, *Quantum deformations of singletons and of free zero-mass fields.* — Found. Phys. **23** (1993), 571–586.
39. M. Flato, P. Hillion, and D. Sternheimer, *Un groupe de type Poincaré associé à l'équation de Dirac-Weyl.* — C. R. Acad. Sci. Paris **264**, 85–88 (1967).
M. Flato and P. Hillion, *Poincaré-like group associated with neutrino physics, and some applications.* — Phys. Rev. **D1** (1970), 1667–1673.
40. M. Flato, C. Piron, J. Gréa, D. Sternheimer, and J. P. Vigier, *Are Bell's inequalities concerning hidden variables really conclusive?* — Helv. Phys. Acta **48**, No. 2 (1975), 219–225.
41. M. Flato and D. Sternheimer, *Topological Quantum Groups, star products and their relations.* — St. Petersburg Mathematical Journal (Algebra and Analysis) **6** (1994), 242–251.
42. A. Froelicher and A. Nijenhuis, *A theorem on stability of complex structures.* — Proc. Nat. Acad. Sci USA **43** (1957), 239–241.
43. J. Fröhlich, *Atomism and quantization*, talk at the Simonfest. Caltech, 27 March 2006 (`http://math.caltech.edu/SimonFest/Abstracts/frohlich_abst.pdf`).
44. C. Fronsdal, *Some ideas about quantization.* — Rep. Math. Phys. **15** (1978), 111–145.
F. Bayen and C. Fronsdal, *Quantization on the sphere.* — J. Math. Phys. **22** (1981), 1345–1349.
45. C. Frønsdal (with an appendix by M. Kontsevich), *Quantization on Curves*, `math-ph/0507021` (to be published in Lett. Math. Phys.).
46. I. Gelfand, V. Retakh, and M. Shubin, *Fedosov manifolds.* — Adv. Math. **136** (1998), 104–140.
47. M. Gerstenhaber, *On the deformation of rings and algebras.* — Ann. Math. **79** (1964), 59–103; (II) *ibid.* **84** (1966) 1–19; and (IV), *ibid.* **99** (1974), 257–276.
48. M. Gerstenhaber, *On the deformation of sheaves of rings.* — Global Analysis (Papers in Honor of K. Kodaira) pp. 149–157, Univ. Tokyo Press, Tokyo (1969).
49. M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, in M. Hazewinkel and M. Gerstenhaber (eds.), *Deformation Theory of Algebras and Structures and Applications.* — NATO ASI Ser. C 247 (1988), 11–264, Kluwer Acad. Publ., Dordrecht.
50. J. W. v. Goethe, *Maximen und Reflexionen.* Nr. 1005, in Werke, 60B (1827–1842). `http://www.wissen-im-netz.info/literatur/goethe/maximen/1-16.htm#1005`.

51. A. Groenewold, *On the principles of elementary quantum mechanics.* — Physica **12** (1946), 405–460.
52. G. 't Hooft, *How does God play dice? (Pre-)determinism at the Planck scale*, an essay in honour of John Bell in: *Quantum [un]speakables* (Vienna, 2000), 307–316, Springer, Berlin (2002) (`hep-th/0104219`).
Determinism beneath Quantum Mechanics. Conf. Proceedings, "Quo Vadis Quantum Mechanics," Philadelphia 2002 (`quant-ph/0212095`).
53. E. Ínönü and E. P. Wigner, *On the contraction of groups and their representations.* — Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 510–524.
54. M. Jimbo, *A q-difference algebra of $\mathcal{U}(\mathfrak{g})$ and the Yang–Baxter equation.* — Lett. Math. Phys. **10** (1985), 63–69.
55. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures.* — Ann. Math. **67** (1958), 328–466.
56. M. Kontsevich, *Deformation quantization of Poisson manifolds.* — Lett. Math. Phys. **66** (2003), 157–216 (`q-alg/9709040`).
Operads and motives in deformation quantization. — Lett. Math. Phys. **48** (1999), 35–72 (`math.QA/9904055`).
57. M. Kontsevich, *Deformation quantization of algebraic varieties.* — Lett. Math. Phys. **56** (2001), 271–294 (`math.AG/0106006`).
58. B. Kostant, *Quantization and unitary representations.* — Lecture Notes in Math. **170** (1970), 87–208, Springer Verlag (Berlin).
On the definition of quantization. — pp. 187–210 in: *Géométrie symplectique et physique mathématique* (Colloq. Int. CNRS, No. 237), Éds. CNRS, Paris 1975.
Graded manifolds, graded Lie theory, and prequantization. Differential geometrical methods in mathematical physics. — Lecture Notes in Math. **570** (1977), 177–306, Springer (Berlin).
59. V. Mathai, R. B. Melrose, and I. M. Singer, *Fractional analytic index*, J. Diff. Geom. (in press); `math.DG/0402329`.
60. R. A. Minlos, *Felix Alexandrovich Berezin (A Brief Scientific Biography).* — Contemporary Mathematical Physics, F. A. Berezin Memorial Volume, R. L. Dobrushin, R. A. Minlos, M. A. Shubin, A. M. Vershik eds. American Mathematical Society Translations, Ser. 2, Volume 175, 1996, 1–13. Reproduced (with minor corrections) in Lett. Math. Phys. **74** (2005) 5–19.
61. J. E. Moyal, *Quantum mechanics as a statistical theory.* — Proc. Cambridge Phil. Soc. **45** (1949), 99–124.
62. F. Nadaud, *Generalized deformations, Koszul resolutions, Moyal Products.* — Reviews Math. Phys. **10** (5) (1998), 685–704. *Thèse*, Dijon (janvier 2000).
63. R. Nest and B. Tsygan, *Algebraic index theorem.* — Comm. Math. Phys. **172** (1995), 223–262.
Algebraic index theorem for families. — Adv. Math. **113** (1995), 151–205.
Formal deformations of symplectic manifolds with boundary. — J. Reine Angew. Math. **481** (1996), 27–54.
64. H. Omori, Y. Maeda, and A. Yoshioka, *Weyl manifolds and deformation quantization.* — Adv. Math. **85** (1991), 225–255.
Existence of a closed star product. — Lett. Math. Phys. **26** (1992), 285–294.
65. Pankaj Sharan, *Star-product representation of path integrals.* — Phys. Rev. D **20** (1979), 414–418.

66. M. Rieffel, *Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance.* — Mem. Amer. Math. Soc. **168**, No. 796 (2004), 67–91 (`math.OA/0108005`).
67. I. E. Segal, *A class of operator algebras which are determined by groups.* — Duke Math. J. **18** (1951), 221–265.
68. D. Sternheimer, *Deformation quantization: Twenty years after.* — pp. 107–145 in: J. Rembieliński, (ed.), Particles, Fields, and Gravitation (Lodz 1998), AIP Press, New York 1998 (`math.QA/9809056`).
In retrospect: A personal view on Moshé Flato's scientific legacy, pp. 31–41 in: *Conférence Moshé Flato 1999*, Math. Phys. Studies **21**, Kluwer Acad. Publ., Dordrecht (2000).
69. D. Sternheimer, *Quantization is deformation.* — pp. 331–352 in: *Noncommutative Geometry and Representation Theory in Mathematical Physics* (J. Fuchs et al. eds.), Contemporary Mathematics 391, Amer. Math. Soc. (2005).
70. D. Sternheimer, *Quantization: deformation and/or functor?* — Lett.Math.Phys. **74** (2005), 293–309.
71. L. N. Trefethen and M. Embree, *Spectra and pseudospectra. The behavior of nonnormal matrices and operators.* xviii+606 pp. Princeton University Press, Princeton, NJ (2005).
72. H. Weyl, *The theory of groups and quantum mechanics.* Dover, New-York 1931, edited translation of *Gruppentheorie und Quantenmechanik*, Hirzel Verlag, Leipzig 1928.
Quantenmechanik und Gruppentheorie. — Z. Physik **46** (1927), 1–46.
73. E. P. Wigner, *Quantum corrections for thermodynamic equilibrium.* — Phys. Rev. **40** (1932), 749–759.
74. N. M. J. Woodhouse, *Geometric quantization* (2nd edition). Oxford Science Publications, Oxford Mathematical Monographs. Oxford University Press, New York (1992).

Institut de Mathématiques de Bourgogne,
Université de Bourgogne, Dijon, France
Department of Mathematics,
Keio University, Yokohama, Japan

E-mail: Daniel.Sternheimer@u-bourgogne.fr

Поступило 27 июня 2006 г.