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Dedicated to our teacher Mehpare Bilhan

FESENKO RECIPROCITY MAP

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In recent papers Fesenko has defined the non-Abelian local reciprocity map for every totally ramified arithmetically profinite (APF) Galois extension of a given local field K, by extending the work of Hazewinkel and Neukirch–Iwasawa. The theory of Fesenko extends the previous non-Abelian generalizations of local class field theory given by Koch–de Shalit and by A. Gurevich. In this paper, which is research-expository in nature, we give a detailed account of Fesenko's work, including all the skipped proofs.

In a series of very interesting papers [1–3], Fesenko defined the non-Abelian local reciprocity map for every totally ramified arithmetically profinite (APF) Galois extension of a given local field K by extending the work of Hazewinkel [8] and Neukirch–Iwasawa [15]. "Fesenko theory" extends the previous non-Abelian generalizations of local class field theory given by Koch and de Shalit in [13] and by A. Gurevich in [7].

In this paper, which is research-expository in nature, we give a very detailed account of Fesenko's work [1–3], thereby complementing those papers by including all the proofs. Let us describe how our paper is organized. In the first part, we briefly review the Abelian local class field theory and the construction of the local Artin reciprocity map, following the Hazewinkel method and the Neukirch-Iwasawa method. In parts 3 and 4, we follow [4–6] and [17] to review

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the theory of APF-extensions over K, and sketch the construction of Fontaine-Wintenberger's field of norms $\mathbb{X}(L/K)$ attached to an APF-extension L/K. In order to do so, in part 2 we briefly review the ramification theory of K. Finally, in part 5, we give a detailed construction of the Fesenko reciprocity map $\Phi_{L/K}^{(\varphi)}$ defined for any totally ramified and APF-Galois extension L over K under the assumption that $\boldsymbol{\mu}_p(K^{\text{sep}}) \subset K$, where $p = \text{char}(\kappa_K)$, and investigate the functorial and ramification-theoretic properties of the Fesenko reciprocity maps defined for totally ramified and APF-Galois extensions over K.

In a related paper (see [10]), we shall extend Fesenko's construction to *any* Galois extension of K (in a fixed K^{sep}), and construct the non-Abelian local class field theory. Thus, we feel that the present paper together with [1–3] should be viewed as the technical and theoretical background, an introduction, as well as an appendix to the paper [10] on "generalized Fesenko theory". A similar theory was announced by Laubie in [14], which is an extension of the paper [13] by Koch and de Shalit. The relationship of the Laubie theory with our generalized Fesenko theory will also be investigated in [10].

Notation. Throughout this work, K denotes a local field (a complete discrete valuation field) with finite residue field $O_K/\mathfrak{p}_K =: \kappa_K$ of $q_K = q = p^f$ elements with p a prime number, where O_K denotes the ring of integers in K with a unique maximal ideal $\mathfrak{p}_K$. Let $\boldsymbol{\nu}_K$ denote the corresponding normalized valuation on K (normalized by $\boldsymbol{\nu}_K(K^\times) = \mathbb{Z}$), and let $\widetilde{\boldsymbol{\nu}}$ be a unique extension of $\boldsymbol{\nu}_K$ to a fixed separable closure K^{sep} of K. For any subextension L/K of K^{sep}/K, the normalized form of the valuation $\widetilde{\boldsymbol{\nu}}\mid_L$ on L will be denoted by $\boldsymbol{\nu}_L$. Finally, we let G_K denote the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$.

§1. Abelian local class field theory

Let K be a local field. We fix a separable closure K^{sep} of K. Let G_K denote the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ of K. By the construction of absolute Galois groups, G_K is a profinite topological group with respect to the Krull topology. Now, let G_K^{ab} denote the maximal Abelian Hausdorff quotient group G_K/G_K' of the topological group G_K, where G_K' denotes the closure of the $1st$-commutator subgroup $[G_K, G_K]$ of G_K.

Recall that the Abelian local class field theory for the local field K establishes a unique natural algebraic and topological isomorphism

$$\alpha_K : \widehat{K^\times} \xrightarrow{\sim} G_K^{ab},$$

where the topological group $\widehat{K^\times}$ is the profinite completion of the multiplicative group $K^\times$, satisfying the following conditions.

(1) Let W_K denote the Weil group of K. Then
$$\alpha_K(K^\times) = W_K^{ab}.$$
(2) For every Abelian extension L/K (always assumed to be a subextension of K^{sep}/K, where a separable closure K^{sep} of K is fixed throughout the remainder of the text), the surjective and continuous homomorphism
$$\alpha_{L/K} : \widehat{K^\times} \xrightarrow{\alpha_K} G_K^{ab} \xrightarrow{\mathrm{res}_L} \mathrm{Gal}(L/K)$$
satisfies
$$\ker(\alpha_{L/K}) = N_{L/K}(\widehat{L^\times}) = \bigcap_{K \underset{\text{finite}}{\subseteq} F \subseteq L} N_{F/K}(\widehat{F^\times}) =: \mathcal{N}_L.$$
(3) For each Abelian extension L/K, the mapping
$$L \mapsto \mathcal{N}_L$$
determines a bijective correspondence
$$\{L/K : \text{Abelian}\} \leftrightarrow \{\mathcal{N} : \mathcal{N} \underset{\text{closed}}{\leq} \widehat{K^\times}\}$$
that satisfies the following conditions: for every Abelian extension L, L_1, and L_2 over K,
  (i) L/K is a finite extension if and only if $\mathcal{N}_L \underset{\text{open}}{\leq} \widehat{K^\times}$ (this is equivalent to $(\widehat{K^\times} : \mathcal{N}_L) < \infty$);
  (ii) $L_1 \subseteq L_2 \Leftrightarrow \mathcal{N}_{L_1} \supseteq \mathcal{N}_{L_2}$;
  (iii) $\mathcal{N}_{L_1L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}$;
  (iv) $\mathcal{N}_{L_1 \cap L_2} = \mathcal{N}_{L_1}\mathcal{N}_{L_2}$.
(4) (Ramification theory)[1]. Let L/K be an Abelian extension. For every integer $0 \leq i \in \mathbb{Z}$ and every real number $\nu \in (i-1, i]$,
$$x \in U_K^i \mathcal{N}_L \Leftrightarrow \alpha_{L/K}(x) \in \mathrm{Gal}(L/K)^\nu,$$
where $x \in \widehat{K^\times}$.
(5) (Functoriality). Let L/K be an Abelian extension.
  (i) For $\gamma \in \mathrm{Aut}(K)$,
$$\alpha_K(\gamma(x)) = \widetilde{\gamma}\alpha_K(x)\widetilde{\gamma}^{-1}$$
  for every $x \in \widehat{K^\times}$, where $\widetilde{\gamma} : K^{ab} \to K^{ab}$ is any automorphism of the field K^{ab} satisfying $\widetilde{\gamma}\mid_K = \gamma$;

[1]We shall review the higher-ramification subgroups $\mathrm{Gal}(L/K)^\nu$ of $\mathrm{Gal}(L/K)$ (in upper numbering) in the next section.

(ii) under the condition $[L:K]<\infty$,

$$\alpha_L(x)\mid_{K^{ab}}=\alpha_K(N_{L/K}(x))$$

for every $x\in\widehat{L^\times}$;

(iii) under the condition $[L:K]<\infty$,

$$\alpha_L(x)=V_{K\to L}(\alpha_K(x))$$

for every $x\in\widehat{K^\times}$, where $V_{K\to L}:G_K^{ab}\to G_L^{ab}$ is the group-theoretic transfer homomorphism (*Verlagerung*).

This unique algebraic and topological isomorphism $\alpha_K:\widehat{K^\times}\to G_K^{ab}$ is called the *local Artin reciprocity map of* K.

There are many constructions of the local Artin reciprocity map of K, including the *cohomological* and *analytical* constructions. Now, in the remaining part of this section, we shall review the construction of the local Artin reciprocity map $\alpha_K:\widehat{K^\times}\to G_K^{ab}$ of K, following Hazewinkel [8] and Iwasawa–Neukirch [12, 15]. As usual, let K^{nr} denote the maximal unramified extension of K. It is well known that K^{nr} is not a complete field with respect to the valuation $\nu_{K^{nr}}$ on K^{nr} induced from the valuation ν_K of K. Let $\widetilde{K}$ denote the completion of K^{nr} with respect to the valuation $\nu_{K^{nr}}$ on K^{nr}. For a Galois extension L/K, put $L^{nr}=LK^{nr}$ and $\widetilde{L}=L\widetilde{K}$. For each $\tau\in\mathrm{Gal}(L/K)$, we choose $\tau^*\in\mathrm{Gal}(L^{nr}/K)$ in such a way that

(1) $\tau^*\mid_L=\tau$;
(2) $\tau^*\mid_{K^{nr}}=\varphi^n$ for some $0<n\in\mathbb{Z}$, where $\varphi\in\mathrm{Gal}(K^{nr}/K)$ denotes the (arithmetic) Frobenius automorphism of K.

The fixed field $(L^{nr})^{\tau^*}=\{x\in L^{nr}:\tau^*(x)=x\}$ of this chosen $\tau^*\in\mathrm{Gal}(L^{nr}/K)$ in L^{nr} will be denoted by Σ_{τ^*}; we have $[\Sigma_{\tau^*}:K]<\infty$.

The Iwasawa–Neukirch mapping

$$\iota_{L/K}:\mathrm{Gal}(L/K)\to K^\times/N_{L/K}(L^\times)$$

is then defined by

$$\iota_{L/K}:\tau\mapsto N_{\Sigma_{\tau^*}/K}(\pi_{\Sigma_{\tau^*}})\mod N_{L/K}(L^\times)$$

for every $\tau\in\mathrm{Gal}(L/K)$, where $\pi_{\Sigma_{\tau^*}}$ denotes any prime element of Σ_{τ^*}.

Suppose now that, moreover, the Galois extension L/K is a totally ramified and finite extension. We introduce a subgroup $\widetilde{V}(L/K)$ in the unit group $U_{\widetilde{L}}=O_{\widetilde{L}}^\times$ of the ring of integers $O_{\widetilde{L}}$ of the local field $\widetilde{L}$ by

$$\widetilde{V}(L/K)=\left\langle u^{\sigma-1}:u\in U_{\widetilde{L}},\ \sigma\in\mathrm{Gal}(L/K)\right\rangle.$$

Then the homomorphism

$$\theta : \mathrm{Gal}(L/K) \to U_{\widetilde{L}}/\widetilde{V}(L/K)$$

defined by

$$\theta : \sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L} \mod \widetilde{V}(L/K)$$

for every $\sigma \in \mathrm{Gal}(L/K)$, makes the triangle

$$\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[dd, "\text{can.}"'] \arrow[dr, "\theta"] & \\
& U_{\widetilde{L}}/\widetilde{V}(L/K) \\
\mathrm{Gal}(L/K)^{ab} \arrow[ur, "\theta_o"'] &
\end{tikzcd}$$

commutative. The quotient $U_{\widetilde{L}}/\widetilde{V}(L/K)$ sits in the Serre short exact sequence

$$1 \to \mathrm{Gal}(L/K)^{ab} \xrightarrow{\theta_o} U_{\widetilde{L}}/\widetilde{V}(L/K) \xrightarrow{N_{\widetilde{L}/\widetilde{K}}} U_{\widetilde{K}} \to 1. \tag{1.1}$$

Let $V(L/K)$ denote the subgroup in the unit group $U_{L^{nr}}$ of the ring of integers $O_{L^{nr}}$ of the maximal unramified extension L^{nr} of the local field L defined by

$$V(L/K) = \left\langle u^{\sigma-1} : u \in U_{L^{nr}},\ \sigma \in \mathrm{Gal}(L/K) \right\rangle.$$

The quotient $U_{L^{nr}}/V(L/K)$ sits in the Serre short exact sequence

$$1 \to \mathrm{Gal}(L/K)^{ab} \xrightarrow{\theta_o} U_{L^{nr}}/V(L/K) \xrightarrow{N_{L^{nr}/K^{nr}}} U_{K^{nr}} \to 1. \tag{1.2}$$

As before, let $\varphi \in \mathrm{Gal}(K^{nr}/K)$ denote the Frobenius automorphism of K. We fix any extension of the automorphism φ of K^{nr} to an automorphism of L^{nr}, denoted again by φ. Now, for any $u \in U_K$, there exists $v_u \in U_{L^{nr}}$ such that $u = N_{L^{nr}/K^{nr}}(v_u)$. Then the relation

$$N_{L^{nr}/K^{nr}}(\varphi(v_u)) = \varphi(N_{L^{nr}/K^{nr}}(v_u)) = \varphi(u) = u,$$

combined with the Serre short exact sequence, yields the existence of $\sigma_u \in \mathrm{Gal}(L/K)^{ab}$ satisfying

$$\theta_o(\sigma_u) = \frac{\sigma_u(\pi_L)}{\pi_L} = \frac{v_u}{\varphi(v_u)}.$$

The Hazewinkel mapping

$$h_{L/K} : U_K/N_{L/K}U_L \to \mathrm{Gal}(L/K)^{ab}$$

is then defined by

$$h_{L/K} : u \mapsto \sigma_u$$

for every $u \in U_K$.

It turns out that if L/K is a totally ramified and finite Galois extension, then the Hazewinkel mapping $h_{L/K} : U_K/N_{L/K}U_L \to \mathrm{Gal}(L/K)^{ab}$ and the Iwasawa–Neukirch mapping $\iota_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L/K}(L^\times)$ are mutually inverse. Thus, by the uniqueness of the local Artin reciprocity map $\alpha_K : \widehat{K^\times} \to G_K^{ab}$ of the local field K, it follows that the Hazewinkel map, the Iwasawa–Neukirch map, and the local Artin map are related to each other as follows:

$$h_{L/K} = \alpha_{L/K}$$

and

$$\iota_{L/K} = \alpha_{L/K}^{-1}.$$

§2. Review of the ramification theory

In this section, we shall review the higher-ramification subgroups in upper-numbering of the absolute Galois group G_K of the local field K, which is necessary in the theory of APF-extensions over K. The main reference that we follow for this section is [16].

For a finite separable extension L/K and any $\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}})$, we introduce

$$i_{L/K}(\sigma) := \min_{x \in O_L} \{\boldsymbol{\nu}_L(\sigma(x) - x)\},$$

put

$$\gamma_t := \#\left\{\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}}) : \ i_{L/K}(\sigma) \geq t+1\right\}$$

for $-1 \leq t \in \mathbb{R}$, and define a function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$, the *Hasse-Herbrand transition function of the extension* L/K, by

$$\varphi_{L/K}(u) := \begin{cases} \int_0^u \frac{\gamma_t}{\gamma_0} dt & \text{if } 0 \leq u \in \mathbb{R}, \\ u & \text{if } -1 \leq u \leq 0. \end{cases}$$

It is well known that $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a continuous, monotone increasing, piecewise linear function, and induces a homeomorphism $\mathbb{R}_{\geq -1} \xrightarrow{\approx} \mathbb{R}_{\geq -1}$. Now, let $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be the mapping inverse to the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$.

Assume that L is a finite Galois extension over K with Galois group $\mathrm{Gal}(L/K) =: G$. The normal subgroup G_u of G defined by

$$G_u = \{\sigma \in G : \ i_{L/K}(\sigma) \geq u+1\}$$

for $-1 \leq u \in \mathbb{R}$ is called the uth ramification group of G in the lower numbering, and has order γ_u. Note that $G_{u'} \subseteq G_u$ for every pair $-1 \leq u, u' \in \mathbb{R}$

satisfying $u \leq u'$. The family $\{G_u\}_{u\in\mathbb{R}_{\geq -1}}$ induces a filtration on G, called the *lower ramification filtration of* G. A *break in the lower ramification filtration* $\{G_u\}_{u\in\mathbb{R}_{\geq -1}}$ of G is defined to be any number $u \in \mathbb{R}_{\geq -1}$ satisfying $G_u \neq G_{u+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$. The function $\psi_{L/K} = \varphi_{L/K}^{-1} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ induces the *upper ramification filtration* $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ on G by setting

$$G^v := G_{\psi_{L/K}(v)},$$

or equivalently, by setting

$$G^{\varphi_{L/K}(u)} = G_u$$

for $-1 \leq v, u \in \mathbb{R}$, where G^v is called the vth upper ramification group of G. A *break in the upper filtration* $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ of G is defined to be any number $v \in \mathbb{R}_{\geq -1}$ satisfying $G^v \neq G^{v+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$.

Remark 2.1. We mention the basic properties of lower and upper ramification filtrations on G. In what follows, F/K denotes a subextension of L/K and H denotes the Galois group $\mathrm{Gal}(L/F)$ corresponding to the extension L/F.

(i) The lower numbering on G passes well to the subgroup H of G in the sense that

$$H_u = H \cap G_u$$

for $-1 \leq u \in \mathbb{R}$;

(ii) and if, furthermore, $H \triangleleft G$, then the upper numbering on G passes well to the quotient G/H:

$$(G/H)^v = G^v H/H$$

for $-1 \leq v \in \mathbb{R}$.

(iii) The Hasse–Herbrand function and its inverse satisfy the transitive law

$$\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}$$

and

$$\psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}.$$

If L/K is an *infinite* Galois extension with Galois group $\mathrm{Gal}(L/K) = G$, which is a topological group under the respective Krull topology, we define the upper ramification filtration $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ on G by the projective limit

$$G^v := \varprojlim_{K\subseteq F\subset L} \mathrm{Gal}(F/K)^v \tag{2.1}$$

defined over the transition morphisms $t_F^{F'}(v) : \mathrm{Gal}(F'/K)^v \to \mathrm{Gal}(F/K)^v$, which are essentially the restriction morphisms from F' to F, defined naturally by the diagram

$$\begin{tikzcd}
\mathrm{Gal}(F/K)^v & & \mathrm{Gal}(F'/K)^v \arrow[ll, "t_F^{F'}(v)"'] \arrow[dl, "\text{can.}"] \\
& \mathrm{Gal}(F'/K)^v\mathrm{Gal}(F'/F)/\mathrm{Gal}(F'/F) \arrow[ul, "\substack{\text{isomorphism}\\ \text{introduced in (ii)}}"] &
\end{tikzcd} \tag{2.2}$$

induced from (ii), as $K \subseteq F \subseteq F' \subseteq L$ runs over all finite Galois extensions F and F' over K inside L. The topological subgroup G^v of G is called the vth ramification group of G in the upper numbering. Note that $G^{v'} \subseteq G^v$ for every pair $-1 \leq v, v' \in \mathbb{R}$ satisfying $v \leq v'$, via the commutativity of the square

$$\begin{tikzcd}
\mathrm{Gal}(F/K)^v & \mathrm{Gal}(F'/K)^v \arrow[l, "t_F^{F'}(v)"'] \\
\mathrm{Gal}(F/K)^{v'} \arrow[u, "\text{inc.}"] & \mathrm{Gal}(F'/K)^{v'} \arrow[l, "t_F^{F'}(v')"'] \arrow[u, "\text{inc.}"']
\end{tikzcd} \tag{2.3}$$

for every chain $K \subseteq F \subseteq F' \subseteq L$ of finite Galois extensions F and F' over K inside L. Observe that

(iv) $G^{-1} = G$ and G^0 is the inertia subgroup of G;
(v) $\bigcap_{v \in \mathbb{R}_{\geq -1}} G^v = \langle 1_G \rangle$;
(vi) G^v is a closed subgroup of G, with respect to the Krull topology, for $-1 \leq v \in \mathbb{R}$.

In this setting, a number $-1 \leq v \in \mathbb{R}$ is said to be a *break in the upper ramification filtration* $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G if v is a break in the upper filtration of some finite quotient G/H for some $H \triangleleft G$. Let $\mathcal{B}_{L/K}$ denote the set of all numbers $v \in \mathbb{R}_{\geq -1}$ that occur as breaks in the upper ramification filtration of G. Then:

(vii) **(Hasse–Arf theorem)** $\mathcal{B}_{K^{ab}/K} \subseteq \mathbb{Z} \cap \mathbb{R}_{\geq -1}$;
(viii) $\mathcal{B}_{K^{\mathrm{sep}}/K} \subseteq \mathbb{Q} \cap \mathbb{R}_{\geq -1}$.

§3. APF-extensions over K

In this section, we shall briefly review a very important class of algebraic extensions over a local field K, called the APF-extensions and introduced by Fontaine and Wintenberger (cf. [5, 6] and [17]). As in the previous section, let $\{G_K^v\}_{v\in\mathbb{R}_{\geq -1}}$ denote the upper ramification filtration of the absolute Galois group G_K of K, and let R^v denote the fixed field $(K^{\mathrm{sep}})^{G_K^v}$ of the vth upper ramification subgroup G_K^v of G_K in K^{sep} for $-1 \leq v \in \mathbb{R}$.

Definition 3.1. An extension L/K is called an *APF-extension* (APF is an abbreviation for "*arithmétiquement profinie*") if one of the following equivalent conditions is satisfied:

(i) $G_K^v G_L$ is open in G_K for every $-1 \leq v \in \mathbb{R}$;
(ii) $(G_K : G_K^v G_L) < \infty$ for every $-1 \leq v \in \mathbb{R}$;
(iii) $L \cap R^v$ is a finite extension over K for every $-1 \leq v \in \mathbb{R}$.

Note that if L/K is an APF-extension, then $[\kappa_L : \kappa_K] < \infty$.

Now, let L/K be an APF-extension. We set $G_L^0 = G_L \cap G_K^0$ and define

$$\varphi_{L/K}(v) = \begin{cases} \int_0^v (G_K^0 : G_L^0 G_K^x)dx & \text{if } 0 \leq v \in \mathbb{R}; \\ v & \text{if } -1 \leq v \leq 0. \end{cases} \tag{3.1}$$

Then the map $v \mapsto \varphi_{L/K}(v)$ for $v \in \mathbb{R}_{\geq -1}$, which is well-defined for the APF-extension L/K, determines a continuous, strictly monotone increasing and piecewise-linear bijection $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$. We denote the inverse of this mapping by $\psi_{L/K} := \varphi_{L/K}^{-1} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$.

Thus, if L/K is a (not necessarily finite) Galois APF-extension, then we can define the higher ramification subgroups in lower numbering $\mathrm{Gal}(L/K)_u$ of $\mathrm{Gal}(L/K)$, for $-1 \leq u \in \mathbb{R}$, by setting

$$\mathrm{Gal}(L/K)_u := \mathrm{Gal}(L/K)^{\varphi_{L/K}(u)}.$$

Remark 3.2. The following should be noted:

(i) in case L/K is a finite separable extension, which is clearly an APF-extension by Definition 3.1, the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ coincides with the Hasse–Herbrand transition function of L/K introduced in the previous section;
(ii) if L/K is a finite separable extension and L'/L is an APF-extension, then L'/K is an APF-extension, and the transitivity rules for the functions $\varphi_{L'/K}, \psi_{L'/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ hold by

$$\varphi_{L'/K} = \varphi_{L/K} \circ \varphi_{L'/L}$$

and by

$$\psi_{L'/K} = \psi_{L'/L} \circ \psi_{L/K}.$$

The next result will be extremely useful.

Lemma 3.3. *Suppose that $K \subseteq F \subseteq L \subseteq K^{\mathrm{sep}}$ is a tower of field extensions in K^{sep} over K. Then:*

(i) *if $[F : K] < \infty$, then L/K is an APF-extension if and only if L/F is an APF-extension;*
(ii) *if $[L : F] < \infty$, then L/K is an APF-extension if and only if F/K is an APF-extension;*
(iii) *if L/K is an APF-extension, then F/K is an APF-extension.*

Proof. For a proof, look at Proposition 1.2.3 in [17]. □

§4. Fontaine–Wintenberger fields of norms

Let L/K be an *infinite APF-extension.* For $0 \leq i \in \mathbb{Z}$, let L_i be an increasing directed family of subextensions in L/K such that

(i) $[L_i : K] < \infty$ for every $0 \leq i \in \mathbb{Z}$;
(ii) $\bigcup_{0 \leq i \in \mathbb{Z}} L_i = L$.

Let

$$\mathbb{X}(L/K)^{\times} = \varprojlim_{i} L_i^{\times}$$

be the projective limit of the multiplicative groups $L_i^{\times}$ with respect to the norm homomorphisms

$$N_{L_{i'}/L_i} : L_{i'}^{\times} \to L_i^{\times},$$

for every $0 \leq i, i' \in \mathbb{Z}$ with $i \leq i'$.

Remark 4.1. The group $\mathbb{X}(L/K)^{\times}$ does not depend on the choice of the increasing directed family of subextensions $\{L_i\}_{0 \leq i \in \mathbb{Z}}$ in L/K satisfying conditions (i) and (ii). Thus,

$$\mathbb{X}(L/K)^{\times} = \varprojlim_{M \in S_{L/K}} M^{\times},$$

where $S_{L/K}$ is the partially ordered family of all finite subextensions in L/K, and the projective limit is with respect to the norm

$$N_{M_2/M_1} : M_2^{\times} \to M_1^{\times},$$

for every $M_1, M_2 \in S_{L/K}$ with $M_1 \subseteq M_2$.

We put
$$\mathbb{X}(L/K) = \mathbb{X}(L/K)^{\times} \cup \{0\},$$
where 0 is a fixed symbol, and define the addition
$$+ : \mathbb{X}(L/K) \times \mathbb{X}(L/K) \to \mathbb{X}(L/K)$$
by the rule
$$(\alpha_M) + (\beta_M) = (\gamma_M),$$
where $\gamma_M \in M$ is defined by the limit
$$\gamma_M = \lim_{\substack{M \subset M' \in S_{L/K} \\ [M':M] \to \infty}} N_{M'/M}(\alpha_{M'} + \beta_{M'}), \tag{4.1}$$
which exists in the local field M, for every $M \in S_{L/K}$.

Remark 4.2. Note that, for $(\alpha_M), (\beta_M) \in \mathbb{X}(L/K)$, the composition law
$$((\alpha_M), (\beta_M)) \mapsto (\alpha_M) + (\beta_M) = (\gamma_M)$$
given by (4.1) is well defined, because L/K is assumed to be an APF-extension (cf. Theorem 2.1.3. in [17]).

This implies the following statement.

Theorem 4.3 (Fontaine–Wintenberger)**.** *Suppose L/K is an APF-extension. Then $\mathbb{X}(L/K)$ is a field under the addition*
$$+ : \mathbb{X}(L/K) \times \mathbb{X}(L/K) \to \mathbb{X}(L/K)$$
defined by (4.1) *and under the multiplication*
$$\times : \mathbb{X}(L/K) \times \mathbb{X}(L/K) \to \mathbb{X}(L/K)$$
defined naturally by the componentwise multiplication on $\mathbb{X}(L/K)^{\times}$. This field $\mathbb{X}(L/K)$ is called the field of norms corresponding to the APF-extension L/K.

Now, in particular, we choose the following specific increasing directed family of subextensions $\{L_i\}_{0 \le i \in \mathbb{Z}}$ in L/K:

(i) L_0 is the maximal unramified extension of K inside L;
(ii) L_1 is the maximal tamely ramified extension of K inside L;
(iii) for $i \geq 2$, the L_i are chosen inductively as finite extensions of L_1 inside L with $L_i \subseteq L_{i+1}$ and $\bigcup_{0 \le i \in \mathbb{Z}} L_i = L$.

Observe that L_0/K is a finite subextension of L/K and, by the definition of tamely ramified extensions, $L_0 \subseteq L_1$, with $[L_1 : K] < \infty$. Thus, for any element $(\alpha_{L_i})_{0 \le i \in \mathbb{Z}}$ of $\mathbb{X}(L/K)$ we have
$$\nu_{L_i}(\alpha_{L_i}) = \nu_{L_0}(\alpha_{L_0}) \tag{4.2}$$

for every $0 \leq i \in \mathbb{Z}$. Therefore, the mapping

$$\boldsymbol{\nu}_{\mathbb{X}(L/K)} : \mathbb{X}(L/K) \to \mathbb{Z} \cup \{\infty\}$$

given by

$$\boldsymbol{\nu}_{\mathbb{X}(L/K)}\left((\alpha_{L_i})_{0\leq i\in\mathbb{Z}}\right) = \nu_{L_0}(\alpha_{L_0}) \tag{4.3}$$

for $(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K)$ is well defined; moreover, it is a discrete valuation on $\mathbb{X}(L/K)$, in view of (4.2).

Theorem 4.4 (Fontaine–Wintenberger)**.** *Let L/K be an APF-extension, and let $\mathbb{X}(L/K)$ be the field of norms attached to L/K. Then*:

(i) *the field $\mathbb{X}(L/K)$ is complete with respect to the discrete valuation $\boldsymbol{\nu}_{\mathbb{X}(L/K)} : \mathbb{X}(L/K) \to \mathbb{Z} \cup \{\infty\}$ defined by* (4.3);
(ii) *the residue class field $\kappa_{\mathbb{X}(L/K)}$ of $\mathbb{X}(L/K)$ satisfies $\kappa_{\mathbb{X}(L/K)} \xrightarrow{\sim} \kappa_L$*;
(iii) *the characteristic of the field $\mathbb{X}(L/K)$ is equal to* $\mathrm{char}(\kappa_K)$.

Proof. For a proof, look at Theorem 2.1.3 in [17]. □

Remark 4.5. As usual, the ring of integers $O_{\mathbb{X}(L/K)}$ of the local field (complete discrete valuation field) $\mathbb{X}(L/K)$ is defined by

$$O_{\mathbb{X}(L/K)} = \left\{(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K) : \boldsymbol{\nu}_{\mathbb{X}(L/K)}\left((\alpha_{L_i})_{0\leq i\in\mathbb{Z}}\right) \geq 0\right\}.$$

Thus, by (4.3) and (4.2), for $\alpha = (\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K)$, the following two conditions are equivalent:

(i) $(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in O_{\mathbb{X}(L/K)}$;
(ii) $\alpha_{L_i} \in O_{L_i}$ for every $0 \leq i \in \mathbb{Z}$.

The maximal ideal $\mathfrak{p}_{\mathbb{X}(L/K)}$ of $O_{\mathbb{X}(L/K)}$ is defined by

$$\mathfrak{p}_{\mathbb{X}(L/K)} = \left\{(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K) : \boldsymbol{\nu}_{\mathbb{X}(L/K)}\left((\alpha_{L_i})_{0\leq i\in\mathbb{Z}}\right) \gneq 0\right\}.$$

By (4.3) and (4.2), for $\alpha = (\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K)$, the following two conditions are equivalent:

(iii) $(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathfrak{p}_{\mathbb{X}(L/K)}$;
(iv) $\alpha_{L_i} \in \mathfrak{p}_{L_i}$ for every $0 \leq i \in \mathbb{Z}$.

The unit group $U_{\mathbb{X}(L/K)}$ of $O_{\mathbb{X}(L/K)}$ is defined by

$$U_{\mathbb{X}(L/K)} = \left\{(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K) : \boldsymbol{\nu}_{\mathbb{X}(L/K)}\left((\alpha_{L_i})_{0\leq i\in\mathbb{Z}}\right) = 0\right\}.$$

Again by (4.3) and (4.2), for $\alpha = (\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in \mathbb{X}(L/K)$, the following two conditions are equivalent:

(v) $(\alpha_{L_i})_{0\leq i\in\mathbb{Z}} \in U_{\mathbb{X}(L/K)}$;
(vi) $\alpha_{L_i} \in U_{L_i}$ for every $0 \leq i \in \mathbb{Z}$.

Let L/K be an infinite APF-extension. Consider the tower

$$K \subseteq F \subseteq L \subseteq E \subseteq K^{\mathrm{sep}}$$

of extensions over K, where $[F : K] < \infty$ and $[E : L] < \infty$. By parts (i) and (ii) of Lemma 3.3, it follows that L/F is an infinite APF-extension satisfying

$$\mathbb{X}(L/K) = \mathbb{X}(L/F),$$

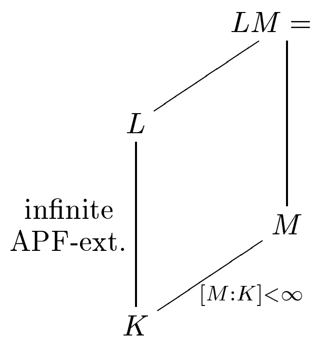
by the definition of field of norms, and E/K is an infinite APF-extension satisfying

$$\mathbb{X}(L/K) \hookrightarrow \mathbb{X}(E/K)$$

under the injective topological homomorphism

$$\varepsilon_{L,E}^{(M)} : \mathbb{X}(L/K) \to \mathbb{X}(E/K),$$

which depends on a finite extension M over K satisfying $LM = E$.



The topological embedding $\varepsilon_{L,E}^{(M)} : \mathbb{X}(L/K) \hookrightarrow \mathbb{X}(E/K)$ is defined as follows. Let $\{L_i\}_{0 \leq i \in \mathbb{Z}}$ be an increasing directed family of subextensions in L/K such that $[L_i : K] < \infty$ for every $0 \leq i \in \mathbb{Z}$ and with $\bigcup_{0 \leq i \in \mathbb{Z}} L_i = L$. Then, clearly, $\{L_i M\}_{0 \leq i \in \mathbb{Z}}$ is an increasing directed family of subextensions in E/K such that $[L_i M : K] < \infty$ for every $0 \leq i \in \mathbb{Z}$ and with $\bigcup_{0 \leq i \in \mathbb{Z}} L_i M = E$. For these two directed families, there exists a sufficiently large positive integer $m = m(M)$, which depends on the choice of M, such that

$$N_{L_j M / L_i M}(x) = N_{L_j / L_i}(x)$$

for $m \leq i \leq j$ and for each $x \in L_j$. Now, the topological embedding $\varepsilon_{L,E}^{(M)}$: $\mathbb{X}(L/K) \hookrightarrow \mathbb{X}(E/K)$ is defined, for every $(\alpha_{L_i})_{0 \leq i \in \mathbb{Z}} \in \mathbb{X}(L/K) - \{0\}$, by

$$\varepsilon_{L,E}^{(M)} : (\alpha_{L_i})_{0 \leq i \in \mathbb{Z}} \mapsto (\alpha'_{L_i M})_{0 \leq i \in \mathbb{Z}},$$

where $\alpha'_{L_iM} \in L_iM$ for every $0 \leq i \in \mathbb{Z}$,

$$\alpha'_{L_iM} = \begin{cases} \alpha_{L_i} & \text{if } i \geq m, \\ N_{L_mM/L_iM}(\alpha_{L_m}) & \text{if } i < m. \end{cases}$$

Thus, under the topological embedding $\varepsilon^{(M)}_{L,E} : \mathbb{X}(L/K) \hookrightarrow \mathbb{X}(E/K)$, $\mathbb{X}(E/K)/\mathbb{X}(L/K)$ can be viewed as an extension of complete discrete valuation fields. At this point, the following remark is in order.

Remark 4.6. Let L/K be an infinite APF-extension and E/L a finite extension. Suppose that M and M' are two finite extensions over K satisfying $LM = LM' = E$. Then the embeddings $\varepsilon^{(M)}_{L,E}, \varepsilon^{(M')}_{L,E} : \mathbb{X}(L/K) \hookrightarrow \mathbb{X}(E/K)$ are the same. Therefore, as a notation, we shall set $\varepsilon^{(M)}_{L,E} = \varepsilon_{L,E}$.

Now, given an infinite APF-extension L/K, this time we let E be a (not necessarily finite) separable extension of L. Let $S^{\text{sep}}_{E/L}$ denote the partially ordered family of all finite separable subextensions in E/L. Then the following is true.

Proposition 4.7.

$$\left\{\mathbb{X}(E'/K); \varepsilon_{E',E''} : \mathbb{X}(E'/K) \hookrightarrow \mathbb{X}(E''/K)\right\}_{\substack{E',E'' \in S^{\text{sep}}_{E/L} \\ E' \subseteq E''}}$$

is an inductive system under the topological embeddings

$$\varepsilon_{E',E''} : \mathbb{X}(E'/K) \hookrightarrow \mathbb{X}(E''/K)$$

for $E', E'' \in S^{\text{sep}}_{E/L}$ with $E' \subseteq E''$.

Let $\mathbb{X}(E, L/K)$ denote the topological field defined by the inductive limit

$$\mathbb{X}(E, L/K) = \varinjlim_{E' \in S^{\text{sep}}_{E/L}} \mathbb{X}(E'/K)$$

over the transition morphisms $\varepsilon_{E',E''} : \mathbb{X}(E'/K) \hookrightarrow \mathbb{X}(E''/K)$ for $E', E'' \in S^{\text{sep}}_{E/L}$ with $E' \subseteq E''$.

The following theorem is central in the theory of fields of norms.

Theorem 4.8 (Fontaine–Wintenberger)**.** *Let L/K be an APF-extension and E/L a Galois extension. Then $\mathbb{X}(E, L/K)/\mathbb{X}(L/K)$ is a Galois extension, and*

$$\mathrm{Gal}\left(\mathbb{X}(E, L/K)/\mathbb{X}(L/K)\right) \simeq \mathrm{Gal}(E/L)$$

canonically.

An immediate and important consequence of this theorem is the following.

Corollary 4.9. *Let L/K be an APF-extension. Then*

$$\mathrm{Gal}(\mathbb{X}(L^{\mathrm{sep}}, L/K)/\mathbb{X}(L/K)) \simeq \mathrm{Gal}(L^{\mathrm{sep}}/L)$$

canonically.

§5. Fesenko reciprocity law

In this section, we shall follow [1–3] to review the Fesenko reciprocity law for the local field K.

We recall the following definition (see [13]).

Definition 5.1. Let $\varphi = \varphi_K \in \mathrm{Gal}(K^{nr}/K)$ denote the Frobenius automorphism of K. An automorphism $\xi \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is called a *Lubin–Tate splitting over* K if $\xi\mid_{K^{nr}} = \varphi$.

Throughout the remainder of the text, we shall fix a Lubin–Tate splitting over the local field K, denoting it simply by φ, or by φ_K if there is fear of confusion. Let K_φ denote the fixed field $(K^{\mathrm{sep}})^\varphi$ of $\varphi \in G_K$ in K^{sep}.

Let L/K be a totally ramified APF-Galois extension satisfying

$$K \subseteq L \subseteq K_\varphi. \tag{5.1}$$

The field of norms $\mathbb{X}(L/K)$ is a local field by Theorem 4.4. Let $\widetilde{\mathbb{X}}(L/K)$ denote the completion $\widehat{\mathbb{X}(L/K)}$ of $\mathbb{X}(L/K)^{nr}$ with respect to the valuation $\boldsymbol{\nu}_{\mathbb{X}(L/K)^{nr}}$, which is a unique extension of the valuation $\boldsymbol{\nu}_{\mathbb{X}(L/K)}$ to $\mathbb{X}(L/K)^{nr}$. As usual, we let $U_{\widetilde{\mathbb{X}}(L/K)}$ denote the unit group of the ring of integers $O_{\widetilde{\mathbb{X}}(L/K)}$ of the complete field $\widetilde{\mathbb{X}}(L/K)$. In this case, there exist isomorphisms

$$\widetilde{\mathbb{X}}(L/K) \simeq \mathbb{F}_p^{\mathrm{sep}}((T))$$

and

$$U_{\widetilde{\mathbb{X}}(L/K)} \simeq \mathbb{F}_p^{\mathrm{sep}}[[T]]^\times,$$

defined by the machinery of Coleman power series (for the details, see Subsection 1.4 in [13]). Thus, the algebraic structures $\widetilde{\mathbb{X}}(L/K)$ and $U_{\widetilde{\mathbb{X}}(L/K)}$ initially seem to depend on the ground field K only. However, as we shall state in Corollary 5.7, the law of composition on the "class formation", which is a certain subquotient of $U_{\widetilde{\mathbb{X}}(L/K)}$, does indeed depend on the $\mathrm{Gal}(L/K)$-module structure of $U_{\widetilde{\mathbb{X}}(L/K)}$.

Remark 5.2. The problem of eliminating this dependence on the Galois-module structure of $U_{\widetilde{\mathbb{X}}(L/K)}$ is closely related to Sen's infinite-dimensional Hodge–Tate theory [11], or more generally, with the p-adic Langlands program.

As in §1, let $\widetilde{K}$ denote the completion of K^{nr} with respect to the valuation $\nu_{K^{nr}}$ on K^{nr}, and let $\widetilde{L} = L\widetilde{K}$. Then $\widetilde{L}/\widetilde{K}$ is an APF-extension, because L/K is an APF-extension, and the corresponding field of norms satisfy

$$\mathbb{X}(\widetilde{L}/\widetilde{K}) = \widetilde{\mathbb{X}}(L/K). \tag{5.2}$$

Now, let

$$\mathrm{Pr}_{\widetilde{K}} : U_{\widetilde{\mathbb{X}}(L/K)} \to U_{\widetilde{K}} \tag{5.3}$$

denote the projection map to the $\widetilde{K}$-coordinate of $U_{\widetilde{\mathbb{X}}(L/K)}$ under the identification described in (5.2). Throughout the text, $U^1_{\widetilde{\mathbb{X}}(L/K)}$ stands for the kernel $\ker(\mathrm{Pr}_{\widetilde{K}})$ of the projection map $\mathrm{Pr}_{\widetilde{K}} : U_{\widetilde{\mathbb{X}}(L/K)} \to U_{\widetilde{K}}$.

Definition 5.3. The subgroup

$$\mathrm{Pr}^{-1}_{\widetilde{K}}(U_K) = \{U \in U_{\widetilde{\mathbb{X}}(L/K)} : \mathrm{Pr}_{\widetilde{K}}(U) \in U_K\}$$

of $U_{\widetilde{\mathbb{X}}(L/K)}$ is called the *Fesenko diamond subgroup of* $U_{\widetilde{\mathbb{X}}(L/K)}$ and is denoted by $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$.

Now, as in [1, 2, 3], we choose an ascending chain of field extensions

$$K = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

in such a way that

(i) $L = \bigcup_{0 \le i \in \mathbb{Z}} E_i$;
(ii) E_i/K is a Galois extension for each $0 \le i \in \mathbb{Z}$;
(iii) E_{i+1}/E_i is cyclic of prime degree $[E_{i+1} : E_i] = p = \mathrm{char}(\kappa_K)$ for each $1 \le i \in \mathbb{Z}$;
(iv) E_1/E_o is cyclic of degree relatively prime to p.

Such a sequence $(E_i)_{0 \le i \in \mathbb{Z}}$ exists (because L/K is a solvable Galois extension) and will be called a *basic ascending chain of subextensions in* L/K. Then, we can construct $\mathbb{X}(L/K)$ by the basic sequence $(E_i)_{0 \le i \in \mathbb{Z}}$ and $\widetilde{\mathbb{X}}(L/K)$ by $(\widetilde{E_i})_{0 \le i \in \mathbb{Z}}$. Note that the Galois group $\mathrm{Gal}(L/K)$ corresponding to the extension L/K acts continuously on $\mathbb{X}(L/K)$ and on $\widetilde{\mathbb{X}}(L/K)$ naturally, if we define the Galois action of $\sigma \in \mathrm{Gal}(L/K)$ on the chain

$$K = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L \tag{5.4}$$

by the action of σ on each E_i for $0 \le i \in \mathbb{Z}$ as

$$K = E_o^\sigma \subset E_1^\sigma = E_1 \subset \cdots \subset E_i^\sigma = E_i \subset \cdots \subset L, \tag{5.5}$$

and respectively on the chain

$$\widetilde{K} = \widetilde{K}E_o \subset \widetilde{E}_1 = \widetilde{K}E_1 \subset \cdots \subset \widetilde{E}_i = \widetilde{K}E_i \subset \cdots \subset \widetilde{L} = \widetilde{K}L \tag{5.6}$$

by the action of σ on the "E_i-part" of each $\widetilde{E}_i$ (note that $E_i \cap K^{nr} = K$)

$$\begin{tikzcd}[column sep=small]
 & \widetilde{K}E_i = \widetilde{E}_i \arrow[dl, dash] \arrow[dd, dash] \\
\widetilde{K} \arrow[dd, dash, "\substack{\text{completion}\\ \text{of max.-ur.-}\\ \text{ext. of } K}"'] & \\
 & E_i \\
K \arrow[ur, dash, "\substack{E_i/K \text{ Galois ext.}\\ {[E_i : K] < \infty}}"'] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}[column sep=small]
 & \widetilde{K}E_i^\sigma = \widetilde{E_i^\sigma} = \widetilde{E}_i \arrow[dl, dash] \arrow[dd, dash] \\
\widetilde{K} \arrow[dd, dash, "\substack{\text{completion}\\ \text{of max.-ur.-}\\ \text{ext. of } K}"'] & \\
 & E_i^\sigma = E_i \\
K \arrow[ur, dash, "\substack{E_i^\sigma/K \text{ Galois ext.}\\ {[E_i^\sigma : K] < \infty}}"'] &
\end{tikzcd}$$

for $0 \leq i \in \mathbb{Z}$ as

$$\widetilde{K} = \widetilde{K}E_o^\sigma \subset \widetilde{K}E_1^\sigma = \widetilde{K}E_1 \subset \cdots \subset \widetilde{K}E_i^\sigma = \widetilde{K}E_i \subset \cdots \subset \widetilde{K}L. \tag{5.7}$$

Therefore, there exist natural continuous actions of $\mathrm{Gal}(L/K)$ on $U_{\mathbb{X}(L/K)}$, on $U_{\widetilde{\mathbb{X}}(L/K)}$, and on $U^\diamond_{\widetilde{\mathbb{X}}(L/K)}$ compatible with the respective topological group structures, so that *we shall always view them as topological* $\mathrm{Gal}(L/K)$*-modules in this text.* Now, we recall the following theorem about norm compatible sequences of prime elements (cf. [13]).

Theorem 5.4 (Koch–de Shalit). *Assume that $K \subseteq L \subset K_\varphi$. Then for any chain*

$$K = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

of finite subextensions of L/K, there exists a unique norm-compatible sequence

$$\pi_{E_o}, \pi_{E_1}, \cdots, \pi_{E_i}, \cdots,$$

where each π_{E_i} is a prime element of E_i for $0 \leq i \in \mathbb{Z}$.

In view of the theorem of Koch and de Shalit, we introduce the *natural* prime element $\Pi_{\varphi;L/K}$ of the local field $\mathbb{X}(L/K)$ (which depends on the fixed Lubin–Tate splitting φ (cf. [13]) as well as on the subextension L/K of K_φ/K) by

$$\Pi_{\varphi;L/K} = (\pi_{E_i})_{0 \leq i \in \mathbb{Z}}.$$

By the theorem of Koch and de Shalit, the prime element $\Pi_{\varphi;L/K}$ of $\mathbb{X}(L/K)$ does *not* depend on the choice of a chain $(E_i)_{0 \leq i \in \mathbb{Z}}$ of finite subextensions of L/K.

Theorem 5.5 (Fesenko). *For each $\sigma \in \mathrm{Gal}(L/K)$, there exists $U_\sigma \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ that solves the equation*

$$U^{1-\varphi} = \Pi^{\sigma-1}_{\varphi;L/K} \tag{5.8}$$

for U. Moreover, the solution set of this equation consists of elements in the coset $U_\sigma.U_{\mathbb{X}(L/K)}$ of U_σ modulo $U_{\mathbb{X}(L/K)}$.

In fact, for the most general form of this theorem and its proof, see [9].

Now, define the arrow

$$\phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \tag{5.9}$$

by

$$\phi^{(\varphi)}_{L/K} : \sigma \mapsto \overline{U}_\sigma = U_\sigma.U_{\mathbb{X}(L/K)}, \tag{5.10}$$

for every $\sigma \in \mathrm{Gal}(L/K)$.

Theorem 5.6 (Fesenko). *The arrow*

$$\phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$$

defined for the extension L/K is injective, and for every $\sigma, \tau \in \mathrm{Gal}(L/K)$ we have

$$\phi^{(\varphi)}_{L/K}(\sigma\tau) = \phi^{(\varphi)}_{L/K}(\sigma)\phi^{(\varphi)}_{L/K}(\tau)^{\sigma}, \tag{5.11}$$

the cocycle condition is satisfied.

We formulate a natural consequence of this theorem. Let $\mathrm{im}(\phi^{(\varphi)}_{L/K}) \subseteq U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ denote the image set of the mapping $\phi^{(\varphi)}_{L/K}$.

Corollary 5.7. *Define a law of composition $*$ on $\mathrm{im}(\phi^{(\varphi)}_{L/K})$ by*

$$\overline{U} * \overline{V} = \overline{U}.\overline{V}^{(\phi^{(\varphi)}_{L/K})^{-1}(\overline{U})} \tag{5.12}$$

for every $\overline{U}, \overline{V} \in \mathrm{im}(\phi^{(\varphi)}_{L/K})$. Then $\mathrm{im}(\phi^{(\varphi)}_{L/K})$ is a topological group under $*$, and the map $\phi^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{im}(\phi^{(\varphi)}_{L/K}), \tag{5.13}$$

where the topological group structure on $\mathrm{im}(\phi^{(\varphi)}_{L/K})$ is defined with respect to the binary operation $*$ described by (5.12).

Now, for each $0 \leq i \in \mathbb{R}$, consider the ith higher unit group $U^i_{\widetilde{\mathbb{X}}(L/K)}$ of the field $\widetilde{\mathbb{X}}(L/K)$, and define the group

$$\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^i = U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \cap U^i_{\widetilde{\mathbb{X}}(L/K)}. \tag{5.14}$$

Theorem 5.8 (Fesenko ramification theorem). *For $0 \leq n \in \mathbb{Z}$, let $\mathrm{Gal}(L/K)_n$ denote the nth higher ramification subgroup of the Galois group $\mathrm{Gal}(L/K)$ corresponding to the APF-Galois subextension L/K of K_φ/K in the lower numbering. Then, we have the inclusion*

$$\begin{aligned}&\phi^{(\varphi)}_{L/K}\left(\mathrm{Gal}(L/K)_n - \mathrm{Gal}(L/K)_{n+1}\right)\\ &\quad\subseteq \left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} - \left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^{n+1} U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}.\end{aligned}$$

Now, let M/K be a Galois subextension of L/K. Thus, there exists a chain of field extensions

$$K \subseteq M \subseteq L \subseteq K_\varphi,$$

where M is a totally ramified APF-Galois extension over K by Lemma 3.3. Let

$$\phi^{(\varphi)}_{M/K} : \mathrm{Gal}(M/K) \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}$$

be the corresponding map defined for the extension M/K.

Now, let

$$K = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

be an ascending chain satisfying $L = \bigcup_{0 \leq i \in \mathbb{Z}} E_i$ and $[E_{i+1} : E_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Then

$$K = E_o \cap M \subseteq E_1 \cap M \subseteq \cdots \subseteq E_i \cap M \subseteq \cdots \subset M$$

is an ascending chain of field extensions satisfying $M = \bigcup_{0 \leq i \in \mathbb{Z}} (E_i \cap M)$ and $[E_{i+1} \cap M : E_i \cap M] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Thus, we can construct $\mathbb{X}(M/K)$ by the sequence $(E_i \cap M)_{0 \leq i \in \mathbb{Z}}$ and $\widetilde{\mathbb{X}}(M/K)$ by the sequence $(\widetilde{E_i \cap M})_{0 \leq i \in \mathbb{Z}}$. Furthermore, for every pair $0 \leq i, i' \in \mathbb{Z}$ satisfying $i \leq i'$, the commutative square

$$\begin{tikzcd} \widetilde{E}_i^\times \arrow[d, "\widetilde{N}_{E_i/E_i \cap M}"'] & \widetilde{E}_{i'}^\times \arrow[l, "\widetilde{N}_{E_{i'}/E_i}"'] \arrow[d, "\widetilde{N}_{E_{i'}/E_{i'} \cap M}"] \\ \widetilde{E_i \cap M}^\times & \widetilde{E_{i'} \cap M}^\times \arrow[l, "\widetilde{N}_{E_{i'} \cap M/E_i \cap M}"'] \end{tikzcd}$$

induces a group homomorphism

$$\widetilde{\mathcal{N}}_{L/M} = \varprojlim_{0 \leq i \in \mathbb{Z}} \widetilde{N}_{E_i/E_i \cap M} : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times} \tag{5.15}$$

defined by

$$\widetilde{\mathcal{N}}_{L/M}\left((\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}}\right) = \left(\widetilde{N}_{E_i/E_i \cap M}(\alpha_{\widetilde{E}_i})\right)_{0 \leq i \in \mathbb{Z}}, \tag{5.16}$$

for every $(\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in \widetilde{\mathbb{X}}(L/K)^{\times}$.

Remark 5.9. The group homomorphism

$$\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times}$$

defined by (5.15) and (5.16) does *not* depend on the choice of an ascending chain

$$K = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

satisfying $L = \bigcup_{0 \leq i \in \mathbb{Z}} E_i$ and $[E_{i+1} : E_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$.

The basic properties of this group homomorphism are the following.

(i) If $U = (u_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in U_{\widetilde{\mathbb{X}}(L/K)}$, then $\widetilde{\mathcal{N}}_{L/M}(U) \in U_{\widetilde{\mathbb{X}}(M/K)}$.

Proof. The definition of the valuation $\nu_{\widetilde{\mathbb{X}}(M/K)}$ of $\widetilde{\mathbb{X}}(M/K)$ and the definition of the valuation $\nu_{\widetilde{\mathbb{X}}(L/K)}$ of $\widetilde{\mathbb{X}}(L/K)$ show that

$$\nu_{\widetilde{\mathbb{X}}(M/K)}\left(\widetilde{\mathcal{N}}_{L/M}(U)\right) = \nu_{\widetilde{\mathbb{X}}(M/K)}\left(\left(\widetilde{N}_{E_i/E_i \cap M}(u_{\widetilde{E}_i})\right)_{0 \leq i \in \mathbb{Z}}\right) = \nu_{\widetilde{K}}(u_{\widetilde{K}}) = 0,$$

as

$$\nu_{\widetilde{\mathbb{X}}(L/K)}(U) = \nu_{\widetilde{K}}(u_{\widetilde{K}}) = 0,$$

since $U \in U_{\widetilde{\mathbb{X}}(L/K)}$. □

(ii) If $U = (u_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, then $\widetilde{\mathcal{N}}_{L/M}(U) \in U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}$.

Proof. The assertion follows by observing that $\mathrm{Pr}_{\widetilde{K}}(U) = u_{\widetilde{K}}$ and $\mathrm{Pr}_{\widetilde{K}}\big(\widetilde{\mathcal{N}}_{L/M}(U)\big) = \widetilde{N}_{E_o/E_o \cap M}(u_{\widetilde{E}_o}) = u_{\widetilde{K}} \in U_K$. □

(iii) If $U = (u_{E_i})_{0 \leq i \in \mathbb{Z}} \in U_{\mathbb{X}(L/K)}$, then $\widetilde{\mathcal{N}}_{L/M}(U) \in U_{\mathbb{X}(M/K)}$.

Proof. The assertion follows by the definition (5.16) of the homomorphism (5.15), combined with the fact that $\widetilde{N}_{E_i/E_i \cap M}(u_{E_i}) = N_{E_i/E_i \cap M}(u_{E_i})$ for every $u_{E_i} \in U_{E_i}$ and every $0 \leq i \in \mathbb{Z}$. □

Thus, the group homomorphism (5.15) defined by (5.16) induces a group homomorphism, which will be called the *Coleman norm map from L to M*,

$$\widetilde{\mathcal{N}}_{L/M}^{\mathrm{Coleman}} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}, \tag{5.17}$$

and is defined by

$$\widetilde{\mathcal{N}}_{L/M}^{\mathrm{Coleman}}(\overline{U}) = \widetilde{\mathcal{N}}_{L/M}(U).U_{\mathbb{X}(M/K)} \tag{5.18}$$

for every $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where as before, $\overline{U}$ denotes the coset $U.U_{\mathbb{X}(L/K)}$ in $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$.

The following theorem was stated in Fesenko's [1–3] without proof. Thus, for completeness, we shall supply a proof of this theorem as well.

Theorem 5.10 (Fesenko). *For the Galois subextension M/K of L/K, the square*

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\phi_{L/K}^{(\varphi)}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \\ {\scriptstyle \mathrm{res}_{\mathrm{M}}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}_{L/M}^{\mathrm{Coleman}}} \\ \mathrm{Gal}(M/K) & \xrightarrow{\phi_{M/K}^{(\varphi)}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}, \end{array} \tag{5.19}$$

where the right vertical arrow

$$\widetilde{\mathcal{N}}_{L/M}^{\mathrm{Coleman}} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}$$

is the Coleman norm map from L to M defined by (5.17) *and* (5.18)*, is commutative.*

Proof. For each $\sigma \in \mathrm{Gal}(L/K)$, we must show that

$$\widetilde{\mathcal{N}}_{L/M}^{\mathrm{Coleman}}\left(\phi_{L/K}^{(\varphi)}(\sigma)\right) = \phi_{M/K}^{(\varphi)}(\sigma\mid_M).$$

Thus, it suffices to prove the congruence

$$\widetilde{\mathcal{N}}_{L/M}(U_\sigma) \equiv U_{\sigma|_M} \pmod{U_{\mathbb{X}(M/K)}},$$

or equivalently, it suffices to prove that

$$\frac{\widetilde{\mathcal{N}}_{L/M}(U_\sigma)}{\widetilde{\mathcal{N}}_{L/M}(U_\sigma)^{\varphi}} = \frac{\Pi_{\varphi;M/K}^{\sigma|_M}}{\Pi_{\varphi;M/K}}.$$

Now, without loss of generality, by Remark 5.9, the ascending chain of extensions

$$K = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

can be chosen as the *basic sequence* introduced at the beginning of this section. Thus, each extension E_i/K is finite and Galois for $0 \leq i \in \mathbb{Z}$. Now, let $U_\sigma = (u_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$. Then, for each $0 \leq i \in \mathbb{Z}$,

$$\frac{\widetilde{N}_{E_i/E_i\cap M}(u_{\widetilde{E}_i})}{\widetilde{N}_{E_i/E_i\cap M}(u_{\widetilde{E}_i})^{\varphi}} = \frac{\widetilde{N}_{E_i/E_i\cap M}(u_{\widetilde{E}_i})}{\widetilde{N}_{E_i/E_i\cap M}(u^{\varphi}_{\widetilde{E}_i})} = \widetilde{N}_{E_i/E_i\cap M}\left(\frac{u_{\widetilde{E}_i}}{u^{\varphi}_{\widetilde{E}_i}}\right).$$

Next, the relation $\frac{u_{\widetilde{E}_i}}{u^{\varphi}_{\widetilde{E}_i}} = \frac{\pi^{\sigma}_{E_i}}{\pi_{E_i}}$, which follows from $\frac{U_\sigma}{U^{\varphi}_\sigma} = \frac{\Pi^{\sigma}_{\varphi;L/K}}{\Pi_{\varphi;L/K}}$, yields

$$\frac{\widetilde{N}_{E_i/E_i\cap M}(u_{\widetilde{E}_i})}{\widetilde{N}_{E_i/E_i\cap M}(u_{\widetilde{E}_i})^{\varphi}} = \widetilde{N}_{E_i/E_i\cap M}\left(\frac{\pi^{\sigma}_{E_i}}{\pi_{E_i}}\right).$$

Thus, by the theorem of Koch and de Shalit, it follows that

$$\widetilde{N}_{E_i/E_i\cap M}\left(\frac{\pi^{\sigma}_{E_i}}{\pi_{E_i}}\right) = \frac{\widetilde{N}_{E_i/E_i\cap M}(\pi_{E_i})^{\sigma}}{\widetilde{N}_{E_i/E_i\cap M}(\pi_{E_i})} = \frac{\pi^{\sigma|_M}_{E_i\cap M}}{\pi_{E_i\cap M}},$$

which proves the formula

$$\frac{\widetilde{\mathcal{N}}_{L/M}(U_\sigma)}{\widetilde{\mathcal{N}}_{L/M}(U_\sigma)^{\varphi}} = \frac{\Pi^{\sigma|_M}_{\varphi;M/K}}{\Pi_{\varphi;M/K}}.$$

The proof is complete. □

Now, let F/K be a finite subextension of L/K. Then, since F is *compatible* with (K,φ) in the sense of [13, p. 89], we may fix the Lubin–Tate splitting over F to be $\varphi_F = \varphi_K = \varphi$. Thus, there exists a chain of field extensions

$$K \subseteq F \subseteq L \subseteq K_\varphi \subseteq F_\varphi,$$

where L is a totally ramified APF-Galois extension over F by Lemma 3.3. So, there exists a mapping

$$\phi^{(\varphi)}_{L/F} : \mathrm{Gal}(L/F) \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)}$$

corresponding to the extension L/F.

For the APF-extension L/F, we fix an ascending chain

$$F = F_o \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset L$$

satisfying $L = \bigcup_{0\leq i\in\mathbb{Z}} F_i$ and $[F_{i+1} : F_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$. We introduce a homomorphism

$$\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/F)^{\times} \to \widetilde{\mathbb{X}}(L/K)^{\times} \tag{5.20}$$

by

$$\begin{aligned}\Lambda_{F/K} : (\alpha_F \xleftarrow{\widetilde{N}_{F_1/F}} \alpha_{F_1} \xleftarrow{\widetilde{N}_{F_2/F_1}} \cdots) \\ \mapsto (\widetilde{N}_{F/K}(\alpha_F) \xleftarrow{\widetilde{N}_{F/K}} \alpha_F \xleftarrow{\widetilde{N}_{F_1/F}} \alpha_{F_1} \xleftarrow{\widetilde{N}_{F_2/F_1}} \cdots)\end{aligned} \tag{5.21}$$

for each $(\alpha_{F_i})_{0\leq i\in\mathbb{Z}} \in \widetilde{\mathbb{X}}(L/F)^\times$.

Remark 5.11. It is clear that the homomorphism

$$\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/F)^\times \to \widetilde{\mathbb{X}}(L/K)^\times$$

defined by (5.20) and (5.21) does *not* depend on the choice of an ascending chain of fields

$$F = F_o \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset L$$

satisfying $L = \bigcup_{0\leq i\in\mathbb{Z}} F_i$ and $[F_{i+1} : F_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$.

The basic properties of this group homomorphism are as follows.

(i) The square

$$\begin{tikzcd} \widetilde{\mathbb{X}}(L/F) \arrow[r, "\Lambda_{F/K}"] & \widetilde{\mathbb{X}}(L/K) \\ \mathbb{X}(L/F) \arrow[r, "\Lambda_{F/K}"] \arrow[u, "\text{inc.}"] & \mathbb{X}(L/K) \arrow[u, "\text{inc.}"'] \end{tikzcd}$$

is commutative.

(ii) If $U = (u_{\widetilde{F}_i})_{0\leq i\in\mathbb{Z}} \in U_{\widetilde{\mathbb{X}}(L/F)}$, then $\Lambda_{F/K}(U) \in U_{\widetilde{\mathbb{X}}(L/K)}$.

(iii) If $U = (u_{\widetilde{F}_i})_{0\leq i\in\mathbb{Z}} \in U^\diamond_{\widetilde{\mathbb{X}}(L/F)}$, then $\Lambda_{F/K}(U) \in U^\diamond_{\widetilde{\mathbb{X}}(L/K)}$.

(iv) If $U = (u_{F_i})_{0\leq i\in\mathbb{Z}} \in U_{\mathbb{X}(L/F)}$, then $\Lambda_{F/K}(U) \in U_{\mathbb{X}(L/K)}$.

Thus, the group homomorphism (5.20) defined by (5.21) induces the group homomorphism

$$\lambda_{F/K} : U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \to U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \tag{5.22}$$

defined by

$$\lambda_{F/K} : \overline{U} \mapsto \Lambda_{F/K}(U).U_{\mathbb{X}(L/K)}, \tag{5.23}$$

for every $U \in U^\diamond_{\widetilde{\mathbb{X}}(L/F)}$, where, as before, $\overline{U}$ denotes the coset $U.U_{\mathbb{X}(L/F)}$ in $U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)}$.

The following theorem was stated in [1–3] without proof. Thus, for completeness, we shall supply a proof of this theorem as well.

Theorem 5.12 (Fesenko). *For the finite subextension F/K of L/K, the square*

$$\begin{tikzcd} \mathrm{Gal}(L/F) \arrow[r, "\phi^{(\varphi)}_{L/F}"] \arrow[d, "\mathrm{inc.}"'] & U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \arrow[d, "\lambda_{F/K}"] \\ \mathrm{Gal}(L/K) \arrow[r, "\phi^{(\varphi)}_{L/K}"] & U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}, \end{tikzcd} \tag{5.24}$$

where the right vertical arrow $\lambda_{F/K} : U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ is defined by (5.22) *and* (5.23), *is commutative.*

Proof. For $\sigma \in \mathrm{Gal}(L/F)$, we have $\phi^{(\varphi)}_{L/F}(\sigma) = U_\sigma.U_{\mathbb{X}(L/F)}$, where $U_\sigma \in U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}$ satisfies

$$\frac{U_\sigma}{U_\sigma^\varphi} = \frac{\Pi^\sigma_{\varphi;L/F}}{\Pi_{\varphi;L/F}}. \tag{5.25}$$

Here, $\Pi_{\varphi;L/F}$ is the norm compatible sequence of primes $(\pi_{F_i})_{0\le i\in\mathbb{Z}}$. Now,

$$\Lambda_{F/K}\left(\frac{U_\sigma}{U_\sigma^\varphi}\right) = \frac{\Lambda_{F/K}(U_\sigma)}{\Lambda_{F/K}(U_\sigma^\varphi)} = \frac{\Lambda_{F/K}(U_\sigma)}{\Lambda_{F/K}(U_\sigma)^\varphi}.$$

On the other hand, $\Lambda_{F/K}(\Pi_{\varphi;L/F}) = \Pi_{\varphi;L/K}$ and $\Lambda_{F/K}(\Pi^\sigma_{\varphi;L/F}) = \Pi^\sigma_{\varphi;L/K}$. Thus, (5.25) yields

$$\frac{\Lambda_{F/K}(U_\sigma)}{\Lambda_{F/K}(U_\sigma)^\varphi} = \frac{\Pi^\sigma_{\varphi;L/K}}{\Pi_{\varphi;L/K}},$$

which shows that

$$\phi^{(\varphi)}_{L/K}(\sigma) = \Lambda_{F/K}(U_\sigma).U_{\mathbb{X}(L/K)} = \lambda_{F/K}(\phi^{(\varphi)}_{L/F}(\sigma)),$$

completing the proof of the commutativity of the square. □

Furthermore, if L/K is a finite extension, then the composition

$$\begin{tikzcd} \mathrm{Gal}(L/K) \arrow[r, "\phi^{(\varphi)}_{L/K}"] \arrow[rr, bend left, "\iota_{L/K}"] & U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \arrow[r, "\mathrm{Pr}_{\tilde{K}}"] & U_K/N_{L/K}U_L, \end{tikzcd}$$

is the Iwasawa–Neukirch map of the extension L/K. Thus, *the mapping $\phi^{(\varphi)}_{L/K}$ defined for L/K is a generalization of the Iwasawa–Neukirch map $\iota_{L/K}$: $\mathrm{Gal}(L/K) \to U_K/N_{L/K}(U_L)$ for the totally ramified APF-Galois subextensions L/K of K_φ/K.*

Likewise, the definition of the Hazewinkel map $h_{L/K} : U_K/N_{L/K}U_L \to \mathrm{Gal}(L/K)^{ab}$ (formulated initially for totally ramified finite Galois extensions L/K) can be extended to the totally ramified APF-Galois subextensions of K_φ/K by generalizing the Serre short exact sequence introduced in (1.1) and (1.2). In order to do so, first we need to assume that the local field K satisfies the condition

$$\boldsymbol{\mu}_p(K^{\mathrm{sep}}) = \{\alpha \in K^{\mathrm{sep}} : \alpha^p = 1\} \subset K, \tag{5.26}$$

where $p = \mathrm{char}(\kappa_K)$.

Remark 5.13. If K is a local field of characteristic $p = \mathrm{char}(\kappa_K)$, the assumption (5.26) on K is satisfied automatically. For the details on the assumption (5.26) on K, we refer the reader to [1–3].

In what follows, as before, we let L/K be a totally ramified APF-Galois extension satisfying (5.1). Under this assumption, there exists a topological $\mathrm{Gal}(L/K)$-submodule $Y_{L/K}$ of $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ such that

(i) $U_{\mathbb{X}(L/K)} \subseteq Y_{L/K}$;
(ii) the composition

$$\Phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\phi^{(\varphi)}_{L/K}} U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \xrightarrow[\text{canonical topol. map}]{c_{L/K}} U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K}$$

is a bijection with the extended Hazewinkel map $H^{(\varphi)}_{L/K} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \to \mathrm{Gal}(L/K)$ as the inverse.

Now, we shall briefly review the constructions of the topological group $Y_{L/K}$ and the extended Hazewinkel map $H^{(\varphi)}_{L/K} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \to \mathrm{Gal}(L/K)$. For the details, we refer the reader to the papers [1–3], which we follow closely.

We fix a *basic* ascending chain

$$K = K_o \subset K_1 \subset \cdots \subset K_i \subset \cdots \subset L \tag{5.27}$$

of subextensions in L/K *once and for all*. Now, we introduce the following notation. For each $1 \le i \in \mathbb{Z}$,

(i) let σ_i be an element of $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$ satisfying $< \sigma_i \mid_{K_i} >= \mathrm{Gal}(K_i/K_{i-1})$;
(ii) let $\widetilde{K}_i = K_i\widetilde{K}$.

By Abelian local class field theory, for each $1 \le k \in \mathbb{Z}$ we have an injective homomorphism

$$\Xi_{K_{k+1}/K_k} : \mathrm{Gal}(K_{k+1}/K_k) \to U_{\widetilde{K}_{k+1}}/U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}} \tag{5.28}$$

defined by

$$\Xi_{K_{k+1}/K_k} : \tau \mapsto \pi_{K_{k+1}}^{\tau-1} U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1} \tag{5.29}$$

for every $\tau \in \mathrm{Gal}(K_{k+1}/K_k)$. Let $\mathrm{im}(\Xi_{K_{k+1}/K_k}) = T_k^{(L/K)} = T_k$ be the isomorphic copy of $\mathrm{Gal}(K_{k+1}/K_k)$ in $U_{\widetilde{K}_{k+1}}/U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}$.

Theorem 5.14 (Fesenko)**.** *Fix $1 \le k \in \mathbb{Z}$. Let*

$$T_k^{(L/K)'} = T_k' = T_k \cap \left(\prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1} \right) / U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}.$$

Then the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & T_k' \arrow[r] & \left(\prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1} \right) / U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1} \arrow[r, "\widetilde{N}_{K_{k+1}/K_k}"] & \prod_{1\le i\le k} U_{\widetilde{K}_k}^{\sigma_i-1} \arrow[r] \arrow[l, bend right, "h_k^{(L/K)}=h_k"'] & 1 \end{tikzcd}$$

splits by a homomorphism

$$h_k : \prod_{1\le i\le k} U_{\widetilde{K}_k}^{\sigma_i-1} \to \left(\prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1} \right) / U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}.$$

This homomorphism is not unique in general.

For each $1 \le k \in \mathbb{Z}$, consider any map

$$g_k^{(L/K)} = g_k : \prod_{1\le i\le k} U_{\widetilde{K}_k}^{\sigma_i-1} \to \prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1}$$

that makes the triangle

$$\begin{tikzcd} & \prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1} \arrow[d, "\pmod{U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}}"] \\ \prod_{1\le i\le k} U_{\widetilde{K}_k}^{\sigma_i-1} \arrow[ur, "g_k"] \arrow[r, "h_k"] & \left(\prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1} \right) / U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1} \end{tikzcd}$$

commutative. Clearly, such a map exists. Now, for every $1 \le i \in \mathbb{Z}$, we choose a mapping

$$f_i^{(L/K)} = f_i : U_{\widetilde{K}_i}^{\sigma_i - 1} \to U_{\widetilde{\mathbb{X}}(L/K_i)} \xrightarrow{\Lambda_{K_i/K}} U_{\widetilde{\mathbb{X}}(L/K)}$$

such that

$$\mathrm{Pr}_{\widetilde{K}_j} \circ f_i = (g_{j-1} \circ \cdots \circ g_i) \mid_{U_{\widetilde{K}_i}^{\sigma_i - 1}}$$

for each $j \in \mathbb{Z}_{>i}$, where $\mathrm{Pr}_{\widetilde{K}_j} : U_{\widetilde{\mathbb{X}}(L/K)} \to U_{\widetilde{K}_j}$ denotes the projection to the $\widetilde{K}_j$-coordinate.

Lemma 5.15 (Fesenko). (i) *Let $z^{(i)} \in \mathrm{im}(f_i) = Z_i^{(L/K)}$ for each $1 \le i \in \mathbb{Z}$. Then the infinite product $\prod_i z^{(i)}$ converges to an element z in $U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}$.*

(ii) *Let*

$$Z_{L/K}(\{K_i, f_i\}) = \left\{ \prod_{1 \le i \in \mathbb{Z}} z^{(i)} : z^{(i)} \in \mathrm{im}(f_i) \right\}.$$

Then $Z_{L/K}(\{K_i, f_i\})$ is a topological subgroup of $U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}$.

Remark 5.16. In fact, $Z_{L/K}(\{K_i, f_i\})$ is a topological subgroup of $U_{\widetilde{\mathbb{X}}(L/K)}^{1}$. Let $z \in Z_{L/K}(\{K_i, f_i\})$ and choose $z^{(i)} \in \mathrm{im}(f_i) \subset U_{\widetilde{\mathbb{X}}(L/K)}$ so that $z = \prod_i z^{(i)}$. It suffices to show that $\mathrm{Pr}_{\widetilde{K}}(z^{(i)}) = 1_K$. For this, let $\alpha^{(i)} \in U_{\widetilde{K}_i}^{\sigma_i - 1}$ be such that $f_i(\alpha^{(i)}) = z^{(i)}$. Thus, $\mathrm{Pr}_{\widetilde{K}_i}(z^{(i)}) = \alpha^{(i)}$. Now, by Hilbert theorem 90, it follows that $\widetilde{N}_{K_i/K}(\alpha^{(i)}) = (\widetilde{N}_{K_{i-1}/K} \circ \widetilde{N}_{K_i/K_{i-1}})(\alpha^{(i)}) = 1_K$, which completes the proof.

Lemma 5.17. *For $1 \le i \in \mathbb{Z}$, let $\sigma = \sigma_i \in \mathrm{Gal}(\widetilde{L}/\widetilde{K})$ be such that $< \sigma \mid_{K_i} >= \mathrm{Gal}(K_i/K_{i-1})$. Let $\tau \in \mathrm{Gal}(L/K)$ be viewed as an element of $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$. Then*

$$\left(U_{\widetilde{K}_i}^{\sigma - 1}\right)^{\tau} = U_{\widetilde{K}_i}^{\sigma - 1}.$$

Proof. Let τ be any element of $\mathrm{Gal}(L/K)$. We regard τ as an element of $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$. Clearly, the conjugate $\tau^{-1}\sigma\tau \in \mathrm{Gal}(\widetilde{L}/\widetilde{K})$ satisfies $< \tau^{-1}\sigma\tau \mid_{K_i} >= \mathrm{Gal}(K_i/K_{i-1})$, because $\left(\tau^{-1}\sigma\tau \mid_{K_i}\right)^n = \mathrm{id}_{K_i}$ yields $(\sigma \mid_{K_i})^n = \mathrm{id}_{K_i}$. Let $0 < d \in \mathbb{Z}$ be such that $\tau^{-1}\sigma\tau \mid_{K_i} = (\sigma \mid_{K_i})^d = (\sigma^d) \mid_{K_i}$. Thus, $\tau^{-1}\sigma\tau\sigma^{-d} \in \mathrm{Gal}(\widetilde{L}/\widetilde{K}_i)$ because $\widetilde{K}_i = \widetilde{K}K_i$. It follows that

$$U_{\widetilde{K}_i}^{\tau^{-1}\sigma\tau - 1} = U_{\widetilde{K}_i}^{\sigma^d - 1}.$$

Since $U_{\widetilde{K}_i}^{\tau^{-1}\sigma\tau-1} = U_{\widetilde{K}_i}^{\tau^{-1}(\sigma-1)\tau} = \left(U_{\widetilde{K}_i}^{\sigma-1}\right)^{\tau}$, the relation

$$\left(U_{\widetilde{K}_i}^{\sigma-1}\right)^{\tau} = U_{\widetilde{K}_i}^{\sigma^d-1}$$

also follows. Now, the inclusion

$$U_{\widetilde{K}_i}^{\sigma^d-1} \subseteq U_{\widetilde{K}_i}^{\sigma-1}$$

is clear, because, for $u \in U_{\widetilde{K}_i}$,

$$\frac{u^{\sigma^d}}{u} = \frac{\left(u^{\sigma^{d-1}}\right)^{\sigma}}{u^{\sigma^{d-1}}} \cdots \frac{(u^{\sigma})^{\sigma}}{u^{\sigma}} \frac{u^{\sigma}}{u}.$$

Thus, for $\tau \in \mathrm{Gal}(L/K)$ we obtain the inclusion $\left(U_{\widetilde{K}_i}^{\sigma-1}\right)^{\tau} \subseteq U_{\widetilde{K}_i}^{\sigma-1}$. Hence,

$$\left(U_{\widetilde{K}_i}^{\sigma-1}\right)^{\tau} = U_{\widetilde{K}_i}^{\sigma-1}$$

for $\tau \in \mathrm{Gal}(L/K)$, which completes the proof. $\square$

Now, let $\tau \in \mathrm{Gal}(L/K)$. Consider the element $\tau^{-1}\sigma_i\tau$ of $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$ for each $1 \leq i \in \mathbb{Z}$. Clearly, $< \tau^{-1}\sigma_i\tau \mid_{K_i} >= \mathrm{Gal}(K_i/K_{i-1})$. By Abelian local class field theory and by Lemma 5.17, the square

$$\begin{tikzcd}
\mathrm{Gal}(K_i/K_{i-1}) \arrow[r, "\Xi_{K_i/K_{i-1}}"] \arrow[d, "\tau\text{-conjugation}"'] & U_{\widetilde{K}_i}/U_{\widetilde{K}_i}^{\sigma_i-1} \arrow[d, "\tau"] \\
\mathrm{Gal}(K_i/K_{i-1}) \arrow[r, "\Xi_{K_i/K_{i-1}}"] & U_{\widetilde{K}_i}/U_{\widetilde{K}_i}^{\sigma_i-1}
\end{tikzcd}$$

is commutative, where the τ-conjugation map $\mathrm{Gal}(K_i/K_{i-1}) \to \mathrm{Gal}(K_i/K_{i-1})$ is defined by $\gamma \mapsto \tau^{-1}\gamma\tau$ for every $\gamma \in \mathrm{Gal}(K_i/K_{i-1})$. It follows that

$$\mathrm{im}\left(\Xi_{K_i/K_{i-1}}\right)^{\tau} = \mathrm{im}\left(\Xi_{K_i/K_{i-1}}\right).$$

Now, by Theorem 5.14, for

$$T_i^{\tau} = T_i = \mathrm{im}(\Xi_{K_{i+1}/K_i})$$

and

$$(T_i')^{\tau} = T_i^{\tau} \cap \left(\prod_{1\leq j\leq i+1} U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_j\tau-1}\right) / U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_{i+1}\tau-1},$$

the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & (T_i')^\tau \arrow[r] & \left(\prod_{1\le j\le i+1} U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_j\tau-1}\right)/U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_{i+1}\tau-1} \arrow[r, "\widetilde{N}_{K_{i+1}/K_i}"] & \prod_{1\le j\le i} U_{\widetilde{K}_i}^{\tau^{-1}\sigma_j\tau-1} \arrow[l, bend right, "(h_i^{(L/K)})^\tau = h_i^\tau"'] \arrow[r] & 1 \end{tikzcd}$$

splits by a homomorphism

$$h_i^\tau : \prod_{1\le j\le i} U_{\widetilde{K}_i}^{\tau^{-1}\sigma_j\tau-1} \to \left(\prod_{1\le j\le i+1} U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_j\tau-1}\right)/U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_{i+1}\tau-1},$$

and which furthermore makes the diagram

$$\begin{tikzcd} 1 \arrow[r] & T_i' \arrow[r] \arrow[d, "\tau"'] & \left(\prod_{1\le j\le i+1} U_{\widetilde{K}_{i+1}}^{\sigma_j-1}\right)/U_{\widetilde{K}_{i+1}}^{\sigma_{i+1}-1} \arrow[r, "\widetilde{N}_{K_{i+1}/K_i}"] \arrow[d, "\tau"'] & \prod_{1\le j\le i} U_{\widetilde{K}_i}^{\sigma_j-1} \arrow[l, bend right, "h_i^{(L/K)} = h_i"'] \arrow[r] \arrow[d, "\tau"'] & 1 \\ 1 \arrow[r] & (T_i')^\tau \arrow[r] & \left(\prod_{1\le j\le i+1} U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_j\tau-1}\right)/U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_{i+1}\tau-1} \arrow[r, "\widetilde{N}_{K_{i+1}/K_i}"] & \prod_{1\le j\le i} U_{\widetilde{K}_i}^{\tau^{-1}\sigma_j\tau-1} \arrow[l, bend left, "(h_i^{(L/K)})^\tau = h_i^\tau"] \arrow[r] & 1 \end{tikzcd}$$

commutative. It follows that there exists a map

$$g_i^\tau : \prod_{1\le j\le i} U_{\widetilde{K}_i}^{\tau^{-1}\sigma_j\tau-1} \to \prod_{1\le j\le i+1} U_{\widetilde{K}_{i+1}}^{\tau^{-1}\sigma_j\tau-1}$$

that makes the diagram

$$
\begin{tikzcd}
 & \prod\limits_{1\le j\le i+1} U^{\tau^{-1}\sigma_j\tau-1}_{\widetilde{K}_{i+1}} \arrow[dd, "(\mathrm{mod}\ U^{\sigma_{i+1}-1}_{\widetilde{K}_{i+1}})"] & \\
 & & \prod\limits_{1\le j\le i+1} U^{\sigma_j-1}_{\widetilde{K}_{i+1}} \arrow[ul, bend right, "\tau"'] \arrow[dd, "(\mathrm{mod}\ U^{\sigma_{i+1}-1}_{\widetilde{K}_{i+1}})"] \\
\prod\limits_{1\le j\le i} U^{\tau^{-1}\sigma_j\tau-1}_{\widetilde{K}_i} \arrow[uur, bend left, "g_i^\tau"] \arrow[r, "h_i^\tau"] & \left(\prod\limits_{1\le j\le i+1} U^{\tau^{-1}\sigma_j\tau-1}_{\widetilde{K}_{i+1}}\right)/U^{\sigma_{i+1}-1}_{\widetilde{K}_{i+1}} & \\
\prod\limits_{1\le j\le i} U^{\sigma_j-1}_{\widetilde{K}_i} \arrow[u, bend right, "\tau"'] \arrow[uurr, bend right, "g_i"'] \arrow[rr, "h_i"] & & \left(\prod\limits_{1\le j\le i+1} U^{\sigma_j-1}_{\widetilde{K}_{i+1}}\right)/U^{\sigma_{i+1}-1}_{\widetilde{K}_{i+1}} \arrow[ul, bend right, "\tau"']
\end{tikzcd}
$$

commutative. Now, for every $1 \le i \in \mathbb{Z}$, choose a mapping

$$f_i^\tau : U^{\sigma_i-1}_{\widetilde{K}_i} \to U_{\widetilde{\mathbb{X}}(L/K)}$$

such that

$$\mathrm{Pr}_{\widetilde{K}_j} \circ f_i^\tau = (g_{j-1}^\tau \circ \cdots \circ g_i^\tau)\mid_{U^{\sigma_i-1}_{\widetilde{K}_i}}$$

for each $j \in \mathbb{Z}_{>i}$. Thus, for $j \in \mathbb{Z}_{>i}$ and $\alpha \in U^{\sigma_i-1}_{\widetilde{K}_i}$ we have

$$\mathrm{Pr}_{\widetilde{K}_j} \circ f_i^\tau(\alpha) = \left(\mathrm{Pr}_{\widetilde{K}_j} \circ f_i(\alpha^{\tau^{-1}})\right)^\tau,$$

which yields the following relation:

$$f_i^\tau(\alpha) = f_i\left(\alpha^{\tau^{-1}}\right)^\tau \tag{5.30}$$

for every $\alpha \in U^{\sigma_i-1}_{\widetilde{K}_i}$. After all these observations, we state an immediate consequence of Lemma 5.17.

Corollary 5.18. *For $\tau \in \mathrm{Gal}(L/K)$ we have*

$$Z_{L/K}(\{K_i, f_i\})^\tau = Z_{L/K}(\{K_i, f_i^\tau\}).$$

Proof. Let $z \in Z_{L/K}(\{K_i, f_i\})$ and choose $z^{(i)} \in \mathrm{im}(f_i) \subset U_{\widetilde{\mathbb{X}}(L/K)}$ such that $z = \prod_i z^{(i)}$. By the continuity of the action of $\mathrm{Gal}(L/K)$ on $U_{\widetilde{\mathbb{X}}(L/K)}$, to prove that $z^\tau \in Z_{L/K}(\{K_i, f_i^\tau\})$ it suffices to show that $\left(z^{(i)}\right)^\tau \in \mathrm{im}(f_i^\tau)$. Now, let $\alpha^{(i)} \in U^{\sigma_i-1}_{\widetilde{K}_i}$ be such that $f_i(\alpha^{(i)}) = z^{(i)}$. Then $\left(z^{(i)}\right)^\tau = f_i\left(\alpha^{(i)}\right)^\tau =$

$f_i\left(((\alpha^{(i)})^{\tau})^{\tau^{-1}}\right)^{\tau} = f_i^{\tau}\left((\alpha^{(i)})^{\tau}\right)$ by (5.30), where $\left(\alpha^{(i)}\right)^{\tau} \in U_{\widetilde{K}_i}^{\sigma_i - 1}$ by Lemma 5.17. Thus, $\left(z^{(i)}\right)^{\tau} \in \operatorname{im}(f_i^{\tau})$. □

Remark 5.19. By [2, p. 71], if $\tau \in \mathrm{Gal}(L/K)$, then $Z_{L/K}(\{K_i, f_i\})$ and $Z_{L/K}(\{K_i, f_i^{\tau}\})$ are algebraically and topologically isomorphic. Thus, Corollary 5.18 indeed defines a continuous action of $\mathrm{Gal}(L/K)$ on $Z_{L/K}(\{K_i, f_i\})$.

Now, we define the topological subgroup $Y_{L/K}(\{K_i, f_i\}) = Y_{L/K}$ of $U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}$ to be

$$Y_{L/K} = \left\{ y \in U_{\widetilde{\mathbb{X}}(L/K)} : y^{1-\varphi} \in Z_{L/K}(\{K_i, f_i\}) \right\}. \tag{5.31}$$

Lemma 5.20. *$Y_{L/K}$ is a topological $\mathrm{Gal}(L/K)$-submodule of $U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}$.*

Proof. Suppose $\tau \in \mathrm{Gal}(L/K)$ and $y \in Y_{L/K}$. Note that $(y^{\tau})^{\varphi} = (y^{\varphi})^{\tau}$, because the action of τ on $y = (u_{\widetilde{K}_i})_{0 \le i \in \mathbb{Z}}$ is defined by the action of τ on the "K_i-part" of $u_{\widetilde{K}_i}$ for each $0 \le i \in \mathbb{Z}$, and the action of φ on $y = (u_{\widetilde{K}_i})_{0 \le i \in \mathbb{Z}}$ is defined by the action of φ on the "$\widetilde{K}$-part" of $u_{\widetilde{K}_i}$ for each $0 \le i \in \mathbb{Z}$. Thus, $\frac{y^{\tau}}{(y^{\tau})^{\varphi}} = \frac{y^{\tau}}{(y^{\varphi})^{\tau}} = \left(\frac{y}{y^{\varphi}}\right)^{\tau} \in Z_{L/K}(\{K_i, f_i\})^{\tau}$. Now, the proof follows from Corollary 5.18 and Remark 5.19. □

Lemma 5.21 (Fesenko)**.** *The mapping*

$$\ell_{L/K}^{(\varphi)} : \mathrm{Gal}(L/K) \to U_{\widetilde{\mathbb{X}}(L/K)}^{1} / Z_{L/K}(\{K_i, f_i\})$$

defined by

$$\ell_{L/K}^{(\varphi)} : \sigma \mapsto \Pi_{\varphi; L/K}^{\sigma - 1} . Z_{L/K}(\{K_i, f_i\})$$

for every $\sigma \in \mathrm{Gal}(L/K)$ is a group isomorphism, where the group operation $*$ on $U_{\widetilde{\mathbb{X}}(L/K)}^{1} / Z_{L/K}(\{K_i, f_i\})$ is defined by

$$\overline{U} * \overline{V} = \overline{U} . \overline{V}^{(\ell_{L/K}^{(\varphi)})^{-1}(\overline{U})}$$

for every

$$\overline{U} = U . Z_{L/K}(\{K_i, f_i\}), \quad \overline{V} = V . Z_{L/K}(\{K_i, f_i\}) \in U_{\widetilde{\mathbb{X}}(L/K)}^{1} / Z_{L/K}(\{K_i, f_i\})$$

with $U, V \in U_{\widetilde{\mathbb{X}}(L/K)}^{1}$.

Now, we introduce the *fundamental exact sequence*

$$1 \to \mathrm{Gal}(L/K) \xrightarrow{\ell_{L/K}^{(\varphi)}} U_{\widetilde{\mathbb{X}}(L/K)} / Z_{L/K}(\{K_i, f_i\}) \xrightarrow{\mathrm{Pr}_{\widetilde{K}}} U_{\widetilde{K}} \to 1$$

as a generalization of the Serre short exact sequence (cf. (1.1) and (1.2)). Thus, for any $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, since $U^{1-\varphi} \in U^{1}_{\tilde{\mathbb{X}}(L/K)}$, there exists a unique $\sigma_U \in \mathrm{Gal}(L/K)$ satisfying

$$U^{1-\varphi}.Z_{L/K}(\{K_i, f_i\}) = \ell^{(\varphi)}_{L/K}(\sigma_U), \tag{5.32}$$

by Lemma 5.21. Next, define the arrow

$$H^{(\varphi)}_{L/K} : U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K} \to \mathrm{Gal}(L/K) \tag{5.33}$$

by

$$H^{(\varphi)}_{L/K} : U.Y_{L/K} \mapsto \sigma_U \tag{5.34}$$

for every $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$. Clearly, this arrow is a well-defined mapping. Indeed, suppose that $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$ satisfy $U \equiv V \pmod{Y_{L/K}}$. Then $\sigma_U = \sigma_V$. In fact, let $Y \in Y_{L/K}$ be such that $U = V.Y$. The definition of $Y_{L/K}$ given in (5.31) forces $Y^{1-\varphi} \in Z_{L/K}(\{K_i, f_i\})$. Thus, since $U^{1-\varphi} = (V.Y)^{1-\varphi} = V^{1-\varphi}Y^{1-\varphi}$, we have $U^{1-\varphi}Z_{L/K}(\{K_i, f_i\}) = V^{1-\varphi}Z_{L/K}(\{K_i, f_i\})$, which shows that $\ell^{(\varphi)}_{L/K}(\sigma_U) = \ell^{(\varphi)}_{L/K}(\sigma_V)$ by (5.32). Now shows Lemma 5.21 shows that $\sigma_U = \sigma_V$.

Lemma 5.22. *Suppose that the local field K satisfies condition* (5.26). *The arrow*

$$H^{(\varphi)}_{L/K} : U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K} \to \mathrm{Gal}(L/K)$$

defined for the extension L/K is a bijection.

Proof. Choose $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$ satisfying $H^{(\varphi)}_{L/K}(U.Y_{L/K}) = H^{(\varphi)}_{L/K}(V.Y_{L/K})$. Then $\sigma_U = \sigma_V$ by the definition (5.34) of the arrow (5.33). Now, (5.32) yields

$$U^{1-\varphi}.Z_{L/K}(\{K_i, f_i\}) = V^{1-\varphi}.Z_{L/K}(\{K_i, f_i\}),$$

which proves that $(V^{-1}U)^{1-\varphi} \in Z_{L/K}(\{K_i, f_i\})$. The fact that $U.Y_{L/K} = V.Y_{L/K}$ follows immediately from (5.31). Now, we choose any $\sigma \in \mathrm{Gal}(L/K)$. By Theorem 5.5, there exists $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$ unique modulo $U_{\mathbb{X}(L/K)}$ (so unique modulo $Y_{L/K}$ because $U_{\mathbb{X}(L/K)} \subseteq Y_{L/K}$) and such that

$$\Pi^{\sigma-1}_{\varphi;L/K}.Z_{L/K}(\{K_i, f_i\}) = U^{1-\varphi}.Z_{L/K}(\{K_i, f_i\}).$$

Thus, by Theorem 5.21 and (5.32),

$$\ell^{(\varphi)}_{L/K}(\sigma) = \ell^{(\varphi)}_{L/K}(\sigma_U),$$

which implies that $\sigma = \sigma_U$ for $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$. $\square$

Next, consider the composition

$$\Phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\phi^{(\varphi)}_{L/K}} U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \xrightarrow{c_{L/K}} U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}. \tag{5.35}$$

Lemma 5.23. (i) $U_{\sigma_U}.Y_{L/K} = U.Y_{L/K}$ *for every* $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$;
(ii) $\sigma_{U_\sigma} = \sigma$ *for every* $\sigma \in \mathrm{Gal}(L/K)$.

Proof. To prove (i), let $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$. Then, by (5.32), there exists a unique $\sigma_U \in \mathrm{Gal}(L/K)$ satisfying

$$U^{1-\varphi}.Z_{L/K}(\{K_i, f_i\}) = \ell^{(\varphi)}_{L/K}(\sigma_U) = \Pi^{\sigma_U - 1}_{\varphi;L/K}.Z_{L/K}(\{K_i, f_i\}). \tag{5.36}$$

The identity on the right-hand side follows from the definition of the mapping $\ell^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to U^{1}_{\tilde{\mathbb{X}}(L/K)}/Z_{L/K}(\{K_i, f_i\})$ given in Lemma 5.21. Now, by Lemma 5.5, for this $\sigma_U \in \mathrm{Gal}(L/K)$ there exists $U_{\sigma_U} \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$ (which is unique modulo $U_{\mathbb{X}(L/K)}$) satisfying

$$U^{1-\varphi}_{\sigma_U} = \Pi^{\sigma_U - 1}_{\varphi;L/K}.$$

Thus,

$$U^{1-\varphi}_{\sigma_U}.Z_{L/K}(\{K_i, f_i\}) = U^{1-\varphi}.Z_{L/K}(\{K_i, f_i\}),$$

by (5.36), which proves that

$$U_{\sigma_U}.Y_{L/K} = U.Y_{L/K} \tag{5.37}$$

by the definition of $Y_{L/K}$ given in (5.31). Moreover, since $U_{\mathbb{X}(L/K)} \subseteq Y_{L/K}$, relation (5.37) does not depend on the choice of U_{σ_U} modulo $U_{\mathbb{X}(L/K)}$. Now, for (ii), let $\sigma \in \mathrm{Gal}(L/K)$. By Lemma 5.5, there exists $U_\sigma \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$ (which is unique modulo $U_{\mathbb{X}(L/K)}$) such that

$$U^{1-\varphi}_{\sigma} = \Pi^{\sigma - 1}_{\varphi;L/K}. \tag{5.38}$$

For any such $U_\sigma \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, there exists a unique $\sigma_{U_\sigma} \in \mathrm{Gal}(L/K)$ satisfying

$$U^{1-\varphi}_{\sigma}.Z_{L/K}(\{K_i, f_i\}) = \ell^{(\varphi)}_{L/K}(\sigma_{U_\sigma})$$

by (5.32). Thus, by (5.38) and Lemma 5.21, it follows that

$$\ell^{(\varphi)}_{L/K}(\sigma_{U_\sigma}) = \Pi^{\sigma-1}_{\varphi;L/K}.Z_{L/K}(\{K_i, f_i\}) = \ell^{(\varphi)}_{L/K}(\sigma),$$

which proves that $\sigma_{U_\sigma} = \sigma$. □

Lemma 5.23 immediately yields

$$H^{(\varphi)}_{L/K} \circ \Phi^{(\varphi)}_{L/K} = \mathrm{id}_{\mathrm{Gal}(L/K)};$$

and

$$\Phi^{(\varphi)}_{L/K} \circ H^{(\varphi)}_{L/K} = \mathrm{id}_{U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}}.$$

The following theorem is a consequence of Lemma 5.22, Lemma 5.23, Theorem 5.6, and the fact that $U_{\mathbb{X}(L/K)}$ is a topological $\mathrm{Gal}(L/K)$-submodule of $Y_{L/K}$.

Theorem 5.24 (Fesenko)**.** *Suppose that the local field K satisfies* (5.26)*. The mapping*

$$\Phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}$$

defined for the extension L/K is a bijection with the inverse

$$H^{(\varphi)}_{L/K} : U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K} \to \mathrm{Gal}(L/K).$$

For every $\sigma, \tau \in \mathrm{Gal}(L/K)$ we have

$$\Phi^{(\varphi)}_{L/K}(\sigma\tau) = \Phi^{(\varphi)}_{L/K}(\sigma)\Phi^{(\varphi)}_{L/K}(\tau)^{\sigma}, \tag{5.39}$$

the cocycle condition is satisfied.

By Corollary 5.7, Theorem 5.24 has the following consequence.

Corollary 5.25. *Define a law of composition $*$ on $U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}$ by*

$$\overline{U} * \overline{V} = \overline{U}.\overline{V}^{(\Phi^{(\varphi)}_{L/K})^{-1}(\overline{U})} \tag{5.40}$$

for every $\overline{U} = U.Y_{L/K}, \overline{V} = V.Y_{L/K} \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}$ with $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$. Then $U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}$ is a topological group under $*$, and the map $\Phi^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\Phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}, \tag{5.41}$$

where the topological group structure on $U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}$ is defined with respect to the binary operation $*$ given by (5.40)*.*

Definition 5.26. Let K be a local field satisfying (5.26). Let L/K be a totally ramified APF-Galois extension satisfying (5.1). The mapping

$$\Phi^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K}$$

defined in Theorem 5.24 is called the *Fesenko reciprocity map for the extension L/K*.

For each $0 \leq i \in \mathbb{R}$, previously we have introduced the groups $\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^i$. For $0 \leq n \in \mathbb{Z}$, let

$$Q^n_{L/K} = c_{L/K}\left(\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \cap \operatorname{im}(\phi^{(\varphi)}_{L/K})\right), \tag{5.42}$$

which is a subgroup of $\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/K}/Y_{L/K}$. Now, the Fesenko ramification theorem, stated in Theorem 5.8, can be reformulated for the reciprocity map $\Phi^{(\varphi)}_{L/K}$ corresponding to the extension L/K as follows.

Theorem 5.27 (Ramification theorem). *Suppose that the local field K satisfies the condition given in* (5.26). *For $0 \leq n \in \mathbb{Z}$, let* $\mathrm{Gal}(L/K)_n$ *denote the nth higher ramification subgroup of the Galois group* $\mathrm{Gal}(L/K)$ *corresponding to the APF-Galois subextension L/K of K_φ/K in the lower numbering. Then we have the inclusion*

$$\Phi^{(\varphi)}_{L/K}\left(\mathrm{Gal}(L/K)_n - \mathrm{Gal}(L/K)_{n+1}\right) \subseteq \left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/K}/Y_{L/K} - Q^{n+1}_{L/K}.$$

Proof. Let $\tau \in \mathrm{Gal}(L/K)_n$. The first half of the proof of Proposition 1 in [2] shows that $\Phi^{(\varphi)}_{L/K}(\tau) \in \left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/K}/Y_{L/K}$. Now, let $\overline{U} = U.Y_{L/K} \in Q^{n+1}_{L/K}$, where $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$. Then, by the definition of $Q^{n+1}_{L/K}$, there exist $V \in \left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^{n+1} U_{\mathbb{X}(L/K)}$ and $\tau \in \mathrm{Gal}(L/K)$ such that $c_{L/K}(\overline{V}) = \overline{U}$ and $\phi^{(\varphi)}_{L/K}(\tau) = \overline{V}$, where $\overline{V} = V.U_{\mathbb{X}(L/K)}$. So, $\Phi^{(\varphi)}_{L/K}(\tau) = \overline{U}$. The second half of the proof of Proposition 1 in [2] now proves that $\tau \in \mathrm{Gal}(L/K)_{n+1}$. $\square$

Let M/K be a Galois subextension of L/K. Thus, there exists a chain of field extensions

$$K \subseteq M \subseteq L \subseteq K_\varphi,$$

where M is a totally ramified APF-Galois extension over the local field K satisfying (5.26) by Lemma 3.3.

The basic ascending chain of subextensions in L/K fixed in (5.27) and restricted to M,

$$K = K_o \cap M \subseteq K_1 \cap M \subseteq \cdots \subseteq K_i \cap M \subseteq \cdots \subseteq L \cap M = M, \tag{5.43}$$

is *almost* a basic ascending chain of subextensions in M/K (in the sense that there may exist elements $0 \leq i \in \mathbb{Z}$ such that $K_i \cap M = K_{i+1} \cap M$). In fact, for each $0 \leq i \in \mathbb{Z}$, the extension $K_i \cap M/K$ is clearly Galois. For each $0 \leq i \in \mathbb{Z}$, consider the surjective homomorphism

$$r_{K_{i+1}\cap M} : \mathrm{Gal}(K_{i+1}/K_i) \twoheadrightarrow \mathrm{Gal}(K_{i+1} \cap M/K_i \cap M)$$

defined by restriction to $K_{i+1} \cap M$,

$$\sigma \mapsto \sigma \mid_{K_{i+1} \cap M}$$

for every $\sigma \in \mathrm{Gal}(K_{i+1}/K_i)$. Since $\mathrm{Gal}(K_{i+1}/K_i)$ is cyclic of prime order $p = \mathrm{char}(\kappa_K)$ (respectively, of order relatively prime to p) for $1 \leq i \in \mathbb{Z}$ (respectively, for $i = 0$), it follows that $\mathrm{Gal}(K_{i+1} \cap M/K_i \cap M)$ is cyclic of order p or 1 (respectively, of order relatively prime to p) for $1 \leq i \in \mathbb{Z}$ (respectively, for $i = 0$). Now we fix this almost basic ascending chain of subextensions in M/K, introduced in (5.43). Observe that, for each $1 \leq i \in \mathbb{Z}$, $\sigma_i \mid_{\widetilde{M}} \in \mathrm{Gal}(\widetilde{M}/\widetilde{K})$ satisfies

$$< (\sigma_i \mid_{\widetilde{M}}) \mid_{K_i \cap M} = \sigma_i \mid_{K_i \cap M} > = \mathrm{Gal}(K_i \cap M/K_{i-1} \cap M),$$

because the restriction map $r_{K_i \cap M} : \mathrm{Gal}(K_i/K_{i-1}) \twoheadrightarrow \mathrm{Gal}(K_i \cap M/K_{i-1} \cap M)$ is a surjective homomorphism and $< \sigma_i \mid_{K_i} > = \mathrm{Gal}(K_i/K_{i-1})$. As usual, we set $\widetilde{K_i \cap M} = (K_i \cap M)\widetilde{K}$. Note that, for each $1 \leq k \in \mathbb{Z}$, the norm map

$$\widetilde{N}_{K_{k+1}/K_{k+1} \cap M} : U_{\widetilde{K}_{k+1}} \to U_{\widetilde{K_{k+1} \cap M}}$$

induces a homomorphism

$$\widetilde{N}^*_{K_{k+1}/K_{k+1} \cap M} : U_{\widetilde{K}_{k+1}}/U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1} \to U_{\widetilde{K_{k+1} \cap M}}/U_{\widetilde{K_{k+1} \cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}$$

defined by

$$\widetilde{N}^*_{K_{k+1}/K_{k+1} \cap M} : u.U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1} \mapsto \widetilde{N}_{K_{k+1}/K_{k+1} \cap M}(u).U_{\widetilde{K_{k+1} \cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}$$

for every $u \in U_{\widetilde{K}_{k+1}}$, because $\widetilde{N}_{K_{k+1}/K_{k+1} \cap M}\left(U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}\right) \subseteq U_{\widetilde{K_{k+1} \cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}$. Thus, the following square (in which) the upper and lower horizontal arrows are defined by (5.28) and (5.29),

$$\begin{array}{ccc}
\mathrm{Gal}(K_{k+1}/K_k) & \xrightarrow{\Xi_{K_{k+1}/K_k}} & U_{\widetilde{K}_{k+1}}/U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1} \\
{\scriptstyle r_{K_{k+1} \cap M}} \big\downarrow & & \big\downarrow {\scriptstyle \widetilde{N}^*_{K_{k+1}/K_{k+1} \cap M}} \\
\mathrm{Gal}(K_{k+1} \cap M/K_k \cap M) & \xrightarrow{\Xi_{K_{k+1} \cap M/K_k \cap M}} & U_{\widetilde{K_{k+1} \cap M}}/U_{\widetilde{K_{k+1} \cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1},
\end{array}$$

is commutative, because $\widetilde{N}_{K_{k+1}/K_{k+1} \cap M}(\pi_{K_{k+1}}) = \pi_{K_{k+1} \cap M}$ by the *norm coherence* of the Lubin–Tate labelling $(\pi_{K'})_{\underbrace{K \subseteq K'}_{[K':K]<\infty} \subset K_\varphi}$. Hence,

$$\widetilde{N}^*_{K_{k+1}/K_{k+1} \cap M}\left(T_k^{(L/K)}\right) = T_k^{(M/K)}.$$

Next, we define an arrow

$$h_k^{(M/K)}:\prod_{1\le i\le k}U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_k\cap M}}\to\left(\prod_{1\le i\le k+1}U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}\right)/U^{\sigma_{k+1}|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}\tag{5.44}$$

that splits the exact sequence

$$\begin{tikzcd}
1 \arrow[r] & T_k^{(M/K)\prime} \arrow[r] & \left(\prod\limits_{1\le i\le k+1}U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}\right)/U^{\sigma_{k+1}|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}} \arrow[r, "\widetilde{N}_{K_{k+1}\cap M/K_k\cap M}"] & \prod\limits_{1\le i\le k}U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_k\cap M}} \arrow[r] \arrow[l, bend right, "h_k^{(M/K)}"'] & 1
\end{tikzcd}\tag{5.45}$$

in such a way that

$$\begin{tikzcd}
\left(\prod\limits_{1\le i\le k+1}U^{\sigma_i-1}_{\widetilde{K}_{k+1}}\right)/U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}} \arrow[r, "\widetilde{N}_{K_{k+1}/K_k}"] \arrow[d, "\widetilde{N}^*_{K_{k+1}/K_{k+1}\cap M}"'] & \prod\limits_{1\le i\le k}U^{\sigma_i-1}_{\widetilde{K}_k} \arrow[d, "\widetilde{N}_{K_k/K_k\cap M}"] \arrow[l, bend right, "h_k^{(L/K)}"'] \\
\left(\prod\limits_{1\le i\le k+1}U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}\right)/U^{\sigma_{k+1}|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}} \arrow[r, "\widetilde{N}_{K_{k+1}\cap M/K_k\cap M}"] & \prod\limits_{1\le i\le k}U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_k\cap M}} \arrow[l, bend left, "h_k^{(M/K)}"]
\end{tikzcd}\tag{5.46}$$

is a commutative square. For this, however, closely following Fesenko (see [1–3]), we review the construction of a splitting

$$h_k^{(L/K)}:\prod_{1\le i\le k}U^{\sigma_i-1}_{\widetilde{K}_k}\to\left(\prod_{1\le i\le k+1}U^{\sigma_i-1}_{\widetilde{K}_{k+1}}\right)/U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}}$$

of the short exact sequence

$$\begin{tikzcd}
1 \arrow[r] & T_k^{(L/K)\prime} \arrow[r] & \left(\prod\limits_{1\le i\le k+1}U^{\sigma_i-1}_{\widetilde{K}_{k+1}}\right)/U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}} \arrow[r, "\widetilde{N}_{K_{k+1}/K_k}"] & \prod\limits_{1\le i\le k}U^{\sigma_i-1}_{\widetilde{K}_k} \arrow[r] \arrow[l, bend right, "h_k^{(L/K)}"'] & 1.
\end{tikzcd}\tag{5.47}$$

The product module $\prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1}$ is a closed $\mathbb{Z}_p$-submodule of $U_{\widetilde{K}_k}^1$. Let $\{\lambda_j\}$ be a system of topological multiplicative generators of the topological $\mathbb{Z}_p$-module $\prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1}$ satisfying the following property. If the torsion $\big(\prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1}\big)_{\text{tor}}$ of the module $\prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1}$ is nontrivial, there exists $\lambda_* \in \{\lambda_j\}$ of order p^m in the torsion of the module, while the remaining λ_j $(j\neq *)$ are topologically independent over $\mathbb{Z}_p$. Now, we define a map

$$h_k^{(L/K)} : \{\lambda_j\} \to \left(\prod_{1\leq i\leq k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1}\right) / U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}$$

on the topological generators $\{\lambda_j\}$ by

$$h_k^{(L/K)} : \lambda_j \mapsto u_j . U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1},$$

where $u_j \in \prod_{1\leq i\leq k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1}$ satisfies $\widetilde{N}_{K_{k+1}/K_k}(u_j) = \lambda_j$. By step 5 of the proof of the Theorem in §3 of [3], it follows that $h_k^{(L/K)}(\lambda_*)^{p^m} \in U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}$. Therefore, the arrow $h_k^{(L/K)}$ extends uniquely to a homomorphism

$$h_k^{(L/K)} : \prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1} \to \left(\prod_{1\leq i\leq k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1}\right) / U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1},$$

which is a splitting of the short exact sequence (5.47). Now, we define

$$h_k^{(M/K)} : \prod_{1\leq i\leq k} U_{\widetilde{K_k\cap M}}^{\sigma_i|_{\widetilde{M}}-1} \to \left(\prod_{1\leq i\leq k+1} U_{\widetilde{K_{k+1}\cap M}}^{\sigma_i|_{\widetilde{M}}-1}\right) / U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}$$

as follows. Observe that

$$\widetilde{N}_{K_k/K_k\cap M} : \prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1} \to \prod_{1\leq i\leq k} U_{\widetilde{K_k\cap M}}^{\sigma_i|_{\widetilde{M}}-1}$$

is a surjective homomorphism, because $\widetilde{N}_{K_k/K_k\cap M} : U_{\widetilde{K}_k} \to U_{\widetilde{K_k\cap M}}$ is a surjective homomorphism. Thus, the collection $\{\widetilde{N}_{K_k/K_k\cap M}(\lambda_j)\}$ is a system of topological multiplicative generators of the topological $\mathbb{Z}_p$-module $\prod_{1\leq i\leq k} U_{\widetilde{K_k\cap M}}^{\sigma_i|_{\widetilde{M}}-1}$. Moreover, note that $\widetilde{N}_{K_k/K_k\cap M}(\lambda_*)^{p^m} = 1$. Therefore, $\widetilde{N}_{K_k/K_k\cap M}(\lambda_*)$ is in the torsion part $\big(\prod_{1\leq i\leq k} U_{\widetilde{K_k\cap M}}^{\sigma_i|_{\widetilde{M}}-1}\big)_{\text{tor}}$. For the remaining λ_j $(j\neq *)$, the collection $\big\{\widetilde{N}_{K_k/K_k\cap M}(\lambda_j)\big\}_{j\neq *}$ is topologically independent over $\mathbb{Z}_p$. Now, following

Fesenko's construction of $h_k^{(L/K)}$, we define a map

$$h_k^{(M/K)} : \left\{\widetilde{N}_{K_k/K_k\cap M}(\lambda_j)\right\} \to \left(\prod_{1\le i\le k+1} U_{\widetilde{K_{k+1}\cap M}}^{\sigma_i|_{\widetilde{M}}-1}\right) / U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}$$

on the topological generators $\left\{\widetilde{N}_{K_k/K_k\cap M}(\lambda_j)\right\}$ by

$$h_k^{(M/K)} : \widetilde{N}_{K_k/K_k\cap M}(\lambda_j) \mapsto \widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j).U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}, \tag{5.48}$$

where $u_j \in \prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1}$ satisfies $\widetilde{N}_{K_{k+1}/K_k}(u_j) = \lambda_j$, and thereby $\widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j)$ satisfies

$$\begin{aligned}\widetilde{N}_{K_{k+1}\cap M/K_k\cap M}\left(\widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j)\right) &= \widetilde{N}_{K_{k+1}/K_k\cap M}(u_j)\\ &= \widetilde{N}_{K_k/K_k\cap M}\left(\widetilde{N}_{K_{k+1}/K_k}(u_j)\right) = \widetilde{N}_{K_k/K_k\cap M}(\lambda_j).\end{aligned}$$

Therefore, the arrow $h_k^{(M/K)}$ extends uniquely to a homomorphism

$$h_k^{(M/K)} : \prod_{1\le i\le k} U_{\widetilde{K_k\cap M}}^{\sigma_i|_{\widetilde{M}}-1} \to \left(\prod_{1\le i\le k+1} U_{\widetilde{K_{k+1}\cap M}}^{\sigma_i|_{\widetilde{M}}-1}\right) / U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1},$$

which is a splitting of the short exact sequence given by (5.45). Indeed, it suffices to show that, for $u_j \in \prod_{1\le i\le k+1} U_{\widetilde{K}_{k+1}}^{\sigma_i-1}$ satisfying $\widetilde{N}_{K_{k+1}/K_k}(u_j) = \lambda_j$, we have

$$\begin{aligned}h_k^{(M/K)} \circ \widetilde{N}_{K_{k+1}\cap M/K_k\cap M} &: \widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j) U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}\\ &\mapsto \widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j) U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1},\end{aligned}$$

which follows from the identities

$$\begin{aligned}h_k^{(M/K)}\left(\widetilde{N}_{K_{k+1}\cap M/K_k\cap M}(\widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j))\right) &= h_k^{(M/K)}(\widetilde{N}_{K_{k+1}/K_k\cap M}(u_j))\\ = h_k^{(M/K)}(\widetilde{N}_{K_k/K_k\cap M}(\widetilde{N}_{K_{k+1}/K_k}(u_j))) &= h_k^{(M/K)}(\widetilde{N}_{K_k/K_k\cap M}(\lambda_j))\end{aligned}$$

and from the definition (5.48) of the arrow $h_k^{(M/K)}$. Moreover, the diagram (5.46) commutes, because

$$\begin{aligned} h_k^{(M/K)}(\widetilde{N}_{K_k/K_k\cap M}(\lambda_j)) &= \widetilde{N}_{K_{k+1}/K_{k+1}\cap M}(u_j)U^{\sigma_{k+1}|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}} \\ &= \widetilde{N}^*_{K_{k+1}/K_{k+1}\cap M}(u_j.U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}}) \\ &= \widetilde{N}^*_{K_{k+1}/K_{k+1}\cap M}(h_k^{(L/K)}(\lambda_j)). \end{aligned}$$

For each $1 \le k \in \mathbb{Z}$, consider any map

$$g_k^{(M/K)} : \prod_{1\le i\le k} U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_k\cap M}} \to \prod_{1\le i\le k+1} U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}} \tag{5.49}$$

that makes the square

$$\begin{tikzcd} \prod_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{K}_k} \arrow[r, "g_k^{(L/K)}"] \arrow[d, "\widetilde{N}_{K_k/K_k\cap M}"'] & \prod_{1\le i\le k+1} U^{\sigma_i-1}_{\widetilde{K}_{k+1}} \arrow[d, "\widetilde{N}_{K_{k+1}/K_{k+1}\cap M}"] \\ \prod_{1\le i\le k} U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_k\cap M}} \arrow[r, dotted, "g_k^{(M/K)}"] & \prod_{1\le i\le k+1} U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}, \end{tikzcd} \tag{5.50}$$

commutative. Note that such a map satisfies

$$h_k^{(M/K)} = g_k^{(M/K)} \mod U^{\sigma_{k+1}|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}.$$

Indeed, by the commutative diagram (5.46), for any $w \in \prod_{1\le i\le k} U^{\sigma_i|_{\widetilde{M}}-1}_{\widetilde{K_k\cap M}}$, there exists $v \in \prod_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{K}_k}$ such that $w = \widetilde{N}_{K_k/K_k\cap M}(v)$, and

$$\begin{aligned} h_k^{(M/K)}(w) &= h_k^{(M/K)}\left(\widetilde{N}_{K_k/K_k\cap M}(v)\right) \\ &= \widetilde{N}^*_{K_{k+1}/K_{k+1}\cap M}\left(h_k^{(L/K)}(v)\right) \\ &= \widetilde{N}^*_{K_{k+1}/K_{k+1}\cap M}\left(g_k^{(L/K)}(v).U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}}\right) \\ &= \widetilde{N}_{K_{k+1}/K_{k+1}\cap M}\left(g_k^{(L/K)}(v)\right).U^{\sigma_{k+1}|_{\widetilde{M}}-1}_{\widetilde{K_{k+1}\cap M}}, \end{aligned}$$

and by the commutativity of the diagram (5.50) we have

$$\widetilde{N}_{K_{k+1}/K_{k+1}\cap M}\left(g_k^{(L/K)}(v)\right) = g_k^{(M/K)}\left(\widetilde{N}_{K_k/K_k\cap M}(v)\right) = g_k^{(M/K)}(w).$$

Thus, the relation

$$h_k^{(M/K)}(w) = g_k^{(M/K)}(w).U_{\widetilde{K_{k+1}\cap M}}^{\sigma_{k+1}|_{\widetilde{M}}-1}$$

follows for every $w \in \prod_{1\leq i\leq k} U_{\widetilde{K_k\cap M}}^{\sigma_i|_{\widetilde{M}}-1}$.

Now, for each $1 \leq i \in \mathbb{Z}$, we introduce the map

$$f_i^{(M/K)} : U_{\widetilde{K_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1} \to U_{\widetilde{\mathbb{X}}(M/K)}$$

by

$$f_i^{(M/K)}(w) = \widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/K)}(v)\right),$$

where $v \in U_{\widetilde{K}_i}^{\sigma_i-1}$ is any element satisfying $\widetilde{N}_{K_i/K_i\cap M}(v) = w \in U_{\widetilde{K_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1}$. Note that if $v' \in U_{\widetilde{K}_i}^{\sigma_i-1}$ is such that $\widetilde{N}_{K_i/K_i\cap M}(v') = w$, then $\widetilde{\mathcal{N}}_{L/M}\big(f_i^{(L/K)}(v)\big) = \widetilde{\mathcal{N}}_{L/M}\big(f_i^{(L/K)}(v')\big)$. In fact, there exists $u \in \ker\big(\widetilde{N}_{K_i/K_i\cap M}\big)$ such that $v' = vu$. Thus, we need to verify that $\widetilde{\mathcal{N}}_{L/M}\big(f_i^{(L/K)}(v)\big) = \widetilde{\mathcal{N}}_{L/M}\big(f_i^{(L/K)}(vu)\big)$. That is, for each $1 \leq j \in \mathbb{Z}$, we must check that

$$\widetilde{N}_{K_j/K_j\cap M}\left(\mathrm{Pr}_{\widetilde{K}_j}(f_i^{(L/K)}(v))\right) = \widetilde{N}_{K_j/K_j\cap M}\left(\mathrm{Pr}_{\widetilde{K}_j}(f_i^{(L/K)}(vu))\right).$$

But for $j > i$ we have

$$\begin{aligned}
\widetilde{N}_{K_j/K_j\cap M}\left(\mathrm{Pr}_{\widetilde{K}_j}(f_i^{(L/K)}(v))\right) &= \widetilde{N}_{K_j/K_j\cap M}\left(g_{j-1}^{(L/K)}\circ\cdots\circ g_i^{(L/K)}(v)\right)\\
&= g_{j-1}^{(M/K)}\circ\cdots\circ g_i^{(M/K)}\left(\widetilde{N}_{K_i/K_i\cap M}(v)\right)\\
&= g_{j-1}^{(M/K)}\circ\cdots\circ g_i^{(M/K)}\left(\widetilde{N}_{K_i/K_i\cap M}(vu)\right)\\
&= \widetilde{N}_{K_j/K_j\cap M}\left(g_{j-1}^{(L/K)}\circ\cdots\circ g_i^{(L/K)}(vu)\right)\\
&= \widetilde{N}_{K_j/K_j\cap M}\left(\mathrm{Pr}_{\widetilde{K}_j}(f_i^{(L/K)}(vu))\right).
\end{aligned}$$

Thus, the map

$$f_i^{(M/K)} : U_{\widetilde{K_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1} \to U_{\widetilde{\mathbb{X}}(M/K)}$$

is well defined. Moreover,

$$\mathrm{Pr}_{\widetilde{K_j\cap M}}\circ f_i^{(M/K)} = \left(g_{j-1}^{(M/K)}\circ\cdots\circ g_i^{(M/K)}\right)|_{U_{\widetilde{K_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1}}$$

for $j > i$. Indeed, for $w \in U^{\sigma_i|_{\widetilde{M}}-1}_{\widehat{K_i\cap M}}$, there exists $v \in U^{\sigma_i-1}_{\widetilde{K}_i}$ with $\widetilde{N}_{K_i/K_i\cap M}(v) = w$, and $f_i^{(M/K)}(w) = \widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/K)}(v)\right)$. That is, the square

$$\begin{tikzcd} U^{\sigma_i-1}_{\widetilde{K}_i} \arrow[r, "f_i^{(L/K)}"] \arrow[d, "\widetilde{N}_{K_i/K_i\cap M}"'] & U_{\widetilde{\mathbb{X}}(L/K)} \arrow[d, "\widetilde{\mathcal{N}}_{L/M}"] \\ U^{\sigma_i|_{\widetilde{M}}-1}_{\widehat{K_i\cap M}} \arrow[r, "f_i^{(M/K)}"] & U_{\widetilde{\mathbb{X}}(M/K)} \end{tikzcd} \tag{5.51}$$

is commutative. Thus,

$$\begin{aligned} \mathrm{Pr}_{\widehat{K_j\cap M}} \circ f_i^{(M/K)}(w) &= \mathrm{Pr}_{\widehat{K_j\cap M}} \circ \widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/K)}(v)\right) \\ &= \widetilde{N}_{K_j/K_j\cap M}\left(\mathrm{Pr}_{\widetilde{K}_j} \circ f_i^{(L/K)}(v)\right) \\ &= \widetilde{N}_{K_j/K_j\cap M}\left((g_{j-1}^{(L/K)} \circ \cdots \circ g_i^{(L/K)})(v)\right) \\ &= \left(g_{j-1}^{(M/K)} \circ \cdots \circ g_i^{(M/K)}\right)\left(\widetilde{N}_{K_i/K_i\cap M}(v)\right), \end{aligned}$$

as desired.

For each $0 < i \in \mathbb{Z}$, let

$$Z_i^{(M/K)} = \mathrm{im}(f_i^{(M/K)}).$$

Then, by Lemma 5.15 or by [2, Lemma 4], for $z^{(i)} \in Z_i^{(M/K)}$ the product $\prod_i z^{(i)}$ converges to an element in $U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}$. Let

$$Z_{M/K}\left(\{K_i \cap M, f_i^{(M/K)}\}\right) = \left\{\prod_i z^{(i)} : z^{(i)} \in Z_i^{(M/K)}\right\},$$

which is a topological subgroup of $U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}$. We introduce the topological $\mathrm{Gal}(M/K)$-submodule $Y_{M/K}\left(\{K_i \cap M, f_i^{(M/K)}\}\right) = Y_{M/K}$ of $U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}$ by

$$Y_{M/K} = \left\{y \in U_{\widetilde{\mathbb{X}}(M/K)} : y^{1-\varphi} \in Z_{M/K}\left(\{K_i \cap M, f_i^{(M/K)}\}\right)\right\}.$$

Lemma 5.28. *The norm map* $\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^\times \to \widetilde{\mathbb{X}}(M/K)^\times$ *introduced by* (5.15) *and* (5.16) *further satisfies*

(i) $\widetilde{\mathcal{N}}_{L/M}\left(Z_{L/K}(\{K_i, f_i^{(L/K)}\})\right) \subseteq Z_{M/K}\left(\{K_i \cap M, f_i^{(M/K)}\}\right)$;

(ii) $\widetilde{\mathcal{N}}_{L/M}(Y_{L/K}) \subseteq Y_{M/K}$.

Proof. Recall that $\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^\times \to \widetilde{\mathbb{X}}(M/K)^\times$ is a continuous mapping.

(i) For any choice of $z^{(i)} \in Z_i^{(L/K)}$, the continuity of the multiplicative arrow $\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^\times \to \widetilde{\mathbb{X}}(M/K)^\times$ yields

$$\widetilde{\mathcal{N}}_{L/M}\Big(\prod_i z^{(i)}\Big) = \prod_i \widetilde{\mathcal{N}}_{L/M}(z^{(i)}),$$

where $\widetilde{\mathcal{N}}_{L/M}(z^{(i)}) \in Z_i^{(M/K)}$ by the commutative square (5.51).

(ii) Now let $y \in Y_{L/K}$. Then $y^{1-\varphi} \in Z_{L/K}\big(\{K_i, f_i^{(L/K)}\}\big)$. Consequently, $\widetilde{\mathcal{N}}_{L/M}(y^{1-\varphi}) = \widetilde{\mathcal{N}}_{L/M}(y)^{1-\varphi} \in Z_{M/K}\big(\{K_i \cap M, f_i^{(M/K)}\}\big)$ by part (i), and the result follows. □

Thus, the norm map $\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^\times \to \widetilde{\mathbb{X}}(M/K)^\times$ defined by (5.16) induces a group homomorphism, which will again be called the *Coleman norm map from L to M*,

$$\widetilde{\mathcal{N}}_{L/M}^{\text{Coleman}} : U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \to U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K}, \tag{5.52}$$

and is defined by

$$\widetilde{\mathcal{N}}_{L/M}^{\text{Coleman}}(\overline{U}) = \widetilde{\mathcal{N}}_{L/M}(U).Y_{M/K} \tag{5.53}$$

for every $U \in U^\diamond_{\widetilde{\mathbb{X}}(L/K)}$, where, as usual, $\overline{U}$ denotes the coset $U.Y_{L/K}$ in $U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K}$.

Let

$$\Phi^{(\varphi)}_{M/K} : \mathrm{Gal}(M/K) \to U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K}$$

be the corresponding Fesenko reciprocity map defined for the extension M/K, where $Y_{M/K} = Y_{M/K}\big(\{K_i \cap M, f_i^{(M/K)}\}\big)$.

Theorem 5.29. *For the Galois subextension M/K of L/K, the square*

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\Phi^{(\varphi)}_{L/K}} & U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \\ {\scriptstyle \mathrm{res}_M}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\ \mathrm{Gal}(M/K) & \xrightarrow{\Phi^{(\varphi)}_{M/K}} & U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K}, \end{array} \tag{5.54}$$

where the right vertical arrow

$$\widetilde{\mathcal{N}}_{L/M}^{\text{Coleman}} : U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \to U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K}$$

is the Coleman norm map from L to M defined by (5.52) *and* (5.53)*, is commutative.*

Proof. It suffices to prove that the square

$$\begin{array}{ccc} U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \\ {\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\ U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K} \end{array}$$

is commutative, which is obvious. Then pasting this square to the square (5.19), we obtain the diagram

$$\begin{array}{ccccc} \mathrm{Gal}(L/K) & \xrightarrow{\phi^{(\varphi)}_{L/K}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \\ {\scriptstyle \mathrm{res}_M}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} & & \big\downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\ \mathrm{Gal}(M/K) & \xrightarrow{\phi^{(\varphi)}_{M/K}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K}, \end{array}$$

and the commutativity of the square (5.54) follows. □

Let F/K be a finite subextension of L/K. Then, since F is *compatible* with (K, φ) in the sense of [13, p. 89], we may fix the Lubin–Tate splitting over F to be $\varphi_F = \varphi_K = \varphi$. Thus, there exists a chain of field extensions

$$K \subseteq F \subseteq L \subseteq K_\varphi \subseteq F_\varphi,$$

where L is a totally ramified APF-Galois extension over F by Lemma 3.3. Since $\boldsymbol{\mu}_p(K^{\text{sep}}) = \boldsymbol{\mu}_p(F^{\text{sep}})$, the inclusion $\boldsymbol{\mu}_p(F^{\text{sep}}) \subset F$ is satisfied. That is, the local field F satisfies the condition given by (5.26).

Now, the basic ascending chain of subextensions in L/K fixed in (5.27), with base changed to F,

$$F = K_oF \subseteq K_1F \subseteq \cdots \subseteq K_iF \subseteq \cdots \subseteq L, \tag{5.55}$$

is *almost* a basic ascending chain of subextensions in L/F, which follows by the isomorphisms $\mathrm{res}_{K_i} : \mathrm{Gal}(K_iF/F) \simeq \mathrm{Gal}(K_i/K_i \cap F)$ and $\mathrm{res}_{K_i} : \mathrm{Gal}(K_{i+1}F/K_iF) \simeq \mathrm{Gal}(K_{i+1}/K_{i+1} \cap K_iF)$ for every $0 \leq i \in \mathbb{Z}$. Moreover, by the primitive element theorem, there exists $0 \leq i_o \in \mathbb{Z}$ such that $F \subseteq K_{i_o}$. If we choose the minimal i_o, the ascending chain (5.55) becomes

$$F = K_oF \subseteq K_1F \subseteq \cdots \subseteq K_{i_o-1}F = K_{i_o} \subset K_{i_o+1} \subset \cdots \subset L.$$

For each $1 \leq i \in \mathbb{Z}$, let σ_i denote the element in $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$ that satisfies

$$< \sigma_i \mid_{K_i} >= \mathrm{Gal}(K_i/K_{i-1}).$$

Now, for $1 \leq i \in \mathbb{Z}$, we introduce elements σ_i^* in $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$ that satisfy

$$< \sigma_i^* \mid_{K_iF} >= \mathrm{Gal}(K_iF/K_{i-1}F)$$

as follows:

(i) for $i > i_o$ we define $\sigma_i^* = \sigma_i$;
(ii) for $i \leq i_o$ we define

$$\sigma_i^* = \begin{cases} \sigma_i & \text{if } K_{i-1}F \subset K_iF; \\ \mathrm{id}_{K_iF} & \text{if } K_{i-1}F = K_iF. \end{cases}$$

Then, clearly, for each $1 \leq i \in \mathbb{Z}$ the elements σ_i^* of $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$ satisfy

$$< \sigma_i^* \mid_{K_iF} >= \mathrm{Gal}(K_iF/K_{i-1}F),$$

and $\sigma_i^* = \sigma_i$ for almost all i. Moreover, for each $1 \leq k \in \mathbb{Z}$, the square

$$\begin{array}{ccc} \mathrm{Gal}(K_{k+1}F/K_kF) & \xrightarrow{\Xi_{K_{k+1}F/K_kF}} & U_{\widetilde{K_{k+1}F}}/U_{\widetilde{K_{k+1}F}}^{\sigma_{k+1}^*-1} \\ {\scriptstyle r_{K_{k+1}}}\downarrow & & \downarrow{\scriptstyle \widetilde{N}^*_{K_{k+1}F/K_{k+1}}} \\ \mathrm{Gal}(K_{k+1}/K_k) & \xrightarrow{\Xi_{K_{k+1}/K_k}} & U_{\widetilde{K}_{k+1}}/U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}, \end{array}$$

is commutative, because $\widetilde{N}_{K_{k+1}F/K_{k+1}}(\pi_{K_{k+1}F}) = \pi_{K_{k+1}}$ by the *norm coherence* of the Lubin-Tate labeling $(\pi_{K'})_{\underbrace{K \subseteq K'}_{[K':K]<\infty} \subset K_\varphi}$. Hence,

$$\widetilde{N}^*_{K_{k+1}F/K_{k+1}}\left(T_k^{(L/F)}\right) = T_k^{(L/K)}.$$

Now, by Theorem 5.14, there exists an arrow

$$h_k^{(L/F)} : \prod_{1\leq i\leq k} U_{\widetilde{K_kF}}^{\sigma_i^*-1} \to \left(\prod_{1\leq i\leq k+1} U_{\widetilde{K_{k+1}F}}^{\sigma_i^*-1}\right) / U_{\widetilde{K_{k+1}F}}^{\sigma_{k+1}^*-1}$$

that splits the exact sequence

$$1 \longrightarrow T_k^{(L/F)'} \longrightarrow \left(\prod_{1\leq i\leq k+1} U_{\widetilde{K_{k+1}F}}^{\sigma_i^*-1}\right) / U_{\widetilde{K_{k+1}F}}^{\sigma_{k+1}^*-1} \underset{\xleftarrow{h_k^{(L/F)}}}{\xrightarrow{\widetilde{N}_{K_{k+1}F/K_kF}}} \prod_{1\leq i\leq k} U_{\widetilde{K_kF}}^{\sigma_i^*-1} \longrightarrow 1. \tag{5.56}$$

We choose an arrow

$$h_k^{(L/K)} : \prod_{1\leq i\leq k} U_{\widetilde{K}_k}^{\sigma_i-1} \to \left(\prod_{1\leq i\leq k+1} U_{\widetilde{K_{k+1}}}^{\sigma_i-1}\right) / U_{\widetilde{K_{k+1}}}^{\sigma_{k+1}-1} \tag{5.57}$$

that splits the exact sequence (5.47) in such a way that

$$\begin{tikzcd}
\left(\prod\limits_{1\le i\le k+1} U^{\sigma_i^*-1}_{\widetilde{K_{k+1}F}}\right)/U^{\sigma_{k+1}^*-1}_{\widetilde{K_{k+1}F}} \arrow[r, "\widetilde{N}_{K_{k+1}F/K_kF}"] \arrow[d, "\widetilde{N}^*_{K_{k+1}F/K_{k+1}}"'] & \prod\limits_{1\le i\le k} U^{\sigma_i^*-1}_{\widetilde{K_kF}} \arrow[d, "\widetilde{N}_{K_kF/K_k}"] \arrow[l, bend right, "h_k^{(L/F)}"'] \\
\left(\prod\limits_{1\le i\le k+1} U^{\sigma_i-1}_{\widetilde{K}_{k+1}}\right)/U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}} \arrow[r, "\widetilde{N}_{K_{k+1}/K_k}"] & \prod\limits_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{K}_k} \arrow[l, bend left, "h_k^{(L/K)}"]
\end{tikzcd} \tag{5.58}$$

is a commutative square. The arrow (5.57) is constructed by following the same lines as in the construction of the arrow (5.44). For each $1 \le k \in \mathbb{Z}$, consider any map

$$g_k^{(L/K)} : \prod_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{K}_k} \to \prod_{1\le i\le k+1} U^{\sigma_i-1}_{\widetilde{K}_{k+1}} \tag{5.59}$$

that makes the following square commutative:

$$\begin{tikzcd}
\prod\limits_{1\le i\le k} U^{\sigma_i^*-1}_{\widetilde{K_kF}} \arrow[r, "g_k^{(L/F)}"] \arrow[d, "\widetilde{N}_{K_kF/K_k}"'] & \prod\limits_{1\le i\le k+1} U^{\sigma_i^*-1}_{\widetilde{K_{k+1}F}} \arrow[d, "\widetilde{N}_{K_{k+1}F/K_{k+1}}"] \\
\prod\limits_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{K}_k} \arrow[r, dotted, "g_k^{(L/K)}"] & \prod\limits_{1\le i\le k+1} U^{\sigma_i-1}_{\widetilde{K}_{k+1}}.
\end{tikzcd} \tag{5.60}$$

Note that such a map satisfies $h_k^{(L/K)} = g_k^{(L/K)} \mod U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}}$. Indeed, by the commutative diagram (5.58), for any $w \in \prod_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{K}_k}$ there exists $v \in \prod_{1\le i\le k} U^{\sigma_i^*-1}_{\widetilde{K_kF}}$ such that $w = \widetilde{N}_{K_kF/K_k}(v)$, and

$$\begin{aligned}
h_k^{(L/K)}(w) &= h_k^{(L/K)}\left(\widetilde{N}_{K_kF/K_k}(v)\right) \\
&= \widetilde{N}^*_{K_{k+1}F/K_{k+1}}\left(h_k^{(L/F)}(v)\right) \\
&= \widetilde{N}^*_{K_{k+1}F/K_{k+1}}\left(g_k^{(L/F)}(v).U^{\sigma_{k+1}^*-1}_{\widetilde{K_{k+1}F}}\right) \\
&= \widetilde{N}_{K_{k+1}F/K_{k+1}}\left(g_k^{(L/F)}(v)\right).U^{\sigma_{k+1}-1}_{\widetilde{K}_{k+1}},
\end{aligned}$$

and by the commutativity of the diagram (5.60) we have

$$\widetilde{N}_{K_{k+1}F/K_{k+1}}\left(g_k^{(L/F)}(v)\right) = g_k^{(L/K)}\left(\widetilde{N}_{K_kF/K_k}(v)\right) = g_k^{(L/K)}(w).$$

Thus, the relation

$$h_k^{(L/K)}(w) = g_k^{(L/K)}(w).U_{\widetilde{K}_{k+1}}^{\sigma_{k+1}-1}$$

follows for every $w \in \prod_{1\le i\le k} U_{\widetilde{K}_k}^{\sigma_i-1}$.

Now, for each $1 \le i \in \mathbb{Z}$, we introduce the map

$$f_i^{(L/K)} : U_{\widetilde{K}_i}^{\sigma_i-1} \to U_{\widetilde{\mathbb{X}}(L/K)}$$

by

$$f_i^{(L/K)}(w) = \Lambda_{F/K}\left(f_i^{(L/F)}(v)\right),$$

where $v \in U_{\widetilde{K_iF}}^{\sigma_i^*-1}$ is any element satisfying $\widetilde{N}_{K_iF/K_i}(v) = w \in U_{\widetilde{K}_i}^{\sigma_i-1}$. Note that if $v' \in U_{\widetilde{K_iF}}^{\sigma_i^*-1}$ is such that $\widetilde{N}_{K_iF/K_i}(v') = w$, then $\Lambda_{F/K}\big(f_i^{(L/F)}(v)\big) = \Lambda_{F/K}\big(f_i^{(L/F)}(v')\big)$. In fact, there exists $u \in \ker\left(\widetilde{N}_{K_iF/K_i}\right)$ such that $v' = vu$. Thus, we must verify that $\Lambda_{F/K}\big(f_i^{(L/F)}(v)\big) = \Lambda_{F/K}\big(f_i^{(L/F)}(vu)\big)$. That is, for each $1 \le j \in \mathbb{Z}$, we must check the identity

$$\mathrm{Pr}_{\widetilde{K}_j}\left(\Lambda_{F/K}\left(f_i^{(L/F)}(v)\right)\right) = \mathrm{Pr}_{\widetilde{K}_j}\left(\Lambda_{F/K}\left(f_i^{(L/F)}(vu)\right)\right). \tag{5.61}$$

Observe for $j > i$ we have

$$\begin{aligned}
\mathrm{Pr}_{\widetilde{K}_j}\left(\Lambda_{F/K}\left(f_i^{(L/F)}(v)\right)\right) &= \widetilde{N}_{K_jF/K_j}\left(\mathrm{Pr}_{\widetilde{K_jF}}\left(\Lambda_{F/K}(f_i^{(L/F)}(v))\right)\right)\\
&= \widetilde{N}_{K_jF/K_j}\left(\mathrm{Pr}_{\widetilde{K_jF}}\left(f_i^{(L/F)}(v)\right)\right)\\
&= \widetilde{N}_{K_jF/K_j}\left(g_{j-1}^{(L/F)}\circ\cdots\circ g_i^{(L/F)}(v)\right)\\
&= g_{j-1}^{(L/K)}\circ\cdots\circ g_i^{(L/K)}\left(\widetilde{N}_{K_iF/K_i}(v)\right)
\end{aligned}$$

because the square (5.60) is commutative. Now, the identity $\widetilde{N}_{K_iF/K_i}(v) = \widetilde{N}_{K_iF/K_i}(vu)$ implies (5.60). Thus, the map

$$f_i^{(L/K)} : U_{\widetilde{K}_i}^{\sigma_i-1} \to U_{\widetilde{\mathbb{X}}(L/K)}$$

is well defined. Moreover,

$$\mathrm{Pr}_{\widetilde{K}_j}\circ f_i^{(L/K)} = \left(g_{j-1}^{(L/K)}\circ\cdots\circ g_i^{(L/K)}\right)|_{U_{\widetilde{K}_i}^{\sigma_i-1}}$$

for $j > i$. Indeed, for $w \in U_{\widetilde{K}_i}^{\sigma_i - 1}$, there exists $v \in U_{\widetilde{K_iF}}^{\sigma_i^* - 1}$ with $\widetilde{N}_{K_iF/K_i}(v) = w$, and $f_i^{(L/K)}(w) = \Lambda_{F/K}\big(f_i^{(L/F)}(v)\big)$. That is, the square

$$\begin{tikzcd} U_{\widetilde{K_iF}}^{\sigma_i^*-1} \arrow[r, "f_i^{(L/F)}"] \arrow[d, "\widetilde{N}_{K_iF/K_i}"'] & U_{\widetilde{\mathbb{X}}(L/F)} \arrow[d, "\Lambda_{F/K}"] \\ U_{\widetilde{K}_i}^{\sigma_i-1} \arrow[r, "f_i^{(L/K)}"] & U_{\widetilde{\mathbb{X}}(L/K)} \end{tikzcd} \tag{5.62}$$

is commutative. Consequently, for $j > i$ we have

$$\begin{aligned} \mathrm{Pr}_{\widetilde{K}_j} \circ f_i^{(L/K)}(w) &= \mathrm{Pr}_{\widetilde{K}_j} \circ \Lambda_{F/K}\left(f_i^{(L/F)}(v)\right) \\ &= \widetilde{N}_{K_jF/K_j}\left(\mathrm{Pr}_{\widetilde{K_jF}} \circ f_i^{(L/F)}(v)\right) \\ &= \widetilde{N}_{K_jF/K_j}\left((g_{j-1}^{(L/F)} \circ \cdots \circ g_i^{(L/F)})(v)\right) \\ &= \left(g_{j-1}^{(L/K)} \circ \cdots \circ g_i^{(L/K)}\right)\left(\widetilde{N}_{K_iF/K_i}(v)\right), \end{aligned}$$

by the commutativity of the diagram (5.60), as desired.

For each $0 < i \in \mathbb{Z}$, let

$$Z_i^{(L/K)} = \mathrm{im}(f_i^{(L/K)}).$$

Then, by Lemma 5.15 or by [2, Lemma 4], for $z^{(i)} \in Z_i^{(L/K)}$ the product $\prod_i z^{(i)}$ converges to an element in $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$. Let

$$Z_{L/K}\left(\{K_i, f_i^{(L/K)}\}\right) = \Big\{\prod_i z^{(i)} : z^{(i)} \in Z_i^{(L/K)}\Big\},$$

which is a topological subgroup of $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$. We introduce the topological $\mathrm{Gal}(L/K)$-submodule $Y_{L/K}\big(\{K_i, f_i^{(L/K)}\}\big) = Y_{L/K}$ of $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ by

$$Y_{L/K} = \left\{y \in U_{\widetilde{\mathbb{X}}(L/K)} : y^{1-\varphi} \in Z_{L/K}\left(\{K_i, f_i^{(L/K)}\}\right)\right\}.$$

Lemma 5.30. *The continuous homomorphism* $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/F)^\times \to \widetilde{\mathbb{X}}(L/K)^\times$ *introduced by* (5.20) *and* (5.21) *further satisfies*

(i) $\Lambda_{F/K}\left(Z_{L/F}(\{K_iF, f_i^{(L/F)}\})\right) \subseteq Z_{L/K}\left(\{K_i, f_i^{(L/K)}\}\right)$;
(ii) $\Lambda_{F/K}(Y_{L/F}) \subseteq Y_{L/K}$.

Proof. (i) For any choice of $z^{(i)} \in Z_i^{(L/F)}$, the continuity of the multiplicative arrow $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/F)^\times \to \widetilde{\mathbb{X}}(L/K)^\times$ yields

$$\Lambda_{F/K}\left(\prod_i z^{(i)}\right) = \prod_i \Lambda_{F/K}(z^{(i)}),$$

where $\Lambda_{F/K}(z^{(i)}) \in Z_i^{(L/K)}$ by the commutative square (5.62).

(ii) Let $y \in Y_{L/F}$. Then, $y^{1-\varphi} \in Z_{L/F}\big(\{K_iF, f_i^{(L/F)}\}\big)$ and $\Lambda_{F/K}(y^{1-\varphi}) = \Lambda_{F/K}(y)^{1-\varphi} \in Z_{L/K}\big(\{K_i, f_i^{(L/K)}\}\big)$ by part (i), and the result follows. □

So, the homomorphism $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/F)^\times \to \widetilde{\mathbb{X}}(L/K)^\times$ defined by (5.21) induces a group homomorphism

$$\lambda_{F/K} : U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/F} \to U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \tag{5.63}$$

defined by

$$\lambda_{F/K}(\overline{U}) = \Lambda_{F/K}(U).Y_{L/K}, \tag{5.64}$$

for every $U \in U^\diamond_{\widetilde{\mathbb{X}}(L/F)}$ (as usual, $\overline{U}$ denotes the coset $U.Y_{L/F}$ in $U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/F}$).

Let

$$\Phi^{(\varphi)}_{L/F} : \mathrm{Gal}(L/F) \to U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/F}$$

be the corresponding Fesenko reciprocity map defined for the extension L/F, where $Y_{L/F} = Y_{L/F}\big(\{K_iF, f_i^{(L/F)}\}\big)$.

Theorem 5.31. *For the finite subextension F/K of L/K, the square*

$$\begin{tikzcd} \mathrm{Gal}(L/F) \arrow[r, "\Phi^{(\varphi)}_{L/F}"] \arrow[d, "\text{inc.}"'] & U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/F} \arrow[d, "\lambda_{F/K}"] \\ \mathrm{Gal}(L/K) \arrow[r, "\Phi^{(\varphi)}_{L/K}"] & U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K}, \end{tikzcd} \tag{5.65}$$

where the right vertical arrow $\lambda_{F/K} : U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/F} \to U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K}$ is defined by (5.63) *and* (5.64)*, is commutative.*

Proof. It suffices to prove that the square

$$\begin{tikzcd} U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \arrow[r, "\text{can.}"] \arrow[d, "\lambda_{F/K}"'] & U^\diamond_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/F} \arrow[d, "\lambda_{F/K}"] \\ U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \arrow[r, "\text{can.}"] & U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \end{tikzcd}$$

is commutative, which is obvious. Then, pasting this square to the square (5.24), we obtain the diagram

$$\begin{tikzcd}
\mathrm{Gal}(L/F) \arrow[r, "\phi^{(\varphi)}_{L/F}"] \arrow[d, "\mathrm{inc.}"'] & U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \arrow[r, "\mathrm{can.}"] \arrow[d, "\lambda_{L/F}"] & U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/Y_{L/F} \arrow[d, "\lambda_{L/F}"] \\
\mathrm{Gal}(L/K) \arrow[r, "\phi^{(\varphi)}_{L/K}"] & U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \arrow[r, "\mathrm{can.}"] & U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K},
\end{tikzcd}$$

and the commutativity of the square (5.65) follows. □

If, moreover, L/K is a finite extension, then the square

$$\begin{tikzcd}
U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/K} \arrow[r, "H^{(\varphi)}_{L/K}"] \arrow[d, "\mathrm{Pr}_{\tilde{K}}"'] & \mathrm{Gal}(L/K) \arrow[d, "\mathrm{mod\ Gal(L/K)}'"] \\
U_K/N_{L/K}U_L \arrow[r, "h_{L/K}"] & \mathrm{Gal}(L/K)^{ab}
\end{tikzcd}$$

commutes. Thus, *the inverse* $H^{(\varphi)}_{L/K} = (\Phi^{(\varphi)}_{L/K})^{-1}$ *of the Fesenko reciprocity map* $\Phi^{(\varphi)}_{L/K}$ *defined for* L/K *is a generalization of the Hazewinkel map for the totally ramified APF-Galois subextensions* L/K *of* K_{φ}/K *under the assumption that the local field* K *satisfies condition* (5.26).

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Department of Mathematics
Istanbul Bilgi University
Kurtuluş Deresi Cad. No. 47, Dolapdere
34440 Beyoğlu, Istanbul
TURKEY
E-mail: ilhan@bilgi.edu.tr

Department of Mathematics
Atilim University
Kizilcaşar Köyü, Incek
06836 Gölbaşı, Ankara
TURKEY
E-mail: eserbest@atilim.edu.tr