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NONPHYSICAL SHEET FOR PERTURBED JACOBIAN MATRICES

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§1. Functional model for periodical Jacobian matrix. Lax–Phillips approach

Let A be three-diagonal hermitian n-periodic Jacobian matrix in $l_2(z)$, with a block-period A_0,

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & & & & \end{pmatrix}, \quad A = \begin{pmatrix} & \bar\alpha & \\ \alpha & A_0 & \bar\alpha \\ & \alpha & A_0 \end{pmatrix}.$$

Then A could be obviously Fourier-represented in $L_2[(0,2\pi), E_n]$ by matrix multiplication operator

$$A(\theta) = A_0 + \begin{pmatrix} & & \overline{\alpha\theta'} \\ & 0 & \\ \alpha\theta' & & \end{pmatrix}$$

$\theta' = e^{i\varphi}$, $0 < \varphi < 2\pi$. The spectrum $\sigma(A)$ of A is defined by solutions of the equation on real axis λ:

$$\frac{1}{2}[\theta + \bar\theta] = \frac{1}{|\alpha|}P(\lambda), \quad \theta = e^{i\varphi + i \arg \alpha}.$$

If the eigenvalues of A_0 are simple and the coupling constant α is small enough, then the spectrum of A is absolutely continuous and consists of n bands γ_s lying near eigenvalues of A_0:

$$\sigma(A) = \bigcup_{s=1}^{n} \gamma_s.$$

The first sheet of spectral variable Λ_+ is a complex plane with n cuts γ_s along the bands. For our considerations we need a double-Riemann surface $\Lambda = \Lambda_+ \cup \Lambda_-$ of two sheets joined by cuts γ_s.

Let us denote by $b_\varkappa(\infty)$ a Blaschke factor on Λ_+ which has only one simple root at infinity (see Appendix), $\varkappa$ is the corresponding automorphity vector $\varkappa = (\varkappa_1, \varkappa_2, \dots, \varkappa_{n-1})$. The function θ defined on the first sheet Λ_+ as a compressing solution of the equation

$$\frac{1}{2}[\theta + \theta^{-1}] = \frac{1}{|\alpha|}P(\lambda), \tag{1}$$

$|\theta(\lambda)| \leqslant 1$, $|\theta|_\Gamma = 1$, should have a root order n at infinity ∞_+, hence

$$(b_\varkappa)^n = \theta$$

up to constant unitary factor which we omit now. It means that $\varkappa n|_{\mathrm{mod}\, 2\pi} = 0$. Let us form an l_2-vector of reproducing kernels $k_\mu(\lambda, \infty)$ for Hardy classes $H^2_{\mu,+}(\Lambda_+)$:

$$\chi = \{\dots, k_\mu(\lambda,\infty), k_{\mu-\varkappa}(\lambda,\infty)b_\varkappa, \dots, k_{\mu-(n-1)\varkappa}b_\varkappa^{n-1}, k_\mu(\lambda,\infty)\theta, k_{\mu-\varkappa}(\lambda,\infty)\theta b_\varkappa, \dots\},$$

which can be regarded as a sequence of n-vectors:

$$\{\theta^l \chi\} = \{\theta^l(k_\mu(\lambda,\infty), k_{\mu-\varkappa}(\lambda,\infty)b_\varkappa, \dots, k_{\mu-(n-1)\varkappa}b_\varkappa^{n-1})\}.$$

The following statement proved in joint paper [1] describes the "functional model" for Jacobian matrix with n-band spectrum.

Theorem 1. *Let us consider the space $L_2(m_\infty)$ of all functions on Γ which are square integrable in respect to the harmonic measure m_∞ on Γ,*

$$m_\infty = \frac{1}{2\pi i}\frac{db_\varkappa}{b_\varkappa} = \frac{1}{2\pi i n}\frac{d\theta}{\theta}.$$

Then the transformation $\mathcal{T}$ for some μ

$$f \xrightarrow{\mathcal{T}} \langle f, \chi\rangle_{l_2} = \sum_{-\infty}^{\infty} \theta^l \langle \mathrm{f}_l, \chi\rangle = \tilde{f}$$

maps l_2 into L_2, m isometrically in such a way that

$$f = \oint \chi \langle f, \chi\rangle_{l_2} dm_\infty,$$

$$Af = A_\mu f = \oint \lambda\chi\langle f, \chi\rangle_{l_2} dm_\infty.$$

Instead of $\chi \equiv \chi_+$ one can use another system $\chi_- = \bar{\chi}_+$. Note that the eigenfunctions χ have a form of Bloch waves with the quasi-momentum exponent θ. They are obviously parametrized by vector μ which is a parameter of the isospectral deformation. Changing μ, we get a family of operators with the same spectrum $\sigma(A_\mu) = \Gamma$. It is obvious, but important that a shift T_n by n steps (by period of A) can be represented in terms of spectral decomposition by the same Bloch waves:

$$T_n = \oint \theta\chi\langle f, \chi\rangle dm = \oint \theta^{-1}\chi_-\langle f, \chi_-\rangle dm.$$

From the spectral representations for both T_n, A and the dispersion equation (1) followes the next analog of Caley equation

$$(T_n + T_n^+) = \frac{2}{|\alpha|}P(A) \equiv 2\mathcal{P}(A).$$

To develop index approach for resonances in a case of band spectrum we need a construction of an evolution group generated by discrete wave equation in corresponding energy-normed space of Cauchy-data.

Let us consider discrete wave equation in $l_2(E_n)$. Denoting by T the shift by one step in $l_2(E_n)$ (which is equivalent to the shift T_n by period n in l_2) and by $\mathcal{T}$ the shift by one step in discrete time, we write the discrete wave equation in form

$$(\mathcal{T} + \mathcal{T}^+)u = (T + T^+)u, \tag{2}$$

The role of Cauchy-data for it are played by vectors in $l_2(E_n) + l_2(E_n)$:

$$\mathbf{u} = \begin{pmatrix} u \\ (\mathcal{T} - \mathcal{T}^+)u \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}. \tag{3}$$

The following result belongs to P. Kurasov (private communication).

Theorem 2 (P. Kurasov). *The quadratic form*

$$|\mathbf{u}|_\varepsilon^2 = \frac{1}{2}[|(T - T^+)u_0|^2_{l_2(E_n)} + |u^1|_{l_2(E_n)}]$$

plays a role of energy norm in a space of all Cauchy-data. The solutions of wave equation with fixed initial Cauchy-data having finite energy norm, exist and are uniquelly defined. The energy norm of the solution is conserved: at any moment it is equal to the energy of initial data. The solution is generally represented in form of linear combination of D'Alambertian waves which dependence of time is trivial

$$\mathcal{T}u = Tu \quad or \quad \mathcal{T}u = T^+u$$

for the waves moving to the right or to the left correspondingly with Cauchy-data

$$\left\{ \begin{pmatrix} u \\ (T - T^+)u \end{pmatrix} \right\} \quad or \quad \left\{ \begin{pmatrix} u \\ (T^+ - T)u \end{pmatrix} \right\}.$$

We will need also the following result, containing in joint paper [1] which gives a base for Lax-Phillips approach to the discrete wave equation. Let us consider the one-step generator for Cauchy-data of the wave equation

$$\mathcal{L}\colon \mathbf{u}(\tau) \longrightarrow \mathbf{u}(\tau + 1),$$

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} T + T^+ & I \\ (T - T^+)^2 & T + T^+ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\mathcal{P}(A) & I \\ 4[\mathcal{P}^2(A) - 1] & 2\mathcal{P}(A) \end{pmatrix}.$$

Theorem 3 (S. Fedorov). *The generator $\mathcal{L}$ is unitary in ε and has a complete orthogonal in energy norm system of eigenfunctions*

$$\Psi_+ = \begin{pmatrix} (-\theta + \theta^{-1})^{-1}\chi_+ \\ \chi_+ \end{pmatrix}, \quad \mathcal{L}\psi_+ = \bar{\theta}\psi_+,$$

$$\Psi_- = \begin{pmatrix} (-\theta + \theta^{-1})^{-1}\chi_- \\ \chi_- \end{pmatrix}, \quad \mathcal{L}\psi_- = \bar{\theta}\psi_-.$$

The spectral representation of $\mathcal{L}$ is given by the following formula

$$\mathbf{u} \longrightarrow \begin{pmatrix} \langle \mathbf{u}, \psi_+ \rangle_\varepsilon \\ \langle \mathbf{u}, \psi_- \rangle_\varepsilon \end{pmatrix} = \begin{pmatrix} \tilde{u}_+ \\ \tilde{u}_- \end{pmatrix},$$

$$\mathbf{u} = \oint (\psi_+ \tilde{u}_+ + \psi_- \tilde{u}_-)dm,$$

$$\mathcal{L}\mathbf{u} \xrightarrow{\mathcal{T}} \theta \begin{pmatrix} \tilde{u}_+ \\ \tilde{u}_- \end{pmatrix}.$$

§2. Resonances for compactely perturbed lattice

Let us consider Jacobian matrix A described in previous section, restricted onto nonnegative halflattice

$$A_+ = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & a_{12} & 0 & 0 & \dots \\ 0 & a_{21} & a_{22} & a_{23} & 0 & \dots \\ 0 & 0 & \dots & & & \end{pmatrix}.$$

To escape difficulties with denoting, we assume A real, $a_{ik} = \bar{a}_{ik}$. Let E be some separable Hilbert space, B — a selfadjoint operator in E and e – a generating vector of B, $|e| = 1$. Denoting the first orth $(1, 0, 0, 0, \dots)$ in $l_2(z_+)$ by e_0, we switch on the "interaction" between A_+ and B, constructing the perturbed operator A_B in $\varepsilon_B = l_2(z_+) \oplus E$ the following way for given $\psi = \psi_+ \oplus \psi_E$:

$$A_B \begin{pmatrix} \Psi_E \\ \psi_+ \end{pmatrix} = \begin{pmatrix} B\psi_E + \alpha e \langle \psi_+, l_0 \rangle_{l_2(z_+)} \\ \alpha e_0 \langle \psi_E, e \rangle + A_+ \psi_+ \end{pmatrix}_{(\alpha = \bar{\alpha})}.$$

Theorem 4. *The operator A_B is selfadjoint in ε_B and its absolutely continuous spectrum $\sigma_a(A_B)$ coincides with $\sigma_a(A) \cup \sigma_a(B)$. The rest part of the spectrum coincides with the support of the singular part of positive measure, defining the following R-type functions on Λ_+:*

$$D(\lambda) = \{a_{10} b_{\varkappa}^{-1} k_{\mu-(n-1)\varkappa} k_{\mu}^{-1} - |\alpha|^2 \langle (B - \lambda)^{-1} e, e \rangle \}^{-1}.$$

The eigenfunctions of the branch of absolutely-continuous spectrum, coinciding with $\sigma_a(A)$, are represented in form of scattered waves which have a z_+ component in $l_2(z_+)$ of the following form

$$\psi_+ = \chi_+ R \chi_+, \quad \psi_E = (B - \lambda I)^{-1} e \langle \psi_+, e_0 \rangle \tag{4}$$

where the reflection coefficient R is equal to

$$-\frac{\overline{k_\mu(\lambda, \infty)}}{k_\mu(\lambda, \infty)} \cdot \frac{\overline{D(\lambda)}}{D(\lambda)}.$$

This part of the spectrum has multiplicity one, corresponding scattered waves ψ are orthogonal and normalized in ε_B and the corresponding spectral projector is equal

$$P = \int \psi \langle \, , \psi \rangle_{\varepsilon_B} dm,$$

where m is the spectral measure of Bloch waves for A, $m = (2\pi i n)^{-1} d\theta|_\theta$.

Proof is basically standart, but tiresome, basing on coordinates resolvent asymptotics and Hilbert identity. In two-band case for details see [2].

We discuss now the wave equation connected with operator A_B

$$(\mathcal{T} + \mathcal{T}^+) u = 2\mathcal{P}(A_B) \tag{5}$$

with the same polynomial which was used in previous section for A in corresponding Caley identity. The Cauchy-data and energy metric are defined in an usual way

$$\mathbf{u} = \begin{pmatrix} u \\ (\mathcal{T} - \mathcal{T}^+) u \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \subset \varepsilon_B + \varepsilon_B,$$

$$|\mathbf{u}|_e = \frac{1}{2} \{ 4 \langle (I - \mathcal{P}^2(A_B)) u_0, u_0 \rangle_{\varepsilon_B} + \langle u_1, u_1 \rangle_{\varepsilon_B} \}. \tag{6}$$

Generally the energy metric could be non positive, since the rest of the spectrum $\sigma(A_B)/\sigma(A)$ could be nontrivial. But we confine our considerations by the part ε_+ of the space ε which is generated by incoming and outgoing subspaces $D_\pm^{\text{right}}$ (or spanned by scattered waves, $\varepsilon_+ = P_+\varepsilon$). The energy metric (6), associated with our problem, is authomatically nonnegative in ε_+. It can be easily checked that the onestep generator $\mathcal{L}_B$ for the time-evolution, described by the wave-equation (3), is given by the formula

$$\mathcal{L}_B = \frac{1}{2}\begin{pmatrix} \mathcal{P}(A_B) & I \\ \mathcal{P}^2(A_B) - 4 & \mathcal{P}(A_B) \end{pmatrix}.$$

It is unitary in energy metric (4).

We consider only the part of $\mathcal{L}_B$ generated by corresponding scattered waves.

Theorem 5. *The distribution Cauchy-data*

$$\Psi_- = \begin{pmatrix} (\bar\theta - \theta)^{-1}\psi \\ \psi \end{pmatrix}$$

with ψ_+ given by formula (4) *are the eigenfunctions of absolutely continuous spectrum of $\mathcal{L}_B$*

$$\mathcal{L}_B\Psi_- = \bar\theta\Psi_-,$$

and the corresponding spectral representation of the part of $\mathcal{L}_B$ is given by the formula

$$\mathcal{L}_{B,-} = \oint \theta\Psi_-\langle\ ,\Psi_-\rangle_\varepsilon dm.$$

The similar spectral representation holds for the other part of $\mathcal{L}_B$, generated by another brabch of distribution Cauchy-data:

$$\Psi_+ = \begin{pmatrix} (\bar\theta - \theta)\bar\psi \\ \bar\psi \end{pmatrix}$$

which are eigenfunctions of $\mathcal{L}_B$ as well,

$$\mathcal{L}_B\Psi_+ = \bar\theta\Psi_B.$$

The corresponding spectral representation of the part of $\mathcal{L}_B$ is given by

$$\mathcal{L}_{B,+} = \oint \theta\Psi_+\langle\ ,\Psi_+\rangle_\varepsilon dm.$$

The minimal part of $\mathcal{L}_B$, containing both $\mathcal{L}_{B,\pm}$ can be written in symmetrical spectral representation J_{sym}:

$$J_{\text{sym}}: U \to \begin{pmatrix} \langle u, \psi_+\rangle_\varepsilon \\ \langle u, \psi_-\rangle_\varepsilon \end{pmatrix} \equiv \tilde u$$

in form of multiplication by θ in corresponding weighted space of square integrable functions

$$L_2\begin{pmatrix} I & \bar R \\ R & I \end{pmatrix}$$

with the norm

$$|\tilde u| = \left\{\oint\left\langle\begin{pmatrix} 1 & \bar R \\ R & I \end{pmatrix}\tilde u, \tilde u\right\rangle dm\right\}^{1/2}.$$

Notice that the suggested spectral representation is an analog of symmetrical spectral representation for Lax-Phillips generators with nontrivial absolutely continuous spectrum, see [3]. It is used for constructing symmetrical variant of Nagy-Foias functional model.

In our case it opens a way for constructing a functional model of family of commuting contractions with absolutely continuous spectrum which could be interpreted as an absolutely continuous spectrum of resonances. We omit this attractive possibility now, and discuss the special case of unitary scattering matrix, $|R||_{\Gamma} = 1$.

Theorem 6. *If the operator B does not have absolutely continuous spectrum, then the reflection coefficient and corresponding scattering matrix are unitary. Scattering matrix is an analytic (-2μ)-authomorphic function on Λ_+, and the incoming and outgoing spaces of data in ε, supported by positive halflattice z_+ are spectrally represented by*

$$H^2_{-,(-\mu)}, \quad S_{-2\mu}H^2_{+,(\mu)} \subset H^2_{+,(-\mu)}.$$

Proof. We calculate here only the spectral representation of D_-^{right},

$$D_-^{\text{right}} = \begin{pmatrix} u_0 \\ (T^+ - T)u_0 \end{pmatrix}, \quad \operatorname{supp} u_0 \subset z_+.$$

The calculation of energy scalar product with distribution Ψ_- gives

$$\langle U, \Psi_- \rangle_\varepsilon = \frac{1}{2}\{\langle (T - T^+)u_0, (T - T^+)(\bar\theta - \theta)^{-1}(\chi_+ + R\chi_-)\rangle_{l_2(z_+)} + \langle (T^+ - T)u_0, \chi_+ + R\chi_- \rangle_{l_2(z_+)}\}.$$

Using the equations $T\chi_+ = \bar\theta\chi_+$, $T\chi_- = \theta\chi_-$, we get

$$\langle u, \psi_- \rangle_\varepsilon = \frac{1}{2}\{\langle u_0, (\bar\theta - \theta)(\chi_+ + R\chi_-)\rangle_{l_2(z_+)} + \langle u_0, (\bar\theta - \theta)\chi_+ + R(\theta - \bar\theta)\chi_- \rangle_{l_2(z_+)}\}$$
$$= (\bar\theta - \theta)\langle u_0, \chi_+ \rangle_{l_2(z_+)} \subset (1 - \theta^2)H^2_{-(-\mu)}.$$

Using the choose of u_0, we see that $\langle u, \psi_- \rangle_\varepsilon$ covers $(1 - \theta^2)H^2_{-(-\mu)}$, and the closure of it coincides with $H^2_{-,(-\mu)}$, since $1 - \theta^2$ is an outer function. The calculation concerning D_+^{right} can be done similarly.

Following Lax-Phillips [4, 5] idea we can compress the unitary evolution group $(\mathcal{L}_B)^n$ onto translation invariant subspace

$$K' = H^2_{+(-\mu)} \ominus S_{-2\mu}H^2_{+(+\mu)}$$

or

$$K = L_2 \ominus [S_{-2\mu}H^2_{+(+\mu)} \oplus H^2_{-(\mu)}]$$

in the second case translation invariant subspace is enlarged due to adding the defect $\mathfrak{N}_0 = L_2 \ominus \{H^2_{-(-\mu)} \oplus H^2_{+(-\mu)}\}$, see appendex. In both cases the restriction gives contracting one-parametrical semigroups $\{T^n\}$, $\{(T')^n\}$,

$$T = P_K\mathcal{L}_B|K, \qquad T' = P_{K'}\mathcal{L}_B|K'.$$

The spectral of generators T, T' correspond to the singularities of inverse scattering matrix $S^{-1} = (\bar R)^{-1}$ (for T') and, possible to some additional isolated eigenvalues, caused by defect $\mathfrak{N}_0$ (for T). Now we confine ourselves by the simplest case T'.

Theorem 13. *The compressed resolvent of $\mathcal{L}_B$*

$$P_{K'}(\mathcal{L}_B - \zeta I)^{-1}|K'$$

coincides for $|\zeta| > 1$, $|\zeta| < 1$ with resolvents of T', $\{(T')^+\}^{-1}$ hence posesses an analytical continuation from the outside of the unit disc onto inside of it with poles at eigenvalues t_ε of

T' and from inside of unit disc to outside of it with the poles at $\{\bar{t}_e\}^{-1}$. The corresponding systems of eigenvectors form Riesz-basis in linear hulls if and only if the Carleson condition for t_e is fulfilled (see appendix).

The system of eigenvectors of T', $(T')^+$ are jointly complex in K'

$$\mathfrak{N}_\alpha + \mathfrak{N}_\alpha'^{+} K'$$

with positive angle between components, if and only if the factors of the reflection coefficient fulfil the Muckenhoupt condition,

$$\sup_\Delta \frac{1}{m\Delta} \int_\Delta |k_\mu^{-1}(\lambda,\infty)D(\lambda)|^2 dm \frac{1}{m(\Delta)} \int_\Delta |k_\mu^{-1}(\lambda,\infty)D(\lambda)|^{-2} dm < \infty.$$

The proof is based on μ-authomorphic version of Muckenhoupt conditions, which will be discussed in other publication.

Now we discuss the problems, concerning analytical continuation of the compressed resolvent of perturbed Jacobian matrix onto nonphysical sheet ω_-, using the previous statements for analytical background. The compressed resolvent of perturbed Jacobian matrix looses many essential properties of resolvent for the same reason, as one of the Friedrichs model. Hence we need to use index approach to study the completeness of resonances and basis properties of them.

Let us consider the completeness and expansion by resonances for perturbed Jacobian matrix which is represented spectrally by multiplication operator in the space of Cauchy data:

$$A_B \sim \tilde{A}_B = \int z\Psi_-\langle\ ,\Psi_-\rangle dm.$$

The resolvent of $\tilde{A}_B$ is obviously an analytic function on the first sheet ω_+ of the Riemann surface and is represented by Caushy kernel

$$\frac{1}{z-\lambda}.$$

Considering compressed resolvent

$$P_{K'}(z-\lambda)^{-1}|K'$$

we should prove that it posesses an analytic continuation onto nonphysical sheet ω_- with singularities defined by scattering matrix.

According to the Stout-Fedorov result see [6, 7] quoted in Appendix the algebra of all bounded analytic functions on ω_+ is generated by two multiplication operators θ^*, θ_1^* by Blaschke products which represent unitary operators in energy space

$$\theta^* \sim \mathcal{L}_B \equiv \mathcal{L}_0,$$
$$\theta_1^* \sim \mathcal{L}_1.$$

Let us denote by T, T_1 the compressions of them onto translation-invariant subspace K'

$$P_{K'}\mathcal{L}_0|K' = T, \qquad P_{K'}\mathcal{L}_1|K' = T_1.$$

From the quoted Shout-Fedorov result followes the next Assertion. For every bounded analytic function Φ on ω_+ represented through θ, θ_1

$$\Phi(z) = \Phi = \Phi(\theta,\theta_1)$$

the compressed multiplication operator is represented as an analytic function of T, T_1:

$$P_{K'}\Phi(\theta,\theta_1)|K' = \Phi(T,T_1).$$

The similar result is valid for the class of bounded analytic functions on ω_-.

Theorem 7. *The compressed resolvent of $\tilde{A}_B$ posesses an analytical continuation onto nonphysical sheet as a linear combination of resolvents T, T^+ with coefficients, including T_1, T_1^+ correspondingly, with the poles at the roots $\{\lambda_s\}$ of scattering matrix and conjugate points $\{\overline{\lambda_s}\}$.*

Proof. Using Hamilton-Caley equation

$$\theta + \theta^{-1} = 2\mathcal{P}(z)$$

we can write the spectrally-represented resolvent of $\tilde{A}_B$ in the following form

$$\frac{1}{z-\lambda} = \frac{\mathcal{P}_{n-1}(z,\lambda)}{\theta+\theta^{-1} - \left[\theta(\lambda)+\theta^{-1}(\lambda)\right]} = \mathcal{P}_{n-1}(z,\lambda)\left[\frac{1}{\theta-\theta(\lambda)} + \frac{\theta(\lambda)}{1-\theta\theta(\lambda)}\right]\frac{\theta\theta(\lambda)}{\theta^2(\lambda)-1}.$$

Here $\mathcal{P}_{n-1}(z,\lambda) = (z-\lambda)^{-1}[\mathcal{P}(z) - \mathcal{P}(\lambda)]2$ is a polynomial of order $n-1$ which is symmetric in respect to the change of variables $z \leftrightarrow \lambda$,

$$\mathcal{P}_{n-1}(z,\lambda) = \sum_{s=0}^{n-1} a_s(z^s + b_s^1 z^{s-1}\lambda + \ldots b_s^1 z\lambda^{s-1} + \lambda^s).$$

The products $\mathcal{P}_{n-1}\theta$, $\mathcal{P}_{n-1}\bar{\theta}$ are bounded analytic functions of z on ω_+, ω_- correspondingly. Hence the following decomposition is true:

$$\frac{1}{z-\lambda} = \frac{\theta(\lambda)}{\theta^2(\lambda)-1}\left[\frac{\mathcal{P}_{n-1}\theta}{1-\theta\theta(\lambda)}\cdot\theta(\lambda) + \mathcal{P}_{n-1} + \frac{\mathcal{P}_{n-1}\bar{\theta}}{1-\bar{\theta}\theta(\lambda)}\cdot\theta(\lambda)\right],$$

where the first term in a square bracket is an analytical function on ω_+, the third term is an analytical function on ω and the middle term is a polynomial in variables λ, z. Using the previous assertion, we get the representation for the compressed resolvent

$$P_{K'}\frac{1}{z-\lambda}|K' = \frac{\theta(\lambda)}{\theta^2(\lambda)-1}\left[\frac{\theta(\lambda)}{1-T\theta(\lambda)}P_{K'}(\mathcal{P}_{n-1}\theta)|K'\right.$$
$$\left. + \frac{\theta(\lambda)}{1-\theta(\lambda)T^+}P_{K'}(\mathcal{P}_{n-1}\bar{\theta})|K' + P_{K'}\mathcal{P}_{n-1}|K'\right].$$

The last term in a bracket is just an operator polynomial in λ. The operator-functions

$$P_{K'}\mathcal{P}_{n-1}\theta|K' \equiv P_{K'}\Phi_+|K',$$
$$P_{K'}\mathcal{P}_{n-1}\bar{\theta}|K' \equiv P_{K'}\Phi_-|K'$$

can be represented as polynomials in λ with operator coefficients

$$P_{K'}\Phi_+|K' = \Phi_+(T, T_1, \lambda),$$
$$P_{K'}\Phi_-|K' = \Phi_-(T^+, T_1^+, \lambda).$$

Hence

$$P_{K'}\frac{1}{z-\lambda}|K' = \frac{\theta(\lambda)}{\theta^2(\lambda)-1}\left[\frac{\theta(\lambda)}{1-T\theta(\lambda)}\Phi_+(T,T_1,\lambda)\right.$$
$$\left. + \frac{\theta(\lambda)}{1-T^+\theta(\lambda)}\Phi_-(T^+,T_1^+,\lambda) + P_{K'}\mathcal{P}_{n-1}P_{K'}\right].$$

The right part of the last equation is obviously analytical function in λ on ω_+, since the factor $\theta(\lambda)$ surpresees possible polynomial poles of order $n-1$ at infinity; the analytical continuation onto ω_- in λ meets poles at the eigenvalues of T, T^+

$$T\mathfrak{G} = \theta_+(\lambda_s)\mathfrak{G} = [\overline{\theta_-}(\bar\lambda_s)]^{-1}\mathfrak{G},$$
$$T^+\mathfrak{G}^+ = \overline{\theta_+(\lambda_s)}\mathfrak{G}^+ = [\theta_-(\bar\lambda_s)]^{-1}\mathfrak{G}^+,$$

localized by the roots λ_s of the scattering matrix.

Thus we see that the poles of the first addenda lie at the roots λ_s of the scattering matrix, the poles of the second ones lie in complex conjugate points. The corresponding residues of the analytically continued resolvent are composed of eigen-functions (or root functions) of T, T^+ respectively.

Let us discuss at last the completeness of the resonances states on the Theorem 13. and the index approach.

According to the general index approach (see above and [8, 9]), we should consider a multiplication operator by analytic function f which posesses two factorizations on the universal cover $\Omega_+ \cup \Omega_-$

$$f = \begin{cases} \Pi f_+^e, & z \in \Omega_+, \\ \theta f_-^e, & z \in \Omega_-, \end{cases}$$

such that in $L_2(\partial\Omega_+)$

$$\overline{fH_-^2} = \theta H_-^2, \qquad \overline{fH_+^2} = \Pi H_+^2.$$

We use this approach for the classes of μ-automorphic functions (on the universal cover ω of the first sheet). The essential difference of the theory of μ-automorphic Hardy classes $H_{+,\mu}^2$ on Ω_+ from the standart ones is that there exist an n-dimensional defect $\mathfrak{N}_0$ ($n = 1 + \text{genus}(\omega_- \cup \omega_+)$):

$$L_{2_{(\mu)}}(\partial\Omega_+) = \overset{\circ}{H}{}_{+(\mu)}^2 \oplus H_{-(\mu)}^2 \oplus \mathfrak{N}_{0(\mu)},$$
$$f_{(-\mu)} = k_\mu^{-1}D.$$

The corresponding Muckenhoupt condition guarantees the completeness of resonance states of T, T^+ in K' (see Theorem 13)

$$\mathfrak{N}_d \dot{+} \mathfrak{N}_d^+ = (H_{+(\mu)}^2 \ominus \Pi_{(-\mu')}H_{+(\mu+\mu')}^2)\dot{+}\theta_{-\mu''}(H_{+(\mu''+\mu)}^2 \ominus \Pi_{(-\mu')}H_{+(\mu'+\mu''+\mu)}^2).$$

Combining the results of Theorems 13, 14, we get the following statement.

Theorem 8. *If the Muckenhoupt condition is fulfilled for the renormalized perturbation determinant $k_\mu^{-1}D$, then the compressed resolvent of perturbed Jacobian matrix can be continued analytically onto nonphysical sheet with poles at the roots λ_s of the scattering matrix and at the conjugate points $\bar\lambda_s$, an a system of resonance state composed of eigenvectors of compressed Lax-Phillips generator T forming a complete system in K.*

Note that additional Carleson condition for $\theta(\lambda_s)$

$$\inf_s \prod_{t\neq s} \left| \frac{\theta(\lambda_t) - \theta(\lambda_s)}{1 - \theta(\lambda_t)\bar\theta(\lambda_s)} \right| > \delta > 0$$

on every sequence λ_s accumulating on spectral bands implies Riesz-basis property for resonances states. This fact followes from harmonic analysis developed in [7] combined with spectral analysis of the generator T similarly to the corresponding fact in [5]. Here $\overset{\circ}{H}{}^2_{+(\mu)}$ is the Hardy class of μ-automorphic functions vanishing at infinity $\infty_+ \subset \omega_+$, $H^0_{-(\mu)} = \overline{H^0_{+(-\mu)}}$, and $\mathfrak{N}_{0(\mu)}$ — a subplace which consists of μ-automorphic functions which have poles on ω_+ and ω_- as well. For $(-\mu)$-automorphic functions the following factorizations are supposed

$$f_{(-\mu)} = \begin{cases} \Pi_{(-\mu')} f^e_{+(-\mu+\mu')}, & z \in \Omega_+, \\ \theta_{(-\mu'')} f^e_{-(-\mu+\mu'')}, & z \in \Omega_-. \end{cases}$$

They imply

$$f_{(-\mu)} H^2_{+(+\mu)} = \Pi_{(-\mu')} H^2_{+(+\mu')},$$
$$f_{(-\mu)} H^2_{-(+\mu)} = \theta_{(-\mu'')} (H^2_{-(+\mu'')} \oplus \mathfrak{N}_{0(+\mu'')}).$$

Similarly, the factorizations for $(\bar{f}_{(-\mu)})^{-1}(\bar{z})$, which correspond to

$$(\bar{f}_{(-\mu)})^{-1} = \begin{cases} \theta_{(-\mu'')} (\bar{f}^e_{-(-\mu+\mu'')})^{-1} \equiv \theta_{(-\mu'')} \hat{f}^e_{+(-\mu+\mu'')}, \\ \Pi_{(-\mu')} (\bar{f}^e_{+(-\mu+\mu')})^{-1} \equiv \Pi_{(-\mu')} \hat{f}^e_{-(-\mu+\mu')}, \end{cases}$$

imply the equations

$$(\bar{f}_{(-\mu)})^{-1} H^2_{+(\mu)} = \theta_{(-\mu'')} H^2_{+(+\mu'')},$$
$$(\bar{f}_{(-\mu)})^{-1} [H^2_{-(\mu)} \oplus \mathfrak{N}_{0(\mu)}] = \Pi_{(-\mu')} (H^2_{-(+\mu')} \oplus \mathfrak{N}_{0(+\mu')}).$$

We use for f the renormalazed perturbation determinant of our scattering problem.

Appendix. Some preliminary facts and assertions from harmonic analysis in the circle and on Riemann surface.

The orthogonal basis in *a* space $L_2(C)$ of all square integrable functions on the unit circle $C = \{ z : |z| = 1 \}$ is formed by powers of independent variable $z = \exp(i\varphi)$, $0 \leqslant \varphi < 2\pi$, $\{z^\nu\}_{\nu=-\infty}^{\nu=\infty}$. The Hardy classes of all functions from $L_2(C)$ which can be continued analytically inside of the unit circle, H^2_+, and outside of it, H^2_-, can be defined as linear hulls

$$H^2_+ = V^\infty_{\nu=0}\{z^\nu\}; \qquad \overset{\circ}{H}{}^2_+ = V^\infty_{\nu=1}\{z^\nu\}; \qquad H^2_- = V^{-1}_{\nu=-\infty}\{z^\nu\} = \overline{(\overset{\circ}{H}{}^2_+)},$$

the reproducing kernel $k_\zeta(z)$ for H^2_+ is given by the formula

$$k_\zeta(z) = (1 - \zeta\bar{z})^{-1}, \qquad |\zeta| < 1,$$

and posesses the property of Cauchy integrals:

$$\langle f, k_\zeta \rangle_{L_2(C)} = f(\zeta), \quad k \in H^2_+.$$

The analytic function S is called inner function in the unit disc $D = \{ z : |z| < 1 \}$, if it is almost everywhere unitary on the unit circle and compressing inside D,

$$|S(z)|\big|_C = 1, \qquad |S(z)|\big|_D < 1.$$

Generally the inner functions are represented as combination of "Blaschke products",

$$B(z) = \prod \frac{\zeta_k - z}{1 - \bar{\zeta}_k z} \frac{\bar{\zeta}_k}{|\zeta_k|}, \quad \sum (1 - |\zeta_k|^2) < \infty,$$

and "singular inner functions" generated by some positive measures m_s supported by unit circle and singular in respect to the Lebesgue measure

$$\theta(z) = \exp\left\{ - \int_C \frac{\zeta + z}{\zeta - z} dm_s \right\}, \quad \operatorname{Var} \bar{m}_s < \infty.$$

Thus every inner function is characterized by the distribution of its zeroes inside D and the density of asymptotic zeroes at the boundary $\partial D = C$ given by m_s:

$$S(z) = z^k B(z)\theta(z).$$

An analytical function f is called outer function in D, if the corresponding multiplication operator is quasiinvertable in H^2_+

$$\overline{fH^2_+} = H^2_+.$$

The outer functions are represented in form

$$f(z) = \exp\left\{ \int \frac{z + \zeta}{\zeta - z} \ln |f(\zeta)| \frac{d\varphi}{2\pi} \right\}.$$

In what followes we consider the analytic functions in D which anyway fulfill the condition

$$\int_C |\ln |f(z)|| d\varphi < \infty.$$

These functions can be factorized in form of the product of inner and outer factors

$$f(z) = S(z) f^e_+(z), \qquad z \in D.$$

The similar representation holds for functions, analytic outside of D. If $f \in H^2_\pm$, then $f^e_\pm \in H^2_\pm$. The shift operator in $L_2(C)$

$$U : f(z) \to z f(z)$$

is unitary and posesses a lattice of invariant subspaces D^s_+, $D^s_+ \subset H^2_+$, parametrized by S, $K = H^2_+ \ominus SH^2_+$:

$$\sigma(T) \equiv \sigma(T_K) = \{\zeta_k\} \cup \operatorname{supp} m_s,$$

the eigenfunctions are given by

$$\psi_e(z) = \frac{S(z)}{\zeta_e - z}, \quad T\psi_e = \zeta_e \psi_e.$$

The biorthogonal system is formed by reproducing kernels which serve eigenfunctions of T^+:

$$\psi^+_e(z) = k_\zeta(z), \quad T^+\psi^+_e = \bar{\zeta}_e \psi^+_e.$$

For multiple zeroes of S the Jordan cell basis is formed by derivatives of ψ, ψ^+, in respect of the spectral parameter ζ.

The system of eigenvectors $\{\psi_e\}$ (as far as ψ^+_e) is complete in K if an only if the singular factor in S is absent. In this case each of systems $\{\psi_e\}, \{\psi^+_e\}$ forms Riesz-basis in K if and only if the following Carleson condition is true

$$\inf_n \prod_{e \neq n} \left| \frac{\zeta_e - \zeta_n}{1 - \bar{\zeta}_e \zeta_n} \right| > 0,$$

the corresponding spectral decomposition for T is given by interpolation series

$$f = \sum \psi_e \langle f, \psi^+_e \rangle (\psi_e(\zeta_e))^{-1}.$$

For properly normalized eigenfunctions systems the additional factor $\psi_e(\zeta_e)$ coincides with Carleson's constant

$$\prod_{n\neq l} \frac{-\zeta_e+\zeta_n}{1-\zeta_e\overline{\zeta_n}} \cdot \frac{\overline{\zeta_n}}{|\zeta_n|}$$

of corresponding, inner functions K

$$K = H^2_+ \ominus SH^2_+.$$

The orthogonal projections on K, SH^2_+ are given by Caushy integrals framed by multiplications on S, S^+ from both sides:

$$P_{SH^2_+} = SP_{H^2_+}S^+ = S\langle *, k_\zeta\rangle S^+,$$
$$P_K = P_{H^2_+} - SH_{H^2_+}S^+.$$

Every simple compressing operator T which is rank one nonunitary

$$\dim(1-T^+T) = 1; \quad T^n \xrightarrow{s} 0, \ n\to\infty,$$

is unitarily equivalent to the corresponding Nagy-Foias functional model generated by some inner function S on the "translation-invariant" subspace $K = H^2_+ \ominus SH^2_+$:

$$T \sim P_K z|K \equiv T_K, \qquad T^n \sim P_k z^n|K, \quad n \geqslant 0.$$

The shifts group $\{z^n\}$ is a unitary dilation of the model contracting semigroup $P_K z^n|K$, $n\geqslant 0$. The spectrum of the generator T (or T_K) coincides with the subset of unit disk $\bar{D}$

$$\sigma_T = \{\, \lambda : s^{-1}(\lambda) \text{ is sungular}\,\}.$$

In a case when the condition

$$T^n \xrightarrow{s} 0$$

is not fulfilled, the absolutely continuous spectrum of T is present. The corresponding spectral analysis see in [10]. All mentioned facts are contained in papers and books [10, 11].

Not so much is known about harmonic analysis of operators on Riemann surfaces. Let Λ be a double surfaces genus r, $r\geqslant 1$. We realize it as a joining of two sheets $\Lambda_+ \cup \Lambda_-$ by $r+1$ two-sided cuts, $\Gamma = \bigcup_{s=1}^{r+1}\gamma_s$. The Green function of the first sheet $\Lambda_+\setminus\Gamma$ with the zero boundary conditions on Γ (Friedrichs extension) is a real part of an analytic function G

$$g(z,\zeta) = \operatorname{Re} G(z,\zeta),$$

defined on the universal cover Ω_+ of Λ_+. The corresponding Blaschke factor defined on Ω as

$$b(z,\zeta) = \exp G(z,\zeta)$$

proves to be μ-automorphic function on Ω in respect to the sheets-overlapping group. It means that, being reduced to Λ b achieves a factor $e^{2\pi i\mu_s}$ when z going around the cut γ_s, $s=1,2,\dots r$, $r+1$

$$b(z,\zeta) = b_\mu(z,\zeta)$$

with only $\mu_1,\dots,\mu_r$ independent $\mu_{2+1} = -\sum_1^r \mu_s\big|_{\bmod Z}$. Here $\mu_s=\mu_s(z)$ is a harmonic function of the parameter ζ on $\Lambda_+\setminus\Gamma$ which fulfilles the following boundary conditions

$$\mu_s(\zeta) = \begin{cases} 1, & \zeta\in\gamma_s, \\ 0, & \zeta\in\gamma_i, \ i\neq s. \end{cases}$$

The corresponding harmonic measure is

$$\frac{1}{2\pi i}\frac{db}{b} = dm$$

and

$$\mu_s(\zeta) = \bar{m}(\gamma_s, \zeta).$$

Generally μ-automorphic functions play crucial role in harmonic analysis on Riemann surface. For given vector $\varkappa$ the Hardy space of all square-integrable functions can be formed on Ω_+ (and on Λ_+), $H^2_{+(\varkappa)}(\Lambda_+)$. The corresponding reproducting kernel $k_\varkappa(z,\lambda)$ is given by averaging over action of overlapping group G of the sheets on Ω_+:

$$k_\varkappa(z,\lambda) = \sum_G e^{2\pi i\langle\varkappa,s\rangle} k(z, G_s\zeta).$$

The $\varkappa$-automorphic inner and outer functions are defined on Ω_+ (and Λ_+) as functions which are transformed under overlapping group's action according to the rule

$$f(G_s z) = e^{2\pi\langle\varkappa,s\rangle} f(z).$$

The invariant subspaces of the multiplication operator by the bounded analytic function θ (0-automorphic on Λ_+) are represented in form

$$D_+ = S_{-\varkappa} H^2_{+(+\varkappa)}(\Lambda_+)$$

where $H^2_{+(+\varkappa)}(\Lambda_+)$ is a space of ($\varkappa$)-automorphic function on Γ which are square-integrable in respect to the fixed harmonic measure m which are analytic on Ω_+ and transformed properly under overlapping group actions, and $S_{-\varkappa}$ is an inner ($\varkappa$)-automorphic function on $\Omega_+(\Lambda_+)$.

The main difference of harmonic analysis on Riemann surface from the harmonic analysis in a unit disc is the nontriviality of overlapping group. The operator of multiplication by bounded analytic (0-automorphic) function in $L_2(\Gamma)$ has a spectrum of multiplicity $\geqslant r+1$. That is why we need another functions, to construct "complete system of quantum obervables" — the family of multiplication operators which is generating family in a double commutant containing multiplication by θ. In a simplest case of two symmetric cuts this family is formed by two functions which are invariant in respect to some automorphisms of the first sheet

$$\Gamma = [-1, -\alpha^2] \cup [\alpha^2, 1];$$

$$G_1 : z \to -z; \qquad G_2 : z \to -\alpha^2/z,$$

$$\theta_0(G_1 z) = \theta_0(z); \qquad \theta_1(G_2 z) = \theta_1(z),$$

$$\theta_1(G_1 z) = -\theta_1(z); \qquad \theta_0(G_2 z) = \theta_0 \frac{\theta_0 - \theta_0(0)}{1 - \theta_0\theta_0(0)}.$$

The sheet's overlapping group is one-parametrical group represented by Mobius transformations, if $\Omega_+ = D$. Then, defining $H^2_\pm$ as linear hull's of basis

$$\{\theta_0^e, \theta_1\theta_0^e\}_{e\in\mathbf{Z}},$$

$$H^2_+(\Lambda) = \bigvee_{e\geqslant 0} \{\theta_0^e, \theta_1\theta_0^e\},$$

$$H^2_-(\Lambda) = \bigvee_{e>0} \{\theta_0^{-e}, \theta_1^{-1}, \theta_1^{-1}\theta_0^{-e}\},$$

$$L_2(\Gamma) = H^2_-(\Lambda_-) \oplus H^2_+(\Lambda_+) \oplus \{1/z\},$$

here the element $1/z = \frac{\theta_1}{\theta_0 - \theta_0(0)} \cdot \text{const}$ is a defect caused by nontrivial (one-parametrical) overlapping group. It is a derivative of the corresponding Abelian differential of the second kind in respect to the harmonic measure m.

Generally for the double Riemann surface genus r there exists r-parametrical overlapping group acting on Ω_+, and the defect of $H^2_+ \oplus H^2_-$ in L_2 is r-dimensional and the base in it is calculated as derivatives of Abelian differential of the second kind in respect to the harmonic measure.

For two cuts the analog of shift group is given by $\{\theta_0^{m_0}\theta_1^{m_1}\}(m_0, m_1) \in Z_2$. This discrete group posesses nontrivial property

$$\theta_1^2 = \theta_0 \frac{\theta_0 - \theta_0(0)}{1 - \theta_0(0)\ \theta_0} \equiv \theta_0 \cdot \theta_0(G_2),$$

caused by specific behaviour of $\theta_{0,1}$ under transformations $G_{1,2}$. But being reduced to some translation-invariant subspace

$$K = H^2_+(\Lambda_+) \ominus S_{-\varkappa} H^2_{+(\varkappa)}(\Lambda_+)$$

it produces contracting semigroups similar to Lax-Phillips ones

$$T_0^{e_0} T_1^{e_1} = P_K \theta_0^{e_0} \theta_1^{e_1} | K.$$

If the inner function $S_{-\varkappa}$ is a Blashke product, then the spectra of T_0, T_1 are discrete and are given by formula $t^s_{0,1} = \theta_{0,1}(\lambda_s)$ where λ_s are the roots of $S_{-\varkappa}$. The corresponding adjoint operators $T^+_{0,1}$ have the complex conjugate eigenvalues and reproducing kernels $k(z, \lambda_s)$ for eigenvectors. The basis property for them is guaranteed by splitted Carleson condition in terms of θ_0

$$\inf_m \prod_{e \neq m} \left| \frac{\theta_0(\lambda_e) - \theta_0(\lambda_m)}{1 - \overline{\theta_0(e)}\theta_0(\lambda_m)} \right| > 0,$$

for $\lambda_e \to [-1, -\alpha^2]$, and for $\lambda_e \to [\alpha^2, 1]$ separately.

The quoted results on harmonic analysis on Riemann surfaces are contained in the book S. Fisher [12] and papers [13, 14, 7, 6]. Two papers [7] are devoted to the harmonic analysis on double Riemann surfaces of higher genus, $r > 1$. This work is done in frames of a grunt N of Academy of Sciences of Russia.

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References

[1] Pavlov B., Fedorov S., *Functional model for Jacobian matrix with baud spectrum*, Probl. Anal., ed. SPB Univ.

[2] Pavlov B., *Lax-Phillips theory and nonphysical sheet for the baud spectrum*, Preprint Mittag-Leffler Inst. no. 36, 1991.

[3] Pavlov B., *Spectral theory of nonselfadjoint differential operators*, Proceedings of the international Congress of Mathematicians, August 16–24, vol. 2, Warszawa, 1983, pp. 1011–1025.

[4] Lax P., Phillips R., *Scattering theory*, Acad. press, New York, 1967.

[5] Pavlov B., Fedorov S., *The group of shifts and harmonic analysis on a Riemann surface of genus one*, Algebra i Analiz 1 (1989), no. 2, 132–168; English transl. in Leningrad. Math. J. 1 (1989).

[6] Stout E. L., *On some algebras of analytic functions on finite open Riemann surfaces*, Math. Zeitschrift **92** (1966), 366–379.
[7] Fedorov S., *Harmonic analysis in a multiply-connected domain* I, II, Mat. Sb. **B 181** (1990), no. 6, 7.
[8] Avron J., Seiler R., Simin B., *Comparisson of dimensions and index of pair of Projectors*, Preprint SFB 288 no. 45, 1993.
[9] Pavlov B., *Nonphysical sheet for Friedrichs model*, St. Petersburg Math. J. **4** (1992), no. 6.
[10] Nikolskij N., *Lectures on shift operator*, Springer, 1986.
[11] Pavlov B., *Dilation theory and spectral analysis of non-selfadjoint differential operators*, Proceedings of the 7th winter school for computing and related topics, Moscow, USSR, 1976.
[12] Fisher S., *Function theory on planar domains*, Whiley, 1983.
[13] Voichik M., *Ideals and invariant subspaces of analytic functions*, Trans. Amer. Math. Soc. **111** (1964), 493–512.
[14] Forelli F., *Bounded holomorphic functions and projections*, Illinois J. Math. **10** (1966), 367–380.

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