Daniel Dement'ev, On the diffraction of high-frequency waves by arbitrary shape cone. Neumann case, *Zap. Nauchn. Sem. POMI*, 1995, Volume 221, 75–82

Daniel Dement'ev

ON THE DIFFRACTION OF HIGH-FREQUENCY WAVES BY ARBITRARY SHAPE CONE. NEUMANN CASE

Dedicated to N. N. Uraltseva on her jubilee

1. Introduction

In this paper we consider a problem of diffraction a plane acoustic wave by a arbitrary shape cone. The spherical wave is created at the result of the scattering by the cone vertex. Our problem consist of a calculation of this wave in a high frequency approximation. The calculations are based on a Smyshlyaev's formula (see [2], [3]). The acoustic wave potential satisfies to Neumann's condition on the boundary of the cone. A same problem (in Dirichlet's boundary condition) was considered in [8]. In this paper we can use the method of calculation near to method used in [8].

2. Description of the problem

Let the diffraction process be described by the classical Helmholtz equation:

$$(\Delta + k^2)u = 0, \quad \text{where} \quad \Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \tag{1}$$

The incident wave is the plane one

$$u^{inc} = e^{-ik(\omega_0, \mathbf{x})}, \quad \mathbf{x} = (x_1, x_2, x_3) \tag{2}$$

Here ω_0 is a unit vector, $(\omega_0, \mathbf{x})$ is the scalar product of ω_0 and $\mathbf{x}$. The vector ω_0 is ortogonal to wave front of the incident wave. The wave u^{inc} propagates in the direction of the vector $-\omega_0$. Let the vertex of the cone Ξ locates at the point C. The incident wave u^{inc} creates a wave G_{diff} scattered by the cone vertex C. The wave G_{diff} has a spherical front. Its center coincides with the cone vertex C. There is a well known formula

$$G_{diff} = 2\pi \frac{e^{ikr}}{kr} f(\omega, \omega_0) + O\big((kr)^{-2}\big) \tag{3}$$

where ω is the direction of the observation ($|\omega| = |\omega_0| = 1$), $r = \sqrt{\sum_{j=1}^{3} x_j^2}$, k – wave number, $f(\omega, \omega_0)$ is a so-called diagram of G_{diff}. The diffraction coefficients for waves scattered by conical points of scatterer are proportional to $f(\omega, \omega_0)$. The $f(\omega, \omega_0)$ has singularities connected with the singularities of a ray field in the domain under consideration. Our calculations are based on the formula for $f(\omega, \omega_0)$ obtained by V. P. Smyshlayev (see [2], [3]).

3. Smyshlyaev's formula

In order to introduce the formula obtained by V. P. Smyshlyaev let us define a Green's function g. Let S^2 be the unit sphere and N be a part of the sphere S^2 outside the cone Ξ, i.e. $N = S^2 \backslash \Xi$ (see Fig. 1).

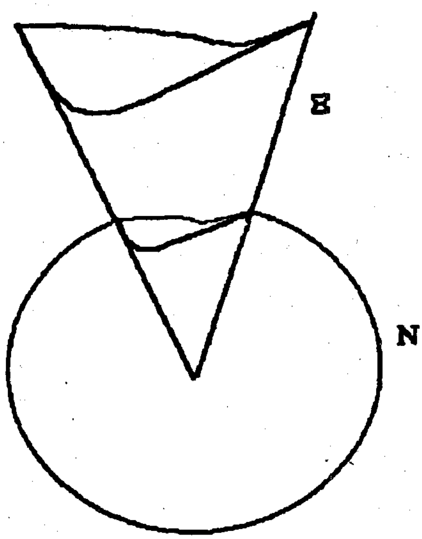

Fig. 1.

Let us assume that $g(\omega, \omega_0)$ is the Green's function for the N, i.e. g is the solution of the problem

$$(\Delta_\omega + \nu^2 - \frac{1}{4})g = \delta(\omega - \omega_0) \quad \omega, \omega_0 \in N \tag{4}$$

$$\frac{\partial g}{\partial \mathbf{m}}\Big|_{\partial N} = 0, \tag{5}$$

where m is a vector tangential to S^2 and ortogonal to ∂N, Δ_ω is the Laplace-Beltrami operator on S^2

$$\Delta_\omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{6}$$

where θ and ϕ are the classical spherical coordinates on S^2. The Smyshlayev formula for $f(\omega, \omega_0)$ in terms of g is:

$$f(\omega, \omega_0) = \frac{i}{\pi}\int\limits_\gamma e^{-i\nu\pi}\, g(\omega, \omega_0, \nu)\, \nu\, d\nu, \tag{7}$$

where γ is a contour on complex plane ν (see Fig. 2)

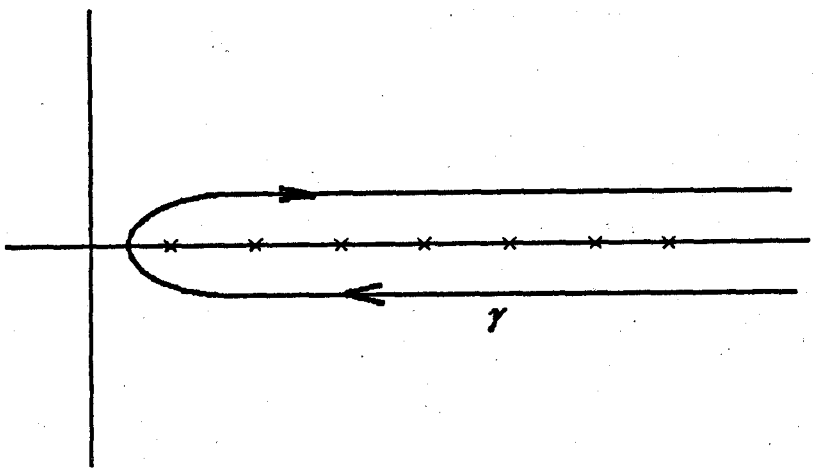

Fig. 2.

The small crosses on the real axis are the poles of function g. The function f has singularities. For example, let

$$\min_{s\in\partial N}(\mathrm{dist}(\omega_0, \mathbf{s}) + \mathrm{dist}(\mathbf{s}, \omega)) = \pi \tag{8}$$

(here $\mathrm{dist}(\mathbf{a}, \mathbf{b})$ is the distance between the points $\mathbf{a}, \mathbf{b}$ ($\mathbf{a}, \mathbf{b} \in S^2$)) on the sphere S^2, then f is singular. We consider the case when the inequality

$$\min_{s\in\partial N}(\mathrm{dist}(\omega_0, \mathbf{s}) + \mathrm{dist}(\mathbf{s}, \omega)) > \pi \tag{9}$$

holds instead of (8). In this case f is regular. V. P. Smyshlayev reduced the formula (7) to a more simple expression (see [2], [3])

$$f(\omega, \omega_0) = \frac{i}{\pi} \int_{-\infty}^{\infty} e^{\tau\pi}\, g_s(\omega, \omega_0, \tau)\, \tau\, d\tau, \tag{10}$$

where g_s is the regular part of the Green's function:

$$g(\omega, \omega_0) = g_0 + g_s = -\frac{1}{4\cosh \pi\tau} P_{-1/2+i\tau}(-\cos\widetilde{\theta}) + g_s, \tag{11}$$

here $\widetilde{\theta} = \operatorname{dist}(\omega, \omega_0)$. The function g_0 is the Green's function for the whole sphere S^2. The function g_s is the solution of the Neumann problem

$$(\Delta_\omega - (\tau^2 + 1/4))g_s = 0 \tag{12}$$

$$\frac{\partial g_s}{\partial \mathbf{m}}\Big|_{\partial N} = -\frac{\partial g_0(\omega, \omega_0)}{\partial \mathbf{m}}\Big|_{\omega\in\partial N} \tag{13}$$

4. Potential theory on the unit sphere

We calculate the function $f(\omega, \omega_0)$ in the case when the inequality (9) is valid. It is convenient to reduce the problem (12)–(13) to Fredholm integral equation of the potential theory and to solve this integral equation using classical methods. We find g_s in the form of the single-layer potential for the operator $\Delta_\omega - (\tau^2 + 1/4)$

$$g_s(\omega, \omega_0) = v = \int_{\partial N} \mu(\psi)\, g_0(\psi, \omega)\, d\sigma, \tag{14}$$

where g_0 is the fundamental solution for the operator $\Delta_\omega - (\tau^2 + 1/4)$ (see (11)), $\psi \in \partial N$, $d\sigma$ is the element of the length of the curve ∂N.

The singularity of the g_0 as $\theta \to 0$ is the same as the singularity of the classical fundamental solution for the Laplace operator $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$:

$$g_0 \sim \frac{1}{2\pi} \log\theta \quad \theta \to 0$$

It allows to prove the theorem on the limiting values for the normal derivation of the single-layer potential

$$\lim_{\substack{\omega' \to \omega\in\partial N \\ \omega' \in \partial N}} -\frac{\partial v(\omega')}{\partial \mathbf{m}_{\omega'}} = -\frac{\partial v_i(\omega)}{\partial \mathbf{m}_\omega} = \frac{1}{2}\mu(\omega) - \int_{\partial N} \mu(\psi) \Big(\frac{\partial g_0(\psi, \omega)}{-\partial m_\omega}\Big)\Big|_{\partial N} d\sigma \tag{15}$$

Where $\frac{\partial}{\partial m}$ is the derivative in the direction of the exterior (for N) normal m to ∂N in the point $\psi \in \partial N$. Normal m is tangent to the sphere S^2. So we can reduce the problem (12–13) to Fredholm integral equation

$$\frac{1}{2}\mu(\omega) - \int_{\partial N} \mu(\psi) \left(\frac{\partial g_0(\psi, \omega)}{-\partial m_\omega} \right)\Big|_{\partial N} d\sigma = \frac{\partial g_0(\omega, \omega_0)}{\partial \mathbf{m}_\omega} \quad \omega \in \partial N \tag{16}$$

It is not difficult to prove that the equation (16) has one and only one solution. If $\mu(\psi)$ is derived from the equation (16) we obtain $v(\omega) \equiv g_s(\omega, \omega_0)$ in the form (14). The kernel of the equation is continuous (but not regular).

5. The stereographic projection

It is not appropriate for our purposes to consider the integral equation since the formulae of the spherical trigonometry are rather complicated. We make classical stereographic projection and reduce our problem to an integral equation on a plane curve $\mathcal{L}$ – the stereographic image of ∂N.

Let us assume that the unit sphere S^2 is given by the equation

$$\xi^2 + \eta^2 + (\zeta - 1)^2 = 1$$

in the coordinate system (ξ, η, ζ). The connection between $(\xi, \eta, \zeta) \in S^2$ and the coordinate system (x, y) on the plane $\mathbf{R}^2$ is expressed by the formulae

$$z = \frac{2(\xi + i\eta)}{2 - \xi}, \; \xi + i\eta = \frac{2z}{z\bar{z} + \eta}, \; \zeta = \frac{2z\bar{z}}{z\bar{z} + 4}, \quad \text{where } z = x + iy, \; i = \sqrt{-1}$$

It is not difficult to obtain the formula for the element of the length on the sphere S^2

$$d\xi^2 + d\eta^2 + d\zeta^2 = 16(4 + z\bar{z})^{-2}(dx^2 + dy^2)$$

The integrand in the formula (16) is

$$\mu(\psi)\frac{1}{4\cosh \tau\pi} P'_{i\tau - 1/2}(-\cos\theta)\frac{\partial(-\cos\theta)}{\partial m_\omega}$$

After rather an easy calculation we get the useful formulae

$$1 - \chi = \frac{8[(x - x_0)^2 + (y - y_0)^2]}{(4 + x^2 + y^2)(4 + x_0^2 + y_0^2)}, \tag{17}$$

$$\frac{\partial \chi}{\partial m_\omega} d\sigma = -\frac{\partial(1 - \chi)}{\partial n} ds = (\chi - 1)\left[-\frac{\frac{\partial}{\partial n} r^2}{r^2} + \frac{\frac{\partial}{\partial n}(4 + x^2 + y^2)}{4 + x^2 + y^2} \right] ds, \tag{18}$$

where $\chi = \cos\theta$, $\theta = \mathrm{dist}(\omega, \psi)$, $r^2 = (x - x_0)^2 + (y - y_0)^2$, $(x_0, y_0), (x, y)$ are the coordinates of the images of ω, ψ on the plane, ds is the element of the length of curve $\mathcal{L}$, $\partial/\partial n$ is the differentiation in the direction of normal to $\mathcal{L}$. We can reduce the integral equation on the plane curve $\mathcal{L}$ using the formulae (17, 18).

6. On calculation $P_{i\tau-\frac{1}{2}}(-\cos\theta)$, $P'_{i\tau-\frac{1}{2}}(-\cos\theta)$

The potential (14), the kernel and the expression in the right hand side of the integral equation (16) depend on $P_{i\tau-\frac{1}{2}}(-\cos\theta)$, $P'_{i\tau-\frac{1}{2}}(-\cos\theta)$. It is necessary to calculate these functions, however some difficulties occur as the parameter τ may be large, and θ is near to 0 or π, because $\theta = 0$ and $\theta = \pi$ are the singular points of the differential equation for $P_{i\tau-\frac{1}{2}}(-\cos\theta)$. We calculate the function $P_{i\tau-\frac{1}{2}}(\cos\theta)$ and $P'_{i\tau-\frac{1}{2}}(\cos\theta)$ in a fixed point using the classical Dirichet-Meler formula. We can find $P_{i\tau-\frac{1}{2}}(\cos\theta)$ in the form:

$$P_{i\tau-\frac{1}{2}}(\cos\theta) = e^{\theta\tau} w_\tau(\theta)$$

Substituting such function in differential equation for $P_{i\tau-\frac{1}{2}}(\cos\theta)$ we transform the equation to the system of two equations for w and w'. The values of $P_{i\tau-\frac{1}{2}}(\cos\theta)$ and $P'_{i\tau-\frac{1}{2}}(\cos\theta)$ in a fixed point are used to get the initial data for w and w'. The functions w and w' do not vary considerably. So it is not difficult to calculate them solving the system by the Runge-Kutta method. We calculate $P_{i\tau-\frac{1}{2}}(\cos\theta)$ and $P'_{i\tau-\frac{1}{2}}(\cos\theta)$ in the small neighborhood of the point $\theta = 0$ using the classical expression for $P_{i\tau-\frac{1}{2}}(\cos\theta)$ in terms of hypergeometrical series. A similar method is used for the neighborhood of the point $\theta = \pi$, however the series appear to be more complicated in that case.

7. The results of the calculations

In the case of the circle cone and the axisymmetrical incident wave there is a explicit formula (see [2, 5]) for $g_s(\omega, \omega_0)$

$$g_s(\omega, \omega_0) = -\frac{1}{4\cos\pi\nu}\frac{P'_{\nu-1/2}(-\cos\theta_1)}{P'_{\nu-1/2}(\cos\theta_1)}P_{\nu-1/2}(\cos\theta)\ , \qquad (19)$$

where $\omega_0 = (0, 0, 1)$, $\omega = (0, 0, \cos\theta)$ is the direction of observation and θ_1 is the cone angle. In the same case the expression for $f(\omega, \omega_0)$ can be easily obtained using the classical Kirchhoff's method .

$$f(\omega, \omega_0) = -i\frac{\sin^2\theta_1\cos\theta_1)}{2\pi\sqrt{(\cos^2\theta_1\ \cos^2\theta - \sin^2\theta_1\sin^2\theta)^3}}(\cos\theta + \sin^2\theta)$$

To test the above method the explicit formula (19) was used. The results of the calculation show that the Kirchoff's approximation can hardly used to calculate $f(\boldsymbol{\omega}, \boldsymbol{\omega}_0)$. Table 1 contains the results of calculation of $f(\boldsymbol{\omega}, \boldsymbol{\omega}_0)$ for the axisymmetrical case ($\boldsymbol{\omega}_0 = (0,0,1)$, $\boldsymbol{\omega} = (0,0,\cos\theta)$) by our method, by the explicit formula and Kirchoff's method.

Table 2 illustrates the results of calculation $f(\boldsymbol{\omega}, \boldsymbol{\omega}_0)$ in case of an elliptical cone

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = x_3^2,$$

where $a = 2\tan(\pi/12)$, $b = 2\tan(\pi/6)$. The wave incidents along the axis $-x_3$ and the point of observation is $\boldsymbol{\omega} = (0,0,\cos\theta)$, $0 \le \theta < \widehat{\theta}$, where $\widehat{\theta}$ is the critical point ($f(\boldsymbol{\omega}, \boldsymbol{\omega}_0)$ has a singularity as $\theta = \widehat{\theta}$). It is seen that $f(\boldsymbol{\omega}, \boldsymbol{\omega}_0)$ increases as θ tends to $\widehat{\theta}$.

There is an explicit formula for $f(\boldsymbol{\omega}, \boldsymbol{\omega}_0)$ in the case of elliptical cone, which is however rather complicated for calculation. Therefore Table 2 is presented here to illustrate the convergence of our method.

Table 1

θ	$\operatorname{Im} f(\omega, \omega_0)$ above method	calculated explicit formula	by Kirchoff method
0	−0.038186	−0.038180	−0.013263
$\pi/24$	−0.038486	−0.038480	−0.013406
$2\pi/24$	−0.039404	−0.039399	−0.013846
$3\pi/24$	−0.041011	−0.041005	−0.014622
$4\pi/24$	−0.043412	−0.043406	−0.015800
$5\pi/24$	−0.046792	−0.046785	−0.017494
$6\pi/24$	−0.051436	−0.051427	−0.019886
$7\pi/24$	−0.057800	−0.057789	−0.023270
$8\pi/24$	−0.066608	−0.066593	−0.028135
$9\pi/24$	−0.079060	−0.079040	−0.035336
$10\pi/24$	−0.097322	−0.097292	−0.046462
$11\pi/24$	−0.125579	−0.125530	−0.064729
$12\pi/24$	−0.172929	−0.172836	−0.097462
$13\pi/24$	−0.262940	−0.262717	−0.164546
$14\pi/24$	−0.477057	−0.476261	−0.337920
$15\pi/24$	−1.329038	−1.322151	−1.091435

ACKNOWLEDGMENTS

This work was supported by the contract DRET ERS 92/ 1441/ A300/ DRET /DS /SR.

Table 2

θ	Number of knots 30	40	50	60	70	80
0	−0.048349	−0.048158	−0.048049	−0.047977	−0.047926	−0.047888
$1\pi/6$	−0.055266	−0.055104	−0.055012	−0.054952	−0.054910	−0.054878
$2\pi/6$	−0.086130	−0.086127	−0.086132	−0.086137	−0.086142	−0.086146
$3\pi/6$	−0.225949	−0.226814	−0.227358	−0.227729	−0.227998	−0.228201

References

1. J. B. Keller, *The geometrical theory of diffraction.* — J. Opt. Soc. Amer. **52** (1962), 116–130.
2. V. P. Smyshlyaev, *On the diffraction of waves by cone at high frequencies.* LOMI preprint E-9-89, Leningrad, 1989.
3. V. P. Smyshlyaev, *Diffraction by cones at high frequencies.* — Wave Motion **12** (1990), 329–339.
4. L. B. Felsen, *Back scattering from wide-angle and narrow cones.* — J. Appl. Phys. **26** (1951), 138–151.
5. L. B. Felsen, *Plane-wave scattering by small-angle cones.* — IRE Trans. Antennas Propagat **5** (1957), 121–129.
6. A. S. Goryainov, *Diffraction of a plane electromagnetic wave propagated along the axis of a cone.* — Radio Eng. Electron **6** (1961), 65–81.
7. J. Cheeger and M. E. Taylor, *On the diffraction of waves by conical singularities.* — Comm. Pure Appl. Math. **35(4)** (1982), 487–529.
8. V. M. Babich, D. B. Dement'ev, and B. A. Samokish, *On the diffraction of high-frequency waves by arbitrary shape cone.* — Wave Motion (in printing).

С.-Петербургский государственный университет

Поступило 19 декабря 1994 г.