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# QUALITATIVE INVESTIGATION OF THE THREE PHASE SOLUTIONS OF SINE–LAPLACE EQUATION

## 1. Introduction

The sine-Laplace (SL) equation

$$\Delta u = \pm \sin u \tag{1}$$

is one of the most interesting real realizations of the sine-Gordon (SG) equation

$$\frac{\partial^2 u}{\partial z_1 \partial z_2} = \sin u \tag{2}$$

and is connected with them through a complex variable substitution. The goal of our investigations is a full description of all real three phase solutions of the SL equation.

For this investigation we need the general formula of multi phase solutions of the SG equation which was given in [1]. The investigation of the three phase solutions of SG equation is given in [2]. The general procedure giving the real solutions of the SL equation and another one is described in [3]. Using this procedure, a class of constant mean curvature $H < 1$ surfaces in three dimensional hyperbolic space was construted in [4]. We apply this method for the description of all real three phase solutions of the SL equation. The first investigations of multi phase solutions of SL equation (two phase solutions) were done in [5, 6] using another method. A preliminary investigation of three phase solutions was done in [19]. The set of our solutions contain smooth and singular solutions. All singularities of the solutions are vortexes. These solutions are interesting in connection with superconductivity and superfluidity [7, 8]. It was found that under some conditions physical systems can have solutions with chain - or lattice - structure of the vortexes [8, 9, 10]. The vortex structure is stable and appears when taking the boundary conditions into account [11, 12].

## 2. Real Solutions of the SL Equation

Let us introduce the main objects for this investigations - the Riemann surfaces and theta functions on them, that have the necessary properties for the reality of the solutions . Let $\Gamma$ be the hyperelliptic curve $\omega^2 = \lambda\prod_{i=1}^{6}(\lambda - E_i)$ given as a two sheeted covering of the Riemann sphere by the function $\lambda$ with ramification at the Weierstrass points $\{P_0, P_\infty, E_i, i = 1, 2, ..., 6\} \in \Gamma$. $\sigma$ is the involution on $\Gamma$ interchanging the sheets. Let $\Gamma$ has an antiholomorphic involution $\tau : \lambda \to 1/\overline{\lambda}$. From the symmetry property of $\tau$, it is easily seen that only 4 types of $\Gamma$ are possible, namely:

$$\text{Type I: } \omega^2 = \lambda\prod_{k=1}^{6}(\lambda - e^{i\varphi_k}), \ E_k = e^{i\varphi_k},$$

$$\text{Type II: } \omega^2 = \lambda(\lambda - E_1)(\lambda - \overline{E}_1^{-1})\prod_{k=1}^{4}(\lambda - e^{i\varphi_k}), \ |E_1| \neq 1,$$

$$\text{Type III: } \omega^2 = \lambda(\lambda - e^{i\varphi_k})(\lambda - e^{-i\varphi_k})\prod_{k=1}^{2}(\lambda - E_k)(\lambda - \overline{E}_k^{-1}), \ |E_1|, |E_2| \neq 1,$$

$$\text{Type IV: } \omega^2 = \lambda\prod_{k=1}^{3}(\lambda - E_k)(\lambda - \overline{E}_k^{-1}), \ |E_k| \neq 1, k = 1, \dots, 3$$

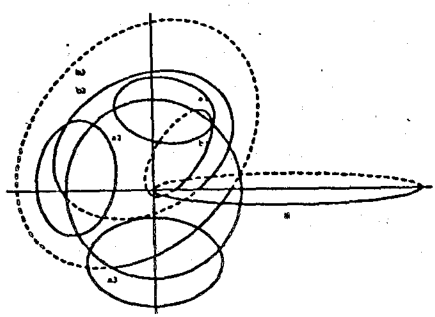
(see Fig. 1).

For the following investigations we need to define a function $\sqrt{\lambda(P)}$ on $\Gamma$. Let us dissect the surface $\Gamma$ along $\mathcal{M}$, where $\mathcal{M}$ is a cut that we build as follows:

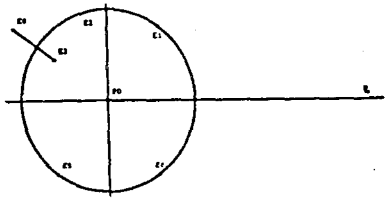
a. Let $\ell_{\surd}(t)$, $t \in [0, 1]$ be a path going from $\lambda = 0$ ($t = 0$, $\ell_{\surd} = 0$) to the point $\lambda = \infty$ ($t = 1$, $\ell_{\surd} = \infty$) on $\mathbb{C}$,

b. lift the $\ell_{\surd}(t)$ on $\Gamma : \mathcal{M}' = \pi_\sigma^{-1}\ell_{\surd}$ ($\pi_\sigma$ is a projection $\pi_\sigma : \Gamma \to \mathbb{C}$).

Type I:



Type II:



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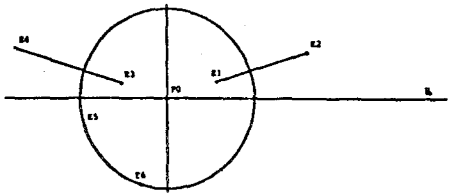

Type IV:

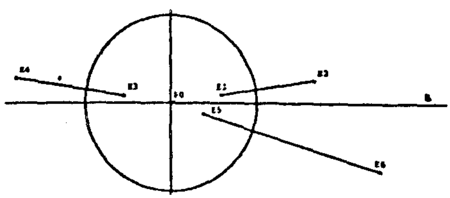

Fig. 1 The four types of the Riemann surfaces giving the real solutions of the SL equation. The canonical basis $\{a, b'\}$ of the homology group $H^1(\Gamma)$ for the Riemann surfaces of type I are given, the integration contours $a_1, a_3, b_1, b_3$ are clockwise and $a_2, B_2, \mathcal{M}$ are anti clockwise. For the Riemann surfaces of type II, III, and IV we get $\{a, b'\}$ after evident transformations.

Consider a closed cycle $\mathcal{M}' - \sigma\mathcal{M}'$ and deform it into a contour $\mathcal{M} \in \Gamma$ so that $\mathcal{M}$ fulfills the conditions:

a. $P_0, P_\infty, E_i \notin \mathcal{M}$ $(i = 1, 2, ..., 6)$, i.e., $\mathcal{M}$ does not pass through the branch points of $\Gamma$;

b. $\pi_\sigma \mathcal{M} \in \mathbb{C}$ has no self intersections.

Then, on the cutted $\widehat{\Gamma} = \Gamma \setminus \mathcal{M}$, $\sqrt{\lambda(P)}$ is a well defined function and we take it as local parameter in the neighborhoods $\mathcal{D}_0, \mathcal{D}_\infty$ of the points

$P_0, P_\infty \in \Gamma$:

$$\mu_0 = \mu_0(P) = \sqrt{\lambda(P)},\ P \in \mathcal{D}_0\ (P_0 \in \mathcal{D}_0)$$

and

$$\mu_\infty = \mu_\infty(P) = \sqrt{\lambda(P)},\ P \in \mathcal{D}_\infty\ (P_\infty \in \mathcal{D}_\infty).$$

Denote by $\{a, b\}$ a canonical basis of the homology group $H^1(\Gamma)$ and by $dU(P)$ the corresponding normalized differentials of first kind:

$$\oint_{a_j} dU_k = 2\pi\imath\delta_{kj},\ k, j = 1, 2, 3.$$

The we get the $B$-matrix of $\Gamma$ as follows:

$$B_{kj} = \oint_{b_j} dU_k,\ k, j = 1, 2, 3.$$

In this notations we can write:

$$\mathcal{M} = (\Delta_1, b) + (\Delta_2, a),\ \Delta_1, \Delta_2 \in \mathbb{Z}^3\ (\text{in}\, H^1(\Gamma)\ ).$$

For any $Z \in \mathbb{C}^3$ we take the useful notation $Z = \left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]$ with $\gamma, \delta$ defined from $Z = 2\pi\imath\gamma + B\delta,\ \gamma,\ \delta \in \mathbb{R}^3$.

It is known [1, 3] that the multi phase solutions of the SG equation have the form

$$u(z_1, z_2) =$$

$$= \frac{2}{\imath} \ln \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](Z(z_1, z_2) + Z^0)}{\Theta\left[\begin{smallmatrix}\alpha+\Delta_1/2\\ \beta+\Delta_2/2\end{smallmatrix}\right](Z(z_1, z_2) + Z^0)} \exp\left\{\frac{\imath\pi}{4}(\Delta_1, \Delta_2) + \frac{\imath\pi}{2}(\Delta_1, \alpha)\right\} \tag{3}$$

with $Z(z_1, z_2) = V_0 z_1 + V_\infty z_2,\ Z \equiv (Z^1, Z^2, Z^3),\ V_0 = -\frac{dU(P)}{d\mu_0}\Big|_{P=P_0}$, $V_\infty = -\frac{dU(P)}{d\mu_\infty}\Big|_{P=P_\infty}$, $Z^0 \in \mathbb{C}^g,\ z_1, z_2 \in \mathbb{C},\ g$ is the genus of $\Gamma$. In our case $g$ equals 3 and the theta function is defined as

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(Z|B) =$$

$$= \sum_{N \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}(B(N + \alpha), (N + \alpha)) + (Z + 2\pi\imath\beta, N + \alpha)\right\}.$$

**Remark.** Here, and henceforth we make the convention that $\widehat{\zeta}$ ($\widehat{\zeta} = \pi\zeta$) denotes an transformation on $\mathbb{C}^3$ and $\zeta^*$ ($\zeta^* dU(P) = dU(\zeta P)$) on differentials induced by some transformation $\zeta$ on $\Gamma$.

After the substitution $z_1 = \pm\overline{z_2}$, i.e., $Z(x,y) = zV_0 \pm \bar{z}V_\infty$, $z = x + \imath y \in \mathbb{C}$, we get a real solution of (1) if the Riemann surface has an anti holomorphic involution $\tau : \lambda \to 1/\overline{\lambda}$ and $Z_0$ is symmetric under the transformation $\tau : Z^0 \mp \tau Z^0 = \left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]$, $\gamma, \delta \in \mathcal{Z}^g$. To each sign "$\mp$" there corresponds one class of solutions, with even argument $\tau Z = Z$ and with odd argument $\tau Z = -Z$ of the theta function. After the necessary transformations of (3) we can write the solution of the SL equation (1) in the form [3]

$$u(x,y) = 4\arg\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(Z(x,y)|B) + \frac{\pi}{2}(\Delta_1, \Delta_2) \tag{4}$$

For the following considerations we must take a concrete basis of $H^1(\Gamma)$. Let $\{a, b'\}$ be chosen as in fig.1.

We take $b = b' + \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} a$. Then we take $\ell_{\surd} = \mathbb{R}_+$ and, consequently, $\mathcal{M} = \left[\begin{smallmatrix}\Delta_1\\ \Delta_2\end{smallmatrix}\right] = \begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 1\end{bmatrix}$. Such a basis of $H^1(\Gamma)$ is very convenient under the application of $\tau$. It can be seen, that

$$\tau a = -a \tag{5}$$

$$\tau b = b + P_\tau a. \tag{6}$$

with

$$P_\tau = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} + d, \tag{7}$$

where $d = \mathrm{diag}(111)$ for Type I; $d = \mathrm{diag}(101)$ for Type II; $d = \mathrm{diag}(010)$ for Type III; $d = \mathrm{diag}(000)$ for Type IV and $\tau^* dU = \overline{dU}$, $Im\overline{B} = -\pi P_\tau \in \pi\mathbb{Z}^{g\times g}$.

**Lemma.** *With the help of the relations* (5)–(7) *we can determine the set of all real three phase solutions of the SL equation:*

(a) *Type I: for even arguments $\tau Z = Z$ the theta function in* (4) *can have only the characteristics*

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1/2 & 1/2\\ 0 & 0 & 0\end{bmatrix}$$

*and for odd arguments of the theta functions the characteristics*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix},$$

(b) Type II:

$$\text{for} \quad \tau Z = Z: \quad \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{for} \quad \tau Z = -Z: \quad \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \end{bmatrix}$$

(c) Type III:

$$\text{for} \quad \tau Z = Z: \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{for} \quad \tau Z = -Z: \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 1/4 \end{bmatrix},$$

(d) Type IV: for $\tau Z = Z$ we have no real solutions,

$$\text{for} \quad \tau Z = -Z: \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

**Notation:** If we consider, for instance, the type I solutions, we see that there are possible 4 different characteristics of theta functions for $\tau Z = Z$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (8)$$

and analogously 4 for $\tau Z = -Z$. But we see, that the last three components of the set of real solutions are simply connected: we can get from one of them all others if we take another numeration for $a_i, b_i$ cycles in the homology basis $\{a, b\}$. This is the reason to take as essentially different solutions one of them only. The Lemma gives the complete classification of all real three phase solutions of the SL equation.

## 3. The Singularities of the Three Phase Solutions of the SL Equation

By the specified form of the solutions (4) we get bounded solutions on a bounded region only. In this case there is a single type of singularities possible, namely that the theta functions vanishes and the function

$u(x, y)$ is not defined in some points. Then the solution is not unique in the neighbourhood of these points. After the rounding this point on some path the function requires an additional contribution of $\pm 4\ 2\pi k$ (this is the topological charge). In our case, for a Riemann curve $\Gamma$ of genius 3, $k$ equals 1 and the topological charge equals $\pm 8\pi$.

For the explicit description of the singular solutions we use the Abel representation of the points $z \in \mathbb{C}^g$. We define a Abel mapping as usual [13]:

$$\mathcal{A}(P) = \int_{P_\infty}^{P} dU, \quad P \in \Gamma, \quad \mathcal{A} : \Gamma \to J(\Gamma) = \mathbb{C}^g / \{2\pi \imath I, B\}$$

and $P_\infty \in \Gamma$ is a basepoint. As it is known, for any $Z \in \mathbb{C}^g$ there exists an divisor $D = \sum_{i=1}^{g} Q_i \in \Gamma$, that $Z = \mathcal{K}_{P_\infty} + \mathcal{A}(D)$, where $\mathcal{A}(D) = \sum_{i=1}^{g} \int_{P_\infty}^{Q_i} du$ , $\mathcal{K}_{P_\infty}$ is a vector of the Riemann constants with the basepoint $P_\infty \in \Gamma$. $\mathcal{K}_{P_\infty}$ is constant for each $Z \in \mathbb{C}^g$ and $D$ and, on the other hand, is a function of the basepoint $P_\infty$ and the homology basis $\{a, b\}$ only.

From the classical Riemann theorem [14] it follows that for $\Theta(Z|B) = 0$ there exists a divisor $D = \sum_{i=1}^{g-1} Q_i$ and all zeros of the theta function are given by

$$Z = \mathcal{K}_{P_\infty} + \mathcal{A}(D).$$

Because $Z$ satisfies $\tau Z = \pm \pi Z$, the divisor $D(x, y)$ has the symmetry property [15]

$$\mathcal{A}(D + \tau_1 D) = 0,$$

where $\tau_1 = \tau$ for even and $\tau_1 = \sigma\tau$ for odd $Z$. Therefore the resulting property for the divisor points $Q_i$ takes the form

$$Q_i \in \mathcal{D} \Longleftrightarrow \left( \{\tau_1 Q_i = \sigma Q_i\} \bigcup \{\exists Q_i' \in \mathcal{D} \, : \, Q_i' = \tau_1 \sigma Q_i\} \right).$$

In our case, we have $g = 3$ and $D = Q_1 + Q_2 + Q_3$. So, the solutions (4) have singularities in the following cases only:

a. if $Q_1 = P_\infty, Q_2 = P_0, Q_3 = \tau_1 \sigma Q_3$

or

b. the points $Q_{1,2} \in \mathcal{D}$ are connected by means of $Q_1 = \sigma Q_2$, $Q_3 = \tau_1 \sigma Q_3$.

In each case the set of zeros of the theta function is a real one dimensional subvariety $S_{Q_3}$ which can be parameterized by the point $Q_3$ lying on the imagine oval of $\tau_1$, i.e., $\tau_1 Q_3 = \sigma Q_3$. The subvariety $S_{Q_3}$ belongs to the 3-dimensional spaces $J_{\pm}(\Gamma) = \{Z : \tau Z = \pm Z\}$ and has an intersection with the plane $\mathcal{L}_{\pm} = \{Z : Z = \bigcup_{x,y \in \mathbb{R}} Z_{\pm}(x,y),\ \tau Z_{\pm} = \pm Z_{\pm}\}$ (i.e., the linear span on the vectors $V_0, V_\infty$ ). The crossing points are the singular points of the solution (4).

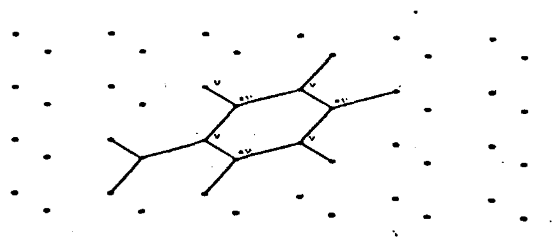

Fig. 2 The singularities of the solution build a chain of vortexes and anti vortexes. In the symmetrical case the singularities form a hexagonal structure..

It is evident that only by $\Gamma$ of the type I we can get smooth solution of (1) if we take $Q_i, i = 1,2,3$ one of each lying on the cut $(E_i, E_{i+1}), i = 1,2,3$. In all other cases we get singularities. The singular points form chains of vortexes (with topological charge $8\pi$) and of anti vortexes ( with topological charge $-8\pi$). The subvariety $S_{Q_3}$ is invariant under the shift by $\tau_1$ on the real period:

$$\omega_k = \oint_l dU_k \in \mathbb{C}^3, k = 1,2,3,\ l \in H^1(\Gamma)$$

and, consequently, the singular points of the chains build an almost periodic structure. After the shift on the vector

$$\omega_\Delta = \frac{1}{2} \oint_{\mathcal{M}} dU = \mathcal{A}(P_0)$$

(for the solutions of the types I, II, and III) we get the chain of anti vortexes. This is the main reason for the solution to have in general the topological charge zero. For the solutions of the type IV, $\omega_\Delta$ does not belong to $J_{\pm}(\Gamma)$, but the function $u(x,y) = -u(-x,-y)$ is odd and, consequently, the topological charge is zero too. In fact, if in the point $(x_0, y_0)$ is a vortex, than in the point $(-x_0, -y_0)$ is an anti vortex.

## 4. PERIODICAL SOLUTIONS OF THE SL EQUATION

It is well known [13] that the lattice of periods of the theta function has the form

$$\Lambda = \{2\pi\imath M + BN\ ,\ M, N \in \mathbb{Z}^g\}.$$

The theta function itself is almost periodic by shifting on some nodes along this lattice, however the solution (4) is a periodic function. So we get a periodic solution only in the case when the plane $\mathcal{L}$ possess as minimum one of the nodes $\Lambda$ (not equal to zero). If two linear independent nodes of $\Lambda$ belong to $\mathcal{L}$ than the solution is some double periodic function. In ref. [16] was shown that one can expect periodicity of solutions of the SG equation if $\Gamma$ possesses one holomorphic involution. This is also true in our case. This involution $s$ acts on $\{a, b\}$ like

$$sa = Sa, sb = S^{-\top}b\ ,\ \text{where } S = -\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and correspondingly

$$s^* dU(p) = (S^\top)^{-1} dU(P), V_0 = S^\top V_\infty. \tag{9}$$

It is easy to prove that due to the involution $s : \lambda \longrightarrow 1/\lambda$ we can reach a one periodic solution for all four types of $\Gamma$. The action $s$ on any point $Z = \left[\begin{smallmatrix}\gamma \\ \delta\end{smallmatrix}\right] \in \mathbb{C}^3$ can be given by [13]:

$$\widehat{s}\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix} = -\begin{bmatrix} \gamma_3 & \gamma_2 & \gamma_1 \\ \delta_3 & \delta_2 & \delta_1 \end{bmatrix}. \tag{10}$$

Because the action $\tau$ in $\mathbb{C}_3$ for the case $Z \in J_+$ has the form

$$\widehat{\tau}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} -\gamma \\ \delta - \Phi\gamma \end{bmatrix},\ \Phi = -\begin{bmatrix} \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \\ 1 & 1 & \epsilon \end{bmatrix} \quad \epsilon = 0, 1. \tag{11}$$

It is evident that $s$ and $\tau$ commute on the subvarieties $J_\pm(\Gamma)$ and $S$ has the eigenvectors

$$q_s^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ q_s^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ q_s^3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

with the eigenvalues $-1$ for $q_s^1$ and $q_s^2$ and $+1$ for $q_s^3$. In the supplementary space $J_-(\Gamma)$, $\tau$ acts according to (9) and the eigenvectors of $S$ have the form

$$r^k = \begin{bmatrix} \gamma_k \\ \Phi\,\delta_k/2 \end{bmatrix},\ k = 1,2,3,\ \gamma_1 = [1\ 0\ 1],\ \gamma_2 = [0\ 1\ 0],\ \gamma_3 = [1\ 0\ -1]$$

with eigenvalues $-1$ for $r_1, r_2$ and $+1$ for $r_3$. Consequently, if the vector $Z_+ \in \mathcal{L}_+$, the plane $\mathcal{L}_+$ is invariant under the action of $S$, we have

$$Z_+(x,y) = Z_+(x,-y),$$

i.e.,, for the eigenvalue $+1$ we have $Z(x,0)$ and for the eigenvalue $-1$ $Z(0,y)$. But we have only one eigenvector $S$ which belongs to $J_+(\Gamma)$ with eigenvalue $+1$, namely $q_s^3$. Therefore we have that one period of the solution is equal to

$$T_1^+ = Z(X_\mathrm{T}, 0) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}. \tag{12}$$

For the imaginary (with respect to $\tau$) vector $Z(x,y) \in \mathcal{L}_-$ we have in the same way

$$Z(x,y) = Z(-x,y)$$

with exactly two eigenvectors of the matrix $S : Z(0,y)$ with eigenvalue $+1$ and $Z(x,0)$ with eigenvalue $-1$. To the space $J_-(\Gamma)$ belongs one eigenvector with eigenvalue $-1$ only, this is $r^3$ and, consequently, there is one period

$$T_1^- = Z(0, y_\mathrm{T}) = \begin{bmatrix} \alpha_3 \\ \Phi\alpha_3/2 \end{bmatrix}.$$

This means

$$T_1^- = Z(0, y_\mathrm{T}) = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

for the solutions of the type I and II and

$$T_1^- = Z(0, y_\mathrm{T}) = \begin{bmatrix} 1 & 0 & -1 \\ 1/2 & 0 & -1/2 \end{bmatrix} tag14$$

for the solution of the type III and IV.

As result we obtain solution (4) with one periodical lattice of singular points consisting of two periodical chains, the chain of vortexes and the chain of anti vortexes. The minimal distances through any two singular points equals $d \leqslant \min(\omega, \omega_\Delta) > 0$. From (7) and (8) we conclude that $\Gamma$ covers the curve $\Gamma_+ = \Gamma/s$,

$$\omega_+^2 = \lambda_+^2 - 3\lambda_+ - 2e,\ \omega_+ = \omega/\lambda^2,\ \lambda_+ = (\lambda+1)/\lambda,\ dU_+ = \left[ dU, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right]$$

and the B-matrix must have some symmetric properties. If we denote (as in [13]):

$$dv_+ = dU_1 - dU_3,\ dW_1 = dU_1 + dU_3,\ dW_2 = \\ = dU_2,\ f_0 = \frac{dv_+}{\sqrt{3}d\sqrt{\lambda_+}^{-1}(P+)}|_{P+=P_\infty^\pm},$$

and

$$\Pi_{11} = \int_{b_1} dW_1,\ \Pi_{12} = \frac{1}{2}\int_{b_2} dW_1,\ \Pi_{22} = \frac{1}{2}\int_{b_2} dW_2,\ \tau_+ = \oint_{b_+} dv_+ \qquad (15)$$

$$a = (\tau_+ + \Pi_{11})/2,\ b = (\tau_+ - \Pi_{11})/2,$$

then the B-matrix has the form

$$B = \begin{bmatrix} a & \Pi_{12} & b \\ \Pi_{12} & 2\Pi_{22} & \Pi_{12} \\ b & \Pi_{12} & a \end{bmatrix}$$

and the Prym matrix $\Pi$ equals

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix}.$$

Now we can degenerate the theta function [13] and turn from three dimensional theta functions to products of one and two dimensional theta functions:

$$\Theta\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}[Z \mid B]$$

$$= \Theta\begin{bmatrix} (\alpha_1-\alpha_3)/2 \\ \beta_1-\beta_3 \end{bmatrix}[Z_+ \mid 2\tau_+]\ \Theta\begin{bmatrix} (\alpha_1+\alpha_3)/2 & \alpha_2 \\ \beta_1+\beta_3 & \beta_2 \end{bmatrix}[Z_- \mid 2\Pi]$$

$$+\Theta\begin{bmatrix} (\alpha_1-\alpha_3+1)/2 \\ \beta_1-\beta_3 \end{bmatrix}[Z_+ \mid 2\tau_+]\ \Theta\begin{bmatrix} (\alpha_1+\alpha_3+1)/2 & \alpha_2 \\ \beta_1+\beta_3 & \beta_2 \end{bmatrix}[Z_- \mid 2\Pi] \qquad (16)$$

where

$$Z_+ = Z^1 - Z^3,\ Z_- = \begin{bmatrix} Z^1 + Z^2 \\ Z^2 \end{bmatrix}.$$

It is evident that the solution $u(x, y)$ (4) with the theta functions given by (16) has a one period. For the Riemann surfaces of type I and IV we can obtain solutions with a second period. We take the Weiserstrass points $E_i, i = 1, 2, ..., 6$ symmetrically to the axes $\arg(\pm\lambda) = k\pi/3$, i.e., $\Gamma$ has an additional symmetry transformation $\varphi : \lambda \to \exp(2\pi\imath/3)\lambda$. Then we conclude that

$$\omega_2 = \lambda(\lambda^6 + 2e\lambda^3 + 1),$$

where the parameter $e = \cos(3\varphi) < 1$ for the Riemann surface of type I and $e = (R^3 + R^{-3})/2$ for type IV.

It is easy to see that for each point of $\mathbb{C}^3$ holds

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_2 & \alpha_3 & \alpha_1 \\ \beta_2 & \beta_3 & \beta_1 \end{bmatrix} \tag{17}$$

and $\varphi$ commutes with $\tau$ so that $\mathcal{L}_\pm$ and $J_\pm(\Gamma)$ are invariant under $\varphi$. This means that for any $Z \in \mathcal{L}_\pm$ we have

$$\widehat{\varphi} Z(z) = Z\left(e^{2\pi\imath/3} z\right). \tag{18}$$

Using this fact together with the relation (16) we conclude that $\varphi$ interchanges the different components of real even (odd) solutions of type I, so that, for instance, in (8) the functions two, three, and four are identified. The first one gives us a smooth solution

$$u(z) = 4 \arg \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (Z(z)|B) + \pi,\ z = x + \imath y$$

with $Z$ even or odd. From the property (18) we have

$$u\left(e^{2\pi\imath/3} z\right) = u(z) \tag{19}$$

i.e., this solution has two periods. For the even cases from (12) and (19) follows that periods equals

$$T_1^+ = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix},\ T_2^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

for the odd case from (12) and (18)

$$T_1^- = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},\ T_2^- = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The same considerations gives us for type IV one $\varphi$-symmetrical solution (there exists one real solution only, it is odd):

$$u(z) = 4 \arg \Theta \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 \end{bmatrix} (Z(z)|B)$$

with periods

$$T_1^- = \begin{bmatrix} 1 & 0 & -1 \\ 1/2 & 0 & -1/2 \end{bmatrix}, T_2^- = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1/2 & 1/2 \end{bmatrix}.$$

This solution is singular.

To get the localization of the singular points we need the value of $\mathcal{K}_{P_\infty}$ for our homology basic $\{a, b\}$ (Fig. 1) and the basepoint $P_\infty$. In this case

$$\mathcal{K}_{P_\infty} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

and the singular points lying at

$$Z_s = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{A}(P_0 + P),\ P \in \{P : |\lambda(P)| = 1\}.$$

In the $(x, y)$-plane the singular points form a hexagonal structure with nodes in the points:

$$M_{nm}^{\nu} = (n + 2/3)\omega_1 + (m + 1/3)\omega_2,\ M_{nm}^{a\nu} = (n + 1/3)\omega_1 + (m + 2/3)\omega_2,$$

$n, m \in \mathbb{Z}$, where $\omega_1 = \imath T_1^-$, $\omega_2 = T_2^-$ , the indexes $\nu$ stay for the points of the vortex and $a\nu$ for the points of the anti vortex (Fig. 2).

We see that this structure is invariant under the rotation $\varphi$. For the solution of type I and type IV with two periods we can degenerate the three dimensional theta function and get a formula for the solution, which contains Jacobi theta functions only.

Naturally, we see that $\Gamma$ covered the elliptic curve

$$\Gamma_0 = \Gamma/\varphi, \pi_\varphi : \Gamma \to \Gamma_0,\ \Gamma = \{(\omega, \lambda) : \omega^2 = \lambda(\lambda^6 + 2e\lambda^3 + 1)\},$$

$$\Gamma_0 = \{(\omega_0, \lambda_0) : \omega_0^2 = \lambda_0(\lambda_0^2 + 2e\lambda_0 + 1)\},\ dv_0 = \left[du, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right],$$

where $\omega_0 = \omega\lambda$ and $\lambda_0 = \lambda^3$. Under this condition the $B$-matrix of $\Gamma$ is symmetrical:

$$B = \begin{bmatrix} h & c & c \\ c & h & c \\ c & c & h \end{bmatrix} \tag{20}$$

where $h$ and $c$ are partially defined in (15) and can be simplified using $\varphi^* dU = \Phi^{-1} dU$:

$$dU_0 = dU_1 + dU_2 + dU_3 = dW_1 + dW_2.$$

For the elliptic curve $\Gamma_0$ we have

$$\oint_{a_0} dU_0 = 2\pi\imath \quad \text{and} \quad \oint_{b_0} dU_0 = \sum_{k=1}^{3} \oint_{b_k} dU_k = h + 2c = \tau_0$$

and in the formula (19) we have than

$$h = (\tau_+ + \Pi_{00})/2 = 2\tau_+ + \tau_0, \; c = \tau_0 - \tau_+.$$

Now we can degenerate the theta function to a sum of products of one dimensional theta functions using the same method as in [13] or [11]. For the two dimensional theta function in (16) we have

$$\begin{aligned} &\Theta\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} [Z \mid B] \\ &= \Theta\begin{bmatrix} (\alpha_1 - \alpha_2)/2 \\ \beta_1 - \beta_2 \end{bmatrix} [Z_b \mid 2\tau_b] \, \Theta\begin{bmatrix} (\alpha_1 + \alpha_2)/2 \\ \beta_1 + \beta_2 \end{bmatrix} [Z_p \mid 2\tau_p] \\ &+ \Theta\begin{bmatrix} (\alpha_1 + \alpha_2 + 1)/2 \\ \beta_1 - \beta_2 \end{bmatrix} [Z_b \mid 2\tau_b] \, \Theta\begin{bmatrix} (\alpha_1 + \alpha_2 + 1)/2 \\ \beta_1 + \beta_2 \end{bmatrix} [Z_p \mid 2\tau_p]. \end{aligned} \tag{21}$$

where $\tau_p = 3\tau_b, \tau_b = 6\tau_0, Z_b = Z_-^1 - Z_-^2, \; Z_p = Z_-^1 + Z_-^2$.

After the necessary transformations we get the final formulas for the two periodical solutions of the type I and IV:

Type I, even case, $\tau Z = Z$:

$$\begin{aligned} u(x,y) = 4 \arg \Bigg\{ &\sum_{\substack{k=\{0,1,2\} \\ n=\{0,1\}}} \Theta\begin{bmatrix} k/3 \\ 0 \end{bmatrix} (0 \mid 3\tau_0) \\ \Theta\begin{bmatrix} m_1/4 + n/2 \\ 0 \end{bmatrix} (t_1 \mid 2\tau_b) \, &\Theta\begin{bmatrix} m_2/4 + k/3 + n/2 \\ 0 \end{bmatrix} (t_2 \mid 2\tau_p) \Bigg\} + \pi \end{aligned} \tag{22}$$

where $m_1, m_2 \in \{0, 1\}$ and $t_1 = \imath f_0 \sqrt{3}\left(y - \sqrt{3}x\right), t_2 = i f_0 3(\sqrt{3}y + x)$,

Type I, odd case, $\tau Z = -Z$

$$\begin{aligned} u(x,y) = 4 \arg \Bigg\{ &\sum_{\substack{k=\{0,1,2\} \\ n=\{0,1\}}} \Theta\begin{bmatrix} k/3 \\ m_1/2 \end{bmatrix} (0 \mid 3\tau_0) \\ \Theta\begin{bmatrix} n/2 \\ m_2/2 \end{bmatrix} (t_1 \mid 2\tau_b) \, &\Theta\begin{bmatrix} k/3 + n/2 \\ m_3/2 \end{bmatrix} (t_2 \mid 2\tau_p) \Bigg\} + \pi \end{aligned} \tag{23}$$

for the symmetrical solutions $m_1 = m_2 = m_3 = 0$ in (22), (23),

Type IV, odd case only, $\tau Z = -Z$

$$u(x,y) = 4\arg\left\{ \sum_{\substack{k=\{0,1,2\}\\ n=\{0,1\}}} \Theta\begin{bmatrix} k/3 \\ 3/4 \end{bmatrix}(0 \mid 3\tau_0) \right. \quad (24)$$

$$\left. \Theta\begin{bmatrix} n_1/4+n/2 \\ 0 \end{bmatrix}(t_1 \mid 2\tau_b)\,\Theta\begin{bmatrix} k/3+n/2 \\ 0 \end{bmatrix}(t_2 \mid 2\tau_p) \right\} + \pi$$

In both odd cases holds $t_1 = \sqrt{3}f_0(x - \sqrt{3}y), t_2 = 3f_0(\sqrt{3}x + y)$.

The solution (22) is a smooth function (Fig. 3), the solutions (23) and (24) have a hexagonal structure of singularities (Fig. 2).

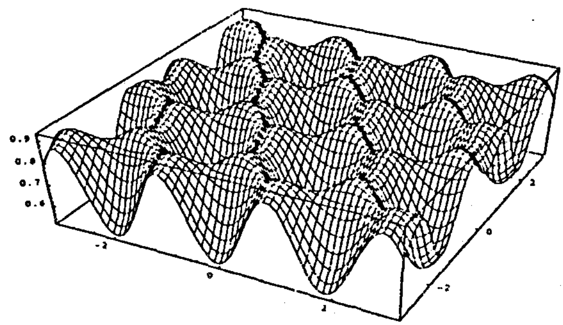


Fig. 3. The hexagonal structure of the smooth solutions of the Type I with even argument for parameter $e = 0.5$.

## THE DESCRIPTION OF THE COMPUTING PROCEDURE

In the cases (22)–(24) we can investigate all properties of the solutions using the well known properties of elliptic functions. In all other cases one needs a method for computing firstly the parameters $B$, $V_0$, $V_\infty$ on some Riemann survace of genus 3 and, secondly, a procedure to calculate effectively the Riemann Theta functions with sufficient precision. The second problem was solved by the investigation of the solutions of KdV and KP equations [17, 18]. However, the calculation of the parameters of the Riemann surfacees was done in [17, 18] using the Schottky uniformization and in [2] using the symmetrical uniformization. These methods are in our case either non effective or non applicable. For this reason it was necessary to write a program for the direct calculation of all necessary parameters directly on the Riemann surface.

In this section we give the prescription of this program. It is written in "*Mathematica*"*. The procedure goes as follows. We fix the position of the ramification points $E_i$, $i = 1, \dots, 7$ which define at once the Riemann surface. At the same time we assign to each ramification point a number so that the line connecting consecutively this points has no self intersections. After that each pair of neighbouring points is bounded by a ellipse with focuses in this points. The distance from a focus point to the apex is taken to be $\frac{1}{10} \min |E_i - E_j|$, $i \neq j$, $i, j = 1, \dots, 7$. The set of this ellipses is used as the integration contours for the differentials

$$\frac{dz}{w}, \frac{zdz}{w}, \frac{z^2dz}{w} \tag{25}$$

(see Fig. 4).

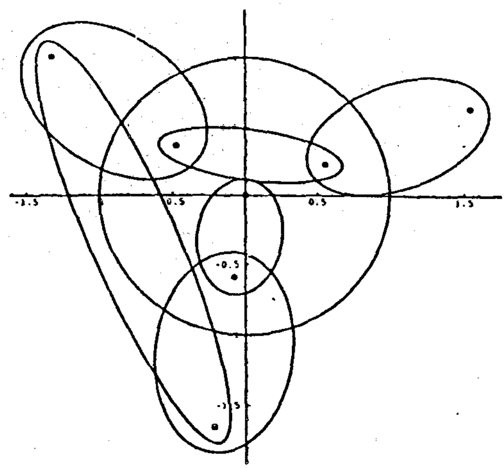


Fig. 4. The integration path for a typical Type IV solution.

Before starting to integrate these differentials along this ellipses, we check wheather the differential on the whole lies on the same sheet of the Riemann surface. If this is not the case, it must be corrected. After that we have to integrate along each ellips one function given in parametric

***Mathematica* is a trademark of Wolfram Research Inc., 100 Trade Center D5rive, Champaign, Il 61820-7237, USA form. Resulting from this integration procedure we obtain a $3 \times 6$ matrix of numbers

$$\left\{\mathcal{I}_{\substack{i=1,2,3 \\ k=1,\ldots,6}}\right\}, \tag{26}$$

whereby the result of the integration of $\frac{dz}{w}$ along the 6 ellipses is $\mathcal{I}_{1k}$, over $\frac{zdz}{w}$ is $\mathcal{I}_{2k}$ and over $\frac{z^2dz}{w}$ is $\mathcal{I}_{3k}$. The ellipses are constructed in such a way, that they have 2 intersection each with another only. From these 6 intersection points we fix one and consider the remaining five. In each of this point we calculate the integrand and compare them with the functions of the other ellipses. If they coincide, i.e., the integration pathes ly on the same sheet of the Riemann surface, we assign to this point the value "1", otherwise "-1". This procedure is done for each of the differentials (25). In addition we compare also the integrated functions (25) in the same points and set up a unique definition for all roots in (25).

After that we turn from the matrices (26) to the matrices $A_c$ and $B_c$ for the canonical basis $\{a, b\}$. The transition matrix $P_1$ in this case has the form:

$$P_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then we normalize the matrices and obtain $A_1 = 2\pi\imath$, $B_1 = 2\pi\imath A_c^{-1} B_c$. By the transformation $B = DB_1 + 2\pi\imath P$, $D = \mathrm{diag}(-1, 1, 1)$ and $P = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ we reach the normalized matrices $A$ and $B$ for the cycles $\{a, b\}$. The vectors $V_0$ and $V_\infty$ equals

$$V_0 = -2DA_c^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{and} V_0 = -2DA_c^{-1} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

In our case they are complex conjugate $V_0 = \overline{V_\infty}$. So, all parameters of our solution are defined and we can construct the function $u(x, y)$ with the help of the developed by the second author program for the calculation of many dimensional Theta functions which was already used in [17, 18].

## References

1. V. A. Kozel, V. P. Kotljarov , — Dokl. Akad. Nauk Ukrai. SSR A No. 10 (1976 ), 878–81 .
2. M. V. Babich, L. A. Bordag , — Communications of JINR 5-85-204 Dubna (1985 ).
3. V. M. Babich , (Engl. transl.1991 Leningrad Math. J. 2 No. 3 507-21) , — Algebra i analis **2** **3** (1990 ).
4. M. V. Babich , *Constant Mean Curvature $H_1$ Surfaces in Hyperbolic Space.* — TU Berlin, FB 3, Preprint Reihe Mathematik No. 316/1992 (1992 ).
5. I. A. Tajmanov , — Mat. zametki **47, No. 1** (1990 ), 147–56 .
6. I. A. Tajmanov , — Mat. zametki **47, No. 3** (1990 ), 100–5 .
7. M. V. Babich, E. Sh. Gutshabash, V. D. Lipovskij A. V. Rybin, M. A. Sall, A. O. Smirnov, , *High temperature superconductivity.* Actual problems Coll. of Scientific Works 3 (Leningrad Isd LGU) .
8. D. R. Tilley, J. Tilley , *Superfluidity and Superconductivity.* New York-London , 1974 .
9. E. P. Gross , — Phys. Rev. **106** (1957 ), 161–200 .
10. D. San-Jam, G. Sarma, E. Thomas , *Superconductivity of the Second Art.* vol. 363 (in Russian), Mir , 1970 .
11. V. K. Tkachenko , — JETF **50, No. 6** (1966 ), 1572–1585 .
12. V. K. Tkachenko , — JETF **56, No. 5** (1969 ), 1763–1765 .
13. J. D. Fay , *Theta Functions on Riemann Surfaces.* — Lect. Not. Math. **352** (1973 ), Springer Verlag .
14. D. Mumford , *Tata lectures on theta.* Boston Birkhäuser , 1983 .
15. M. V. Babich , — Algebra i Analis **3, No. 1** (1991 ), 57–66 .
16. A. I. Bobenko , — Funkz. analis **18, No. 3** (1984 ), 74–75 .
17. A. I. Bobenko, L. A. Bordag , — Zap. Nauchn. Semin. LOMI **165** (1987 ), 31–41 .
18. A. I. Bobenko, L. A. Bordag , — J. Phys. A Math. Gen. **22** (1989 ), 1259–1274 .
18. M. V. Babich, L. A. Bordag , *The Three Phase Solutions of Sine-Laplace Equation.* Preprint NTZ 11/92, Leipzig University .

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